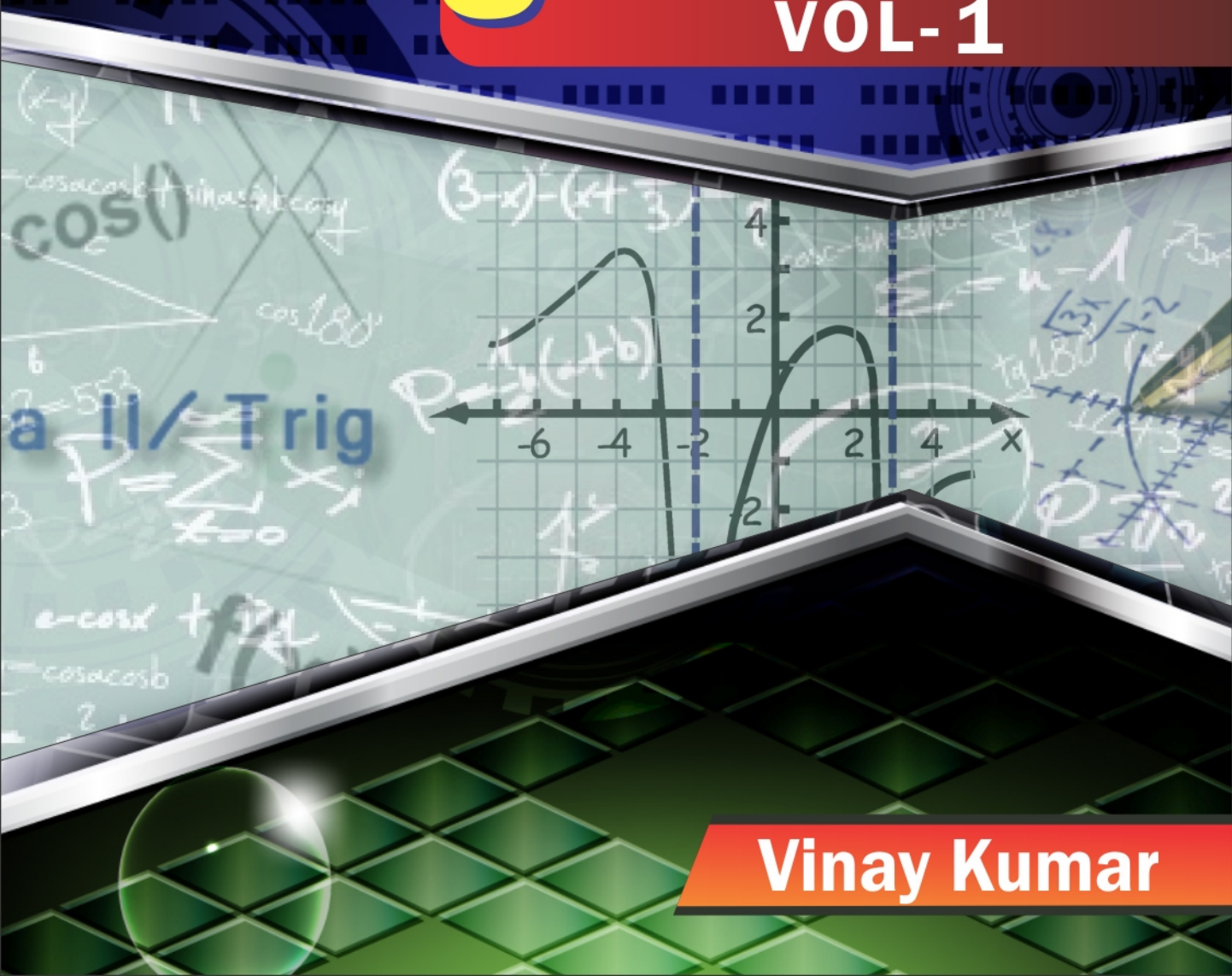
McGraw Hill Education WE Series

Comprehensive Algebra

VOL-1



Vinay Kumar

www.pdfworld.net

McGRAW HILL EDUCATION WINNING WE EDGE SERIES

Comprehensive Algebra

VOL-1

pdfworld.net

www.pdfworld.net

www.pdfworld.net

About the Author

Vinay Kumar (VKR) graduated from IIT Delhi in Mechanical Engineering.

Presently, he trains JEE aspirants at VKR classes, Kota, Rajasthan.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

McGRAW HILL EDUCATION WINNING WE EDGE SERIES

Comprehensive Algebra VOL-1

Vinay Kumar
B.Tech., IIT Delhi

pdfworld.net

McGraw Hill Education (India) Private Limited
NEW DELHI

McGraw Hill Education Offices

New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

www.pdfworld.net

www.pdfworld.net

McGraw Hill Education (India) Private Limited

Published by McGraw Hill Education (India) Private Limited,
P-24, Green Park Extension, New Delhi 110 016.

Comprehensive Algebra Vol-1

Copyright © 2013, McGraw Hill Education (India) Private Limited.
No part of this publication may be reproduced or distributed in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise or stored in a database or retrieval system without the prior written permission of the publishers. The program listings (if any) may be entered, stored and executed in a computer system, but they may not be reproduced for publication.

This edition can be exported from India only by the publishers,
McGraw Hill Education (India) Private Limited.

ISBN (13): 978-1-25-906422-7
ISBN (10): 1-25-906422-0

Vice President and Managing Director—McGraw-Hill Education: India: *Ajay Shukla*
Deputy General Manager—Test Prep and School: *Tanmoy Roychowdhury*
Publishing Manager—Test Prep: *K N Prakash*
Assistant Sponsoring Editor—*Bhavna Malhotra*
Asst Manager (Developmental Editing): *Anubha Srivastava*
Asst Manager—Production: *Medha Arora*
Production Executive: *Dharmender Sharma*
Product Specialist: *Vikas Sharma*
General Manager—Production: *Rajender P. Ghansela*
Manager—Production: *Reji Kumar*

Information contained in this work has been obtained by McGraw Hill Education (India), from sources believed to be reliable. However, neither McGraw Hill Education (India) nor its authors guarantee the accuracy or completeness of any information published herein, and neither McGraw Hill Education (India) nor its authors shall be responsible for any errors, omissions, or damages arising out of use of this information. This work is published with the understanding that McGraw Hill Education (India) and its authors are supplying information but are not attempting to render engineering or other professional services. If such services are required, the assistance of an appropriate professional should be sought.

Typeset at Script Makers, 19, A1-B, DDA Market, Paschim Vihar, New Delhi 110 063, and text and cover printed at Rajkamal Electric Press, Plot No.2, Phase IV, HSIIDC, Kundli, Sonepat, Haryana - 131028

Cover Designer: Rajesh Pandey

RQXLCRCORYRDA

www.pdfworld.net

www.pdfworld.net

Preface

Algebra is a vital topic of mathematics and comprises nearly 25% of questions asked in JEE. A thorough understanding and mastery of this topic is therefore essential for success in IIT-JEE and other engineering entrance examinations.

Due to the vastness of the topic, it has been divided into two volumes.

Comprehensive Algebra Vol-1 for JEE (Main & Advanced) has been conceived and developed as a textbook-cum-practice book. Starting off with an indepth treatment of the theory portion, it is supplemented with illustrative examples and practice problems (graded according to difficulty levels).

There are 5 chapters in this volume. Each chapter is dealt thoroughly, first with an extensive approach to theory followed by illustrative examples, practice problems, etc.

The collection of carefully selected Objective Questions will help in imparting practice in problem solving. The review exercises comprise problems of moderate level of difficulty while the Target exercises are tougher problems. For tutors and trainers, the book would be of great help as it has numerous problems graded by difficulty levels which can be given as assignments.

My sincere thanks are due to all my teachers who have been my constant motivation and have given valuable suggestions and recommendations. I am especially thankful to Mr Devendra Kumar, Mr Shashyesh Bhardwaj and Mr Avinash Kumar Singh for their support and cooperation in bringing out this book.

Suggestions for improvement are always welcome and shall be gratefully acknowledged.

Vinay Kumar

www.pdfworld.net

www.pdfworld.net

pdfworld.net

www.pdfworld.net

www.pdfworld.net

Contents

Preface *v*

1. Basics, Modulus and Logarithm **1.1–1.138**

1.1 Real Number System 1.1
1.2 Complex Number System 1.11
1.3 Intervals 1.14
1.4 Mathematical Induction 1.17
1.5 Divisibility Test 1.19
1.6 Ratio and Proportion 1.21
1.7 Polynomials 1.23
1.8 Function 1.25
1.9 Linear Function 1.27
1.10 Linear Inequalities 1.29
1.11 Method of Intervals 1.31
1.12 Rational Function 1.36
1.13 Partial Fractions 1.39
1.14 Irrational Function 1.43
1.15 Irrational Inequalities 1.47
1.16 Modulus Function 1.50
1.17 Modulus Inequalities 1.55
1.18 Properties of Modulus 1.59
1.19 Graphs of Modulus Functions 1.61
1.20 Exponential and Logarithmic Function 1.68
1.21 Laws of Logarithm 1.72
1.22 Exponential Equation 1.80
1.23 Exponential Inequalities 1.83
1.24 Logarithmic Equation 1.85

www.pdfworld.net

www.pdfworld.net

1.25 Logarithmic Inequalities 1.91
1.26 Greatest Integer Function 1.98

Target Problems for JEE Advanced 1.102
Objective Exercises 1.114
Review Exercises for JEE Advanced 1.124
Target Exercises for JEE Advanced 1.126
Previous Years' Questions (JEE Advanced) 1.128
Answers 1.130

2. Quadratic Equations **2.1–2.115**

2.1 Introduction 2.1
2.2 Quadratic Equation 2.2
2.3 Sum and Product of Roots 2.6
2.4 Identity 2.9
2.5 Transformation of Equation 2.10
2.6 Symmetric Functions of Roots 2.12
2.7 Polynomial Function of Root 2.14
2.8 Nature of Roots 2.15
2.9 Rational Roots 2.18
2.10 Common Roots of Quadratic Equations 2.20
2.11 Graph of Quadratic Function 2.23
2.12 Quadratic Inequality 2.29
2.13 Range of Quadratic Function 2.35
2.14 Rational Function 2.37
2.15 Resolution of a Second Degree Expression in X and Y 2.40
2.16 Location of Roots 2.42
2.17 Solving Inequalities Based on Location of Root 2.49
2.18 Polynomials 2.51
2.19 Remainder Theorem 2.52
2.20 Roots of a Polynomial Equation 2.54
2.21 Polynomial Equation Reducible to Quadratic Equation 2.59
2.22 Relation between Roots and Coefficients 2.62
2.23 Transformation of Polynomial Equation 2.64
2.24 Symmetric Functions of Roots 2.66
2.25 Trigonometrical Method of Solving Cubic Equation 2.68
2.26 Common Roots in Polynomial Equations 2.70
2.27 Multiple Roots 2.71
2.28 Intermediate Value Theorem 2.72
2.29 Descartes' Rule of Signs 2.73
2.30 Algebraic Interpretation of Rolle's Theorem 2.75
Target Problems for JEE Advanced 2.77
Objective Exercises 2.95

pdfworld.net

www.pdfworld.net

www.pdfworld.net

Review Exercises for JEE Advanced 2.104
Target Exercises for JEE Advanced 2.106
Previous Years' Questions (JEE Advanced) 2.108
Answers 2.111

3. Sequence and Series **3.1–3.112**

3.1 Introduction 3.1
3.2 Series 3.6
3.3 Arithmetic Progression (A.P.) 3.9
3.4 Properties of A.P. 3.16
3.5 Geometric Progression (G.P.) 3.17
3.6 Infinite Geometric Series 3.23
3.7 Harmonic Progression (HP) 3.29
3.8 Arithmetic, Geometric & Harmonic Means 3.31
3.9 Arithmetico-geometric Progression 3.38
3.10 Summation of Series 3.40
3.11 Recurring Series 3.45
3.12 Method of Subtraction 3.46
3.13 Method of Differences 3.49
3.14 Summation of Terms Having Factors in A.P. 3.56
3.15 Method of Undetermined Coefficients 3.60
3.16 Miscellaneous Methods of Summation 3.61
3.17 A.M. $\geq$ G.M. $\geq$ H.M. Inequalities 3.63
3.18 Miscellaneous Inequalities 3.70

Target Problems for JEE Advanced 3.78
Objective Exercises 3.90
Review Exercises for JEE Advanced 3.100
Target Exercises for JEE Advanced 3.102
Previous Years' Questions (JEE Advanced) 3.104
Answers 3.108

4. Permutation and Combination **4.1–4.113**

4.1 Introduction 4.1
4.2 Fundamental Principles of Counting 4.2
4.3 Factorial 4.7
4.4 Permutation 4.8
4.5 Permutation with Repetition 4.13
4.6 Further Illustrations 4.15
4.7 Combination 4.18
4.8 Permutations of Alike Objects 4.27
4.9 Permutation under Restrictions 4.29
4.10 Non-Consecutive Selection 4.36
4.11 Combination under Restrictions 4.38
4.12 Combination of Alike Objects 4.43

www.pdfworld.net

www.pdfworld.net

4.13 Division into Groups 4.45
4.14 Inclusion–Exclusion Principle 4.50
4.15 Circular Permutation 4.51
4.16 Linear Equations with Unit Coefficients 4.55
4.17 Applications of Coefficient Method 4.59
4.18 Factors of Natural Number 4.70
4.19 Derangements 4.73
4.20 Number of Subsets 4.76
4.21 Number of Functions 4.77
4.22 Geometrical Problems 4.78

Target Problems for JEE Advanced 4.84
Objective Exercises 4.94
Review Exercises for JEE Advanced 4.104
Target Exercises for JEE Advanced 4.106
Previous Years' Questions (JEE Advanced) 4.108
Answers 4.111

5. Binomial Theorem **5.1–5.84**

5.1 Introduction 5.1
5.2 Binomial Theorem 5.4
5.3 General Term of a Binomial Expansion 5.7
5.4 Middle Term(s) and Rational Terms 5.11
5.5 Ratio of Two Consecutive Terms 5.12
5.6 Numerically Greatest Term 5.15
5.7 Divisibility Using Binomial Theorem 5.18
5.8 Analysing Integral and Fractional Part of a Number 5.21
5.9 Comparison Between Two Numbers 5.23
5.10 Summation of Series Involving Binomial Coefficients 5.24
5.11 Algebraic Method 5.31
5.12 Differentiation Method 5.33
5.13 Integration Method 5.36
5.14 Series Involving Product of Binomial Coefficients 5.39
5.15 Series Involving Product of Binomial Coefficients with Variable Upper Indices 5.44
5.16 Binomial Theorem for Rational Index 5.45
5.17 Multinomial Theorem 5.49
5.18 Exponential Series 5.51
5.19 Logarithmic Series 5.53

Target Problems for JEE Advanced 5.55
Objective Exercises 5.66
Review Exercises for JEE Advanced 5.75
Target Exercises for JEE Advanced 5.77
Previous Years' Questions (JEE Advanced) 5.79
Answers 5.83

www.pdfworld.net

www.pdfworld.net

1 Basic, Modulus and Logarithm

1.1 REAL NUMBER SYSTEM

Natural Numbers

The concept of natural numbers originated from the necessity to count objects. We can compare two natural numbers and see which of the two is the greater. All natural numbers arranged in the increasing order form a series of natural numbers, the first number being unity, the second number two, the third number three, and so on. Each natural number has a place of its own in the series. We shall denote the series of natural numbers by the letter N.

Thus, the counting numbers 1, 2, 3, 4,..... are called natural numbers.

$$N = \{1, 2, 3, 4, ...\}$$

Whole Numbers

Let us consider the number, zero. It is designated as 0. Zero is not a natural number and is considered to be a predecessor of all the natural numbers. The set of natural numbers including zero is called set of whole numbers. It is designated as W. Thus W = {0, 1, 2,.....}.

Integers

In the set of natural numbers, for instance, we cannot subtract 5 from 3. It is therefore necessary to extend the set of natural numbers. We shall introduce new numbers into consideration, natural numbers with the minus sign, i.e. numbers of the form –m, where m is a natural number, and shall call them negative integers.

The set of numbers consisting of all natural numbers, zero, and all negative numbers, is known as the set of integers and is denoted by the letter I or Z.

The numbers ..... –3, –2, –1, 0, 1, 2, 3..... are called integers.

Thus, I (or Z) = {...–3, –2, –1, 0, 1, 2, 3.....}

1. Natural numbers are sometimes called positive integers and are also denoted by I^+ or Z^+.
2. The set of whole numbers is also called as the set of non-negative integers.
3. The set of negative integers, denoted by I^- and consists of {....., –3, –2, –1}

www.pdfworld.net

www.pdfworld.net

4. The set of non-positive integers is $\{\ldots, -3, -2, -1, 0\}$
5. Integers which are divisible by 2 are called **even integers**.
   i.e. $0, \pm 2, \pm 4, \ldots$
6. Integers, which are not divisible by 2 are called as **odd integers**.
   i.e. $\pm 1, \pm 3, \pm 5, \pm 7 \ldots$

Prime and Composite Numbers

The set of natural numbers consists of unity, prime, and composite numbers. A natural number which is larger than unity is a prime number if it has no divisors except for unity and itself. A natural number which is larger than unity is a composite number if it has at least one divisor different from unity and itself.

Let 'p' be a natural number, 'p' is said to be prime if it has exactly two distinct factors, namely 1 and itself.

i.e. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,.....

Let 'a' be a natural number, 'a' is said to be to be composite if, it has atleast three distinct factors.

1. '1' is neither prime nor composite.
2. '2' is the only even prime number.
3. Numbers which are not prime are composite numbers (except 1)
4. '4' is the smallest composite number.

The Fundamental Theorem of Arithmetic

For every natural number p>1 there is a unique factoring.

1. If a number m is a divisor of numbers n_1 and n_2, then m is a divisor of the sum $n_1 + n_2$.
2. If a number m is a divisor of numbers n_1 and n_2, and $n_1 > n_2$, then the number m is a divisor of the difference $n_1 - n_2$.
3. Let a and b be two integers. A positive integer d is said to be the greatest common divisor (G.C.D.) of a and b if
   (i) it is a common divisor of a and b ;
   (ii) every common divisor of a and b is a divisor of d.
4. L.C.M. (p, m) x G.C. D. (p, m) = p.m.

Co-prime Numbers

Two natural numbers (not necessarily prime) are **co-prime**, if their G.C.D. is unity.

For example (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8) are co-prime numbers.

These numbers are also called **relatively prime** numbers.

1. Two distinct prime numbers are always co-prime but converse need not be true.
2. Consecutive numbers are always co-prime numbers.
3. If natural numbers p_1 and p_2 are co-prime and a natural number p is divisible both by p_1 and by p_2, then p is divisible by the product p_1p_2.

Twin Prime Numbers

If the difference between two prime numbers is two, then the numbers are called as twin prime numbers.

For example {3, 5}, {5, 7}, {11, 13}}, {17, 19}, {29, 31} are twin prime numbers.

Theorem

There are infinitely many prime numbers.

Proof :

We shall show that the number of primes in N is infinite.

Suppose the number of primes in N is finite. Let $\{p_1, p_2, \ldots, p_n\}$ be the set of primes in N such that $p_1 < p_2 < \ldots < p_n$. Consider $n = 1 + p_1p_2 \ldots p_n$. Clearly n is not divisible by any one of $p_1, p_2, \ldots,$

www.pdfworld.net

www.pdfworld.net

p_n. Hence n itself is a prime on n has a prime divisor other than $p_1, p_2....p_n$. This contradicts that the set of primes is $\{p_1, p_2....,p_n\}$. Therefore the number of primes in N is infinite.

Definition

Let a, b and m be integers, m > 0. Then we say that **a is congruent to b modulo m** if (a – b) is divisible by m. We denote this by $a \equiv b \pmod{m}$.

For example $6 \equiv 1 \pmod{5}$, because $5 \mid (6 - 1)$;

q | r implies that q divides r

$10 \equiv 0 \pmod{5}$, because $5 \mid (10 - 0)$;

$5 \equiv 0 \pmod{5}$, because $5 \mid (5 - 0)$;

$7 \equiv 2 \pmod{5}$, because $5 \mid (7 - 2)$;

$13 \equiv 7 \pmod{6}$

1. An even number a can be written in the form a = 2q, where q is an integer.
2. An odd number a can be written in the form a = 2q +1, where q is an integer.
3. An integer a which is exactly divisible by three can be written in the form a = 3q, where q is an integer.
4. An integer a which is not exactly divisible by three can be written in one of the following forms :
   $a = 3\ell + 1$ or $a = 3n + 2$, where ℓ and n are integers.
5. An integer a which is exactly divisible by a natural number k can be written in the form a = kq, where q is an integer.
6. An integer a which is not exactly divisible by a natural number k can be written in the form a = kq + r, where r is one of the numbers 1, 2, ..., (k – 1), and q is an integer.

In accordance with the divisibility of integers by a given natural number k, the set of integers can be divided into k **classes**.

For instance, if k = 2, then the set of all integers can be divided into two classes, e.g. even numbers and odd numbers.

The set of all integers can also be divided into three classes:

(a) numbers which are multiples of the number three, i.e. numbers of the form 3q, where q is an integer,

(b) numbers which when divided by three yield unity as the remainder, i.e. numbers of the form $3\ell + 1$, where l is an integer,

(c) numbers which when divided by three yield two as the remainder, i.e. numbers of the form 3n + 2, where n is an integer.

It is now clear how we can divide the set of integers into 4 classes, 5 classes, and so on.

We find that division of integers into classes helps us to solve problems.

Common Fractions

The number which is equal to the kth part of unity (k is a natural number larger than unity) is designated as $\frac{1}{k}$. If we take that part m times (m is natural number), then the resulting new number is designated as $\frac{m}{k}$.

$\frac{p}{q}$ is a fraction or a quotient of the natural numbers p and q, the number p being the numerator of the fraction and q the denominator.

Let us consider fractions with a minus sign, i.e. numbers of the form, where m and n are natural numbers. The fraction $-\frac{m}{n}$ is sometimes called a number opposite to the fraction $\frac{m}{n}$.

Rational Numbers

Let us now consider the set of rational numbers which consists of all positive fractions, zero, and all fractions with the minus sign. We can assume that each number belonging to this set is the ratio of an integer to a natural number. We shall therefore assume that this set consists of numbers of the form $\frac{p}{q}$, where q is a

www.pdfworld.net

www.pdfworld.net

natural number and p is an integer. The set of rational numbers is denoted by the letter Q.

Decimal Number System

Ten symbols are introduced in this system, which are called digits, the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9 denoting the first nine natural numbers and the symbol 0 denoting zero. In this number system, the number ten is designated as 10 and each natural number p is represented as

$p = a_n . 10^n + a_{n-1} . 10^{n-1} + ... + a_2 . 10^2 + a_1 . 10 + a_0$, where n is a number from the set of whole numbers, a_n is one of the numbers 1, 2, 3, ..., 9.

Another notation is usually used to write the number p which is based on the principle of the place values of digits.

The notation 2705 means that the number consists of two thousands, seven hundreds, a zero tens, and five units. i.e.

$2705 = 2.10^3 + 7.10^2 + 0.10 + 5$.

Terminating Decimal

We can write, for instance, the fractions

$$\frac{3721}{100}, \frac{21}{10000}, \frac{131}{1000} \text{ as } 37.21, 0.0021, 0.131.$$

A fraction written in this form is known as a terminating or finite decimal fraction.

Any terminating decimal fraction can be easily changed to a common fraction.

For example, $0.34 = \frac{34}{100} = \frac{17}{50}$.

Any fraction $\frac{p}{q}$, where the natural number q does not have any prime divisors other than 2 and 5 can be written as a terminating decimal fraction.

Non-terminating Repeating Decimal

A non-terminating repeating decimal fraction is a decimal in which the decimal point is followed by infinitely many digits, with one digit or an ordered set of digits, beginning with a certain place after the decimal point, recurring.

For example, 4.27131313..... is a non-terminating repeating decimal fraction with the period 13.

$$4.21131313.... = 4.27\overline{13},$$

$$0.454545.... = 0.\overline{45}.$$

To change a non-terminating repeating decimal fraction to a common fraction, we must subtract the number preceeding the first period from the number preceeding the second period and make this difference the numerator, and write the digit 9 in the denominator as many time as there are digits in the period, and add as many zeros after the nines as there are digits between the decimal point and the first period. For example, $0.117\overline{20}$

$$= \frac{11720 - 1172}{90000} = \frac{10548}{90000} = \frac{1172.9}{9.10000}$$

$$= \frac{293.4}{4.2500} = \frac{293}{2500}$$

$$0.\overline{45} = \frac{45 - 0}{99} = \frac{5.9}{9.11} = \frac{5}{11},$$

$$4.27\overline{13} = \frac{42713 - 427}{9900} = \frac{42286}{9900}$$

$$= \frac{2.21143}{2.4950} = \frac{21143}{4950}$$

Irrational Numbers

A need arises for introducing new numbers, distinct from rational numbers, such as, for instance, the number whose square is 2. The symbol ≈ is read " is approximately equal to". The reason that $\sqrt{2} \approx 1.414214$ is that $(1.414114)^2$ is not exactly 2.

Earlier we considered non-terminating periodic decimal fractions. We shall now extend this concept by introducing new numbers which we shall call non-terminating decimal fractions.

Every rational number can be written as a non-terminating periodic decimal fraction. It is therefore natural to term an irrational number a

www.pdfworld.net

www.pdfworld.net

number which can be written as a non-terminating non-periodic decimal fraction.

Irrational numbers are numbers which cannot be expressed in p/q form and their set is denoted by Q^c or Q'. (i.e. complementary set of Q). These numbers cannot be expressed as recurring decimals. For example, $\sqrt{2}$, $1+\sqrt{3}$, e, π, etc. are irrational numbers.

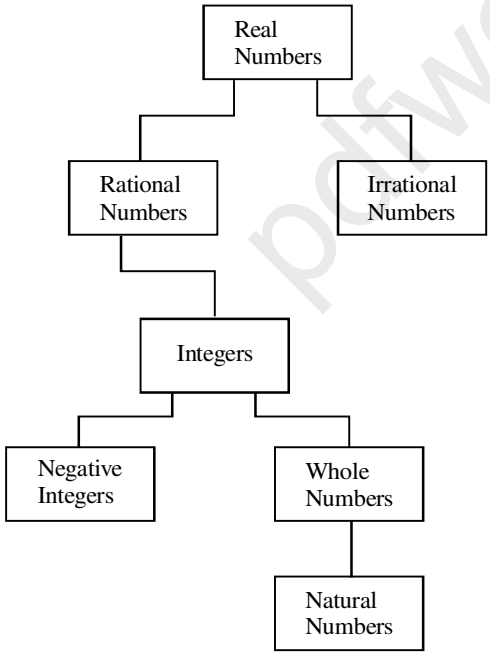
$e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$.

Real Numbers

When we combine the set of all irrational numbers with the set of all rational numbers, we obtain the set of all real numbers.

The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus, $R = Q \cup Q^c$.

The figure shows a diagram of the subsets of the real numbers.



The sets that make up the real numbers are summarized in the following Table. We refer to these sets as subsets of the real numbers, meaning that all elements in each subset are also elements in the set of real numbers.

➤ **Example 1.** List the numbers in the set

$$\left\{-7,\ -\frac{3}{4},\ 0,\ 0.\overline{6},\ \sqrt{5},\ \pi,\ 6\frac{1}{4},\ 7.3,\ \sqrt{81}\right\}$$

that belong to each subset of the real numbers: (i) Natural numbers, (ii) Whole numbers, (iii) Integers, (iv) Rational numbers, (v) Irrational numbers, and (vi) Real numbers.

➤ ***Solution***

(i) Natural numbers : The only natural number in the set is $\sqrt{81}$ since $\sqrt{81} = 9$. (9 multiplied by itself is 81.)

(ii) Whole numbers : The elements of the set that are whole numbers are 0 and $\sqrt{81}$.

(iii) Integers : The elements of the set that are integers are $\sqrt{81}$, 0, and –7,

(iv) Rational numbers : The rational numbers are

$$-7\ \ 6\frac{1}{4}\left(6\frac{1}{4}=\frac{24}{4}\right),\ \ 7.3\left(7.3=7\frac{3}{10}=\frac{73}{10}\right)$$

and $\sqrt{81}\left(\sqrt{81}=\frac{9}{1}\right)$.

(v) Irrational numbers : The irrational numbers in the set are $\sqrt{5}(\sqrt{5} \approx 2.236)$ and π ($\pi \approx 3.14$). Both $\sqrt{5}$ and π neither terminate nor have repeating patterns.

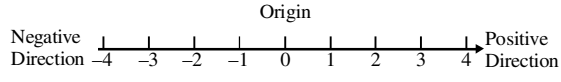
(vi) Real numbers : All the numbers in the set are real numbers.

The Real Number Line

The real number line is a graph used to represent the set of real numbers.

Real numbers can be represented as points of a line which is called as real number line.

An arbitrary point, called the origin, is labeled 0; units to the right of the origin are positive and units to the left of the origin are negative.



The real number line

www.pdfworld.net

www.pdfworld.net

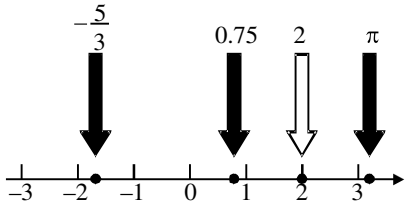
Important Subsets of the Real Numbers

Name	*Description*	*Examples*
Natural Numbers	{1, 2, 3, 4, 5,....} These numbers are used for counting and are also called the counting numbers.	87 2,369 $\sqrt{36}$ (or 6)
Whole numbers	{0, 1, 2, 3, 4, 5,.....} The whole numbers add 0 to the set of natural numbers.	0, 87 $\frac{18}{3}$ (or 6)
Integers	{..., –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5,....} The integers add the opposites of the natural numbers to the set of whole numbers.	–87, –14 $-\sqrt{25}$ (or –5) 0, 14 87
Rational numbers	These numbers can be expressed as an integer divided by a nonzero integer : $\frac{a}{b}$; a and b are integers ; $b \neq 0$. In decimal form, rational numbers either terminate or repeat.	$\frac{3}{4}$ (or 0.75) (a = 3, b = 4) $-\frac{3}{4} = -0.2727.... = -0.27$ (a = –3, b = 11) $87 = \frac{87}{1}$ (a = 87, b = 1)
Irrational numbers	This is the set of numbers whose decimal representations do not repeat and do no terminate. Irrational numbers divided by an integer.	$\sqrt{2} \approx 1.414214$ $-\sqrt{3} \approx -1.73205$ $\pi \approx 3.142$ $-\frac{\pi}{2} \approx -1.571$

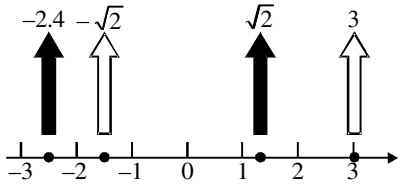
Figure illustrates that every real numbers corresponds to a point on the number line and every point on the number line corresponds to a real number. For this reason, we say that there is a one-to-one correspondence between all the real numbers and all points on a real number line. The real number corresponding to a particular point on the line is called the coordinate of the point. If you draw a point on the real number line corresponding to a real number, you are plotting the real number.

www.pdfworld.net

www.pdfworld.net



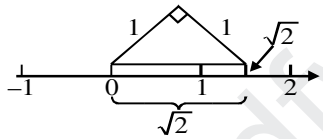
Every real number corresponds to exactly one point on the real numbers line.



Every point on the real numbers line corresponds to exactly one real number.

A one-to-one correspondence between real numbers and points on a number line

STUDY TIP A right triangle with two sides of length 1 and a third side of length $\sqrt{2}$ can be used to plot $\sqrt{2}$ on a number line.

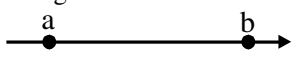


Ordering the Real Numbers

All real numbers follow the **law of trichotomy** i.e. if there are two real numbers a and b then either a = b or a < b or a > b.

The real number line is useful in demonstrating the order of real numbers. We say that the real number a is less than real number b, written a < b, if a is to the left of b on the number line. Equivalently, b is greater than a, written b > a, if b is to the right of a on the number line (Figure)

The symbols < and > are sometimes combined with an equal sign.



a < b

a is less than b : a is to the left of b

The symbols ≤ and ≥ can be understood with the following examples :

Symbols	*Meaning*	*Examples*
a ≤ b	a is less than or equal to b.	3 ≤ 7 (because 3 < 7) 7 ≤ 7 (because 7 = 7)
b ≥ a	b is greater than or equal to a.	7 ≥ 3 (because 7 > 3) –5 ≥ –5 (because –5 = –5)

n-th Root of a Number

Let a and b be real numbers, and let n ≥ 2 be a positive integer. If $a = b^n$, then b is the n-th root of a. If n = 2, then the root is a square root. If n = 3, then the root is a cube root.

Some numbers have more than one nth root. For example, both 5 and –5 are square roots of 25. The **principal nth root** of a number is defined as follows :

Let a be a real number that has atleast one nth root. The principal nth root of a is the nth root that has the same sign as a. It is denoted by a radical symbol: $\sqrt[n]{a}$

The positive integer n is the index of the radical, and the number a is the radicand. If n = 2, we omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$.

CAUTION $\sqrt{\ }$ symbol stands for the positive square root only.

Let us evaluate the expressions involving radicals :

$\sqrt{49} = 7$ because $7^2 = 49$.

$\sqrt[3]{\frac{125}{64}} = \frac{5}{4}$ because $\left(\frac{5}{4}\right)^3 = \frac{5^3}{4^3} = \frac{125}{64}$.

$\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

$\sqrt[4]{-81}$ is not a real number because there is a no real number that can be raised to the fourth power to produce –81.

Here are some generalizations about the n-th roots of a real number

(i) If a is a positive real number and n is a positive even integer, then a has exactly two (real) n-th roots. We denote these roots by $\sqrt[n]{a}$ and $-\sqrt[n]{a}$

(ii) If a is any real number and n is an odd integer, then a has only one (real) n-th root, which is denoted by $\sqrt[n]{a}$.

www.pdfworld.net

www.pdfworld.net

(iii) If a is a negative real number and n is an even integer, then a has no (real) n-th rot.

(iv) $\sqrt[n]{0} = 0$.

(v) Integers such as 1, 4, 9, 16, 25, and 49 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

(vi) If a is not a perfect n-th power, then $\sqrt[n]{a}$ is called a surd of the n-th order. Two surds of the same order are said to be like surds; otherwise unlike surds.

(vii) In an expression of the form $\frac{a}{\sqrt{b}+\sqrt{c}}$, the denominator can be rationalized by multiplying numerator and the denominator by $\sqrt{b}-\sqrt{c}$ which is called the conjugate of $\sqrt{b}+\sqrt{c}$.

(viii) If $x+\sqrt{y} = a+\sqrt{b}$ where x, y, a, b are rationals, then x = a and y = b.

1. Integers are rational numbers, but converse need not be true.
2. Negative of an irrational number is an irrational number.
3. Sum of a rational number and an irrational number is always an irrational number
   e.g. $2+\sqrt{3}$
4. The product of a non zero rational number & an irrational number will always be an irrational number.
5. If $a \in Q$ and $b \notin Q$, then ab = rational number, only if a = 0.
6. Sum, difference, product and quotient of two irrational numbers need not be an irrational number (it may be a rational number also).

➤ **Example 2.** Every number is of one of the forms 5n, 5n ± 1, 5n ± 2.

➤ ***Solution*** For if any number is divided by 5, the remainder is one of the numbers 0, 1, 2, 5 – 2, 5 – 1.

➤ **Example 3.** Every square number is of one the forms 5n, 5n ± 1.

➤ ***Solution*** The square of every number is of one of the forms $(5m)^2$, $(5m \pm 1)^2$, $(5m \pm 2)^2$. If those are divided by 5, the remainders are 0, 1, 4; and, since 4 = 5 – 1, the forms are 5n, 5n + 1, and 5n – 1.

➤ **Example 4.** Consider any three consecutive natural numbers. prove that the cube of the largest cannot be the sum of the cubes of other two.

➤ ***Solution*** Let n – 1, n, n + 1 be the consecutive numbers.

If the equation $(n+1)^3 = n^3 + (n-1)^3$
that is, $n^3 + 3n^2 + 3n + 1 = n^3 + n^3 - 3n^2 + 3n - 1$,
were satisfied,
then $2 = n^2(n-6)$
But the right side is positive only if n > 6,
and in that case
$n^2(n-6) > 36$
and therefore cannot be true.

➤ **Example 5.** Prove that the product of four consecutive integers is one less than a perfect square.

➤ ***Solution*** Let the four consecutive integers be n, n + 1, n + 2, n + 3.

If 1 is added to their product, we have

$$n(n+1)(n+2)(n+3)+1$$
$$= [n(n+3)][(n+1)(n+2)]+1$$
$$= (n^2+3n)(n^2+3n+2)+1$$
$$= (n^2+3n)^2 + 2(n^2+3n) + 1 = (n^2+3n+1)^2$$

Therefore, the product of the four numbers is one less than the square of the integer $n^2 + 3n + 1$.

➤ **Example 6.** If x and y are prime numbers which satisfy $x^2 - 2y^2 = 1$, solve for x and y.

➤ ***Solution*** $x^2 - 2y^2 = 1$ gives $x^2 = 2y^2 + 1$ and hence x must be an odd number. If x = 2n + 1, then $x^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2y^2 + 1$.

Therefore $y^2 = 2n(n+1)$. This means that y^2 is even and hence y is an even integer. Now, y is also a prime implies that y = 2. This gives x = 3. Thus the only solution is x = 3, y = 2.

➤ **Example 7.** Consider any three consecutive natural numbers the smallest of which is greater than 3.

www.pdfworld.net

www.pdfworld.net

Then the square of the largest cannot be the sum of the squares of the other two.

➤ ***Solution*** Let n, n + 1, n + 2 be three consecutive natural numbers with n > 3.

$(n+2)^2 - [n^2 + (n+1)^2]$
$= 3 + 2n - n^2,$
$= -(n+1)(n-3),$
which cannot be zero since n > 3.

➤ **Example 8.** Prove that $\sqrt[3]{2}$ cannot be represented in the form $p + \sqrt{q}$, where p and q are rational (q > 0 and is not a perfect square).

➤ ***Solution*** Put $\sqrt[3]{2} = p + \sqrt{q}$.

Hence $2 = p^3 + 3pq + (3p^2 + q)\sqrt{q}$
Since q is not a perfect square, $3p^2 + q = 0$ must be true, which is impossible.

➤ **Example 9.** Show that $\sqrt{2} + \sqrt{5}$ is irrational.

➤ ***Solution*** Suppose $\sqrt{2} + \sqrt{5} = x = p/q$ is a rational number with $p, q \in Z$.

Then $(x - \sqrt{2})^2 = 5$ i.e. $x^2 - 2\sqrt{2}x + 2 = 5$
Hence $x^2 - 3 = 2\sqrt{2}x$, which gives $\sqrt{2} = (x^2 - 3)/(2x)$, a rational number.
This contradicts the fact that $\sqrt{2}$ is irrational.
So $\sqrt{2} + \sqrt{5}$ is irrational.

➤ **Example 10.** Prove that $\log_3 5$ is irrational.

➤ ***Solution*** Let $\log_3 5$ is rational.

$\therefore \quad \log_3 5 = \frac{p}{q}$; where p and q are co-prime numbers

$\Rightarrow \quad 3^{p/q} = 5 \quad \Rightarrow \quad 3^p = 5^q.$

which is not possible, hence our assumption is wrong and $\log_3 5$ is irrational.

➤ **Example 11.** Simplify (make the denominator rational) $\frac{12}{3 + \sqrt{5} - 2\sqrt{2}}$.

➤ ***Solution*** The expression

$$= \frac{12(3+\sqrt{5}+2\sqrt{2})}{(3+\sqrt{5})^2 - (2\sqrt{2})^2} = \frac{12(3+\sqrt{5}+2\sqrt{2})}{6+6\sqrt{5}}$$

$$= \frac{2(3+\sqrt{5}+2\sqrt{2})(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)}$$

$$= \frac{2(2+2\sqrt{5}+2\sqrt{10}-2\sqrt{2})}{4}$$

$$= 1 + \sqrt{5} + \sqrt{10} - \sqrt{2}$$

➤ **Example 12.** Find the factor which will rationalise

$$\sqrt{3} + \sqrt[3]{5}$$

➤ ***Solution*** Let $x = 3^{1/2}$ and $y = 5^{1/3}$. The L.C.M. of the denominators of the indices 2 and 3 is 6.
Hence x^6 and y^6 are rational.
Now $x^6 + y^6 = (x + y)(x^5 - x^4 y + x^3 y^2 - x^2 y^3 + xy^4 - y^5)$
Hence the rationalizing factor required
$= x^5 - x^4 y + x^3 y^2 - x^2 y^3 + xy^4 - y^5$
where $x = 3^{1/2}$ and $y = 5^{1/3}$.

➤ **Example 13.** Find the square root of $7 + 2\sqrt{10}$

➤ ***Solution*** Let $\sqrt{7 + 2\sqrt{10}} = \sqrt{x} + \sqrt{y}$

Squaring, $x + y + 2\sqrt{xy} = 7 + 2\sqrt{10}$

Hence, x + y = 7 and xy = 10. These two relations give x = 5, y = 2

Thus, $\sqrt{7 + 2\sqrt{10}} = \sqrt{5} + \sqrt{2}$

➤ **Example 14.** If $a + b\sqrt{p} + c\sqrt{q} = 0$ where a, b, c are rationals and $\sqrt{p}$, $\sqrt{q}$ are unlike surds, then prove that a = 0, b = 0, c = 0.

➤ ***Solution*** By transposing and squaring we can show that $2ab\sqrt{p} = c^2 q - a^2 - b^2 p$.

If $ab \neq 0$, the left-hand side would be irrational and the right-hand side would be rational; which is impossible. Therefore ab = 0, and consequently a = 0 or b = 0.

If a = 0, then $b\sqrt{p} + c\sqrt{q} = 0$; and if $b \neq 0$, then $\sqrt{p}/\sqrt{q} = -c/b$; so that $\sqrt{p}$ and $\sqrt{q}$ would be like surds; which is not the case. Therefore b = 0 and c = 0.
If b = 0 then $a + c\sqrt{q} = 0$, and therefore a = 0 and c = 0.

➤ **Example 15.** If $a + b\sqrt[3]{p} + c\sqrt[3]{p^2} = 0$ where a, b, c, p are rationals and p is not a perfect cube, then prove that a, b, c are all zero.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Multiplying the given equation by $\sqrt[3]{p}$ we have $cp + a\sqrt[3]{p} + b\sqrt[3]{p^2} = 0$,
and eliminating the terms containing

$$\sqrt[3]{p^2}\ ,\ (b^2 - ac)\ \sqrt[3]{p} = c^2p - ab.$$

since $\sqrt[3]{p}$ is irrational, it follows that $b^2 = ac$ and $c^2p = ab$,
therefore $c^4p^2 = a^2b^2 = a^3c$.

if $c \neq 0$ we should have $p^2 = \left(\frac{a}{c}\right)^3$, so that $\sqrt[3]{p^2}$ would be rational, which is not the case. Hence $c = 0$, and therefore also $a = 0$ and $b = 0$.

➤ **Example 16.** Find the natural number n for which the fraction $\frac{15n^2 + 8n + 6}{n}$ is a natural number.

➤ ***Solution*** We have $\frac{15n^2 + 8n + 6}{n} = 15n + 8 + \frac{6}{n}$.
Since $15n + 8$ is a natural number, and $\frac{6}{n}$ is a natural number only for $n = 1, n = 2, n = 3, n = 6$, for these values of n the given fraction is a natural number.

➤ **Example 17.** Find the integral solutions of the equation $xy = 2x - y$.

➤ ***Solution*** From the given equation we find $y = \frac{2x}{x+1}$, where $x \neq -1$.
Let us rewrite the fraction as follows :

$$y = \frac{(2x+2)-2}{x+1} = 2 - \frac{2}{x+1}$$

Hence it follows that the value of y will be an integer only if the fraction $\frac{2}{x+1}$ is an integer which is possible for an integral x when $x + 1 = \pm 1$ or $x + 1 = \pm 2$.
Substituting the corresponding x into the formula $y = \frac{2x}{x+1}$, we find all the integral solutions of the given equation; $x_1 = 0, y_1 = 0; x_2 = -2, y_2 = 4; x_3 = 1, y_3 = 1; x_4 = -3, y_4 = 3$.

Practice Problems [A]

1. Indicate which numbers in the given sets are (a) Natural numbers (b) Whole numbers (c) Integers (d) Rational numbers (e) Irrational numbers.

(i) $\left\{-10, -\sqrt{2}, -\frac{3}{4}, 0, \frac{4}{5}, \sqrt{4}, \pi, 7, \frac{18}{2}, 100\right\}$

(ii) $\left\{-\sqrt[3]{8}, \frac{0}{3}, \sqrt[3]{7}, \sqrt{\frac{4}{9}}, 1.\overline{126}\right\}$

2. Prove that if a natural number ends in the digit 7, it cannot be the square of an integer.

3. For what natural numbers n is the fraction $(3n + 4)/5$ an integer.

4. Determine for what natural numbers n the number $n^4 + 4$ is a composite number.

5. If x and y are rational numbers such that $\sqrt{xy}$ is irrational then what is $\sqrt{x} + \sqrt{y}$?

6. Prove that the number $[nm(n - m)]$ is even for any integers n and m.

7. Prove that the sum of four successive natural numbers cannot be a prime number.

8. Find the number equal to $3.1\overline{45}$ when expressed as a rational number in lowest terms.

9. Prove that $\log_2 7$ is an irrational number.

10. Prove that the expression $\frac{(2x^2 + 2xy)(x - 1)}{x^2 - x + xy - y}$ takes on even values for any integral x.

11. Express with a rational denominator $\frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

12. Simplify : $\frac{2 + \sqrt{3}}{\sqrt{2} + \sqrt{2 + \sqrt{3}}} + \frac{2 - \sqrt{3}}{\sqrt{2} - \sqrt{2 - \sqrt{3}}}$

13. The positive integers p, q & r are all primes. If $p^2 - q^2 = r$ then the set of all possible values of p, q and r.

14. Find the integral pair(s) (x , y) whose sum is equal to their product.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

15. A printer numbers the pages of a book starting with 1 and uses 3189 digits in all. How many pages does the book have?

16. Prove that, if $n > 4$, then the number $1! + 2! + 3! + ... n!$ is never a square.

1.2 COMPLEX NUMBER SYSTEM

Some of the numbers that appear in algebra, particularly as the solutions of equations, are not real numbers. In this section, we expand the set of real numbers to include the square roots of negative numbers.

The Imaginary Unit i

Since the square of a real number is never negative, there is no real number x such that $x^2 = -1$. To provide a setting in which such equations have solutions, mathematicians invented an expanded system on numbers, the complex numbers. The imaginary number i, defined to be a solution to the equation $x^2 = -1$, is the basis of this new set.

The imaginary unity i is defined as

$$i = \sqrt{-1} \quad \text{where} \quad i^2 = -1.$$

Using the imaginary unit i, we can express the square root of any negative number as a real multiple of i.

For example, $\sqrt{-25} = i\sqrt{25} = 5i$.

We can check this result by squaring 5i and obtaining $-25.(5i)^2 = 5^2i^2 = 25(-1) = -25$

A new system of numbers, called complex numbers, is based on adding multiples of i, such as 5i, to the real numbers.

Complex Numbers

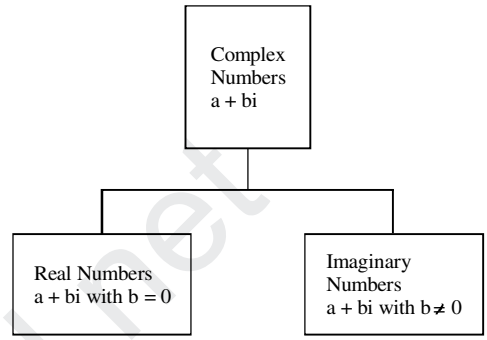
The set of all numbers in the form a + bi with real numbers a and b, and i, the imaginary unity, is called the set of complex numbers.

The real number a is called the real part, and the real number b is called the imaginary part of the complex number a + bi.

Every real number is a complex number in the form a + bi with b = 0.

If a = 0 and $b \neq 0$, then the complex number bi is called a purely imaginary number.

The set of complex numbers is denoted by C. Note that $N \subset W \subset I \subset Q \subset R \subset C$.



The Complex number system

CAUTION A complex number is said to be simplified if it is expressed in the standard form a + bi.

If b is a radical, we usually write i before b. For example, we write $7 + i\sqrt{5}$ rather than $7 + \sqrt{5}i$, which could easily be confused with $7 + \sqrt{5}i$.

Equality of Complex Numbers

Expressed in standard form, two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

a + bi = c + di if and only if a = c and b = d.

Powers of i

The fact that $i^2 = -1$ can be used to find higher powers of i.

$i^1 = i$	$i^5 = i$	$i^9 = i$	$i^{13} = i$
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$	$i^{14} = -1$
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$	$i^{15} = -i$
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1$	$i^{16} = 1$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

We observe that the powers of i rotate through the four numbers i, –1, –i, and 1.

Although we can use the rotating pattern to simplify higher powers of i, an easier method is to write these powers in terms of i^2 and then replace i^2 by –1.

➤ **Example 1.** Simplify $i^{12} + i^{39} + i^{50} + i^{1883}$.

➤ ***Solution*** $i^{12} = (i^2)^6 = (-1)^6 = 1$

$i^{39} = i^{38}\, i = (i^2)^{19} i = (-1)\, i = -i$

$i^{50} = (i^2)^{25} = (-1)^{25} = -1$

$i^{1883} = i^{1882} i = (i^2)^{941} i = (-1)^{941}\, i = (-1)i = -i$

Adding them we get –2i.

Operations with Complex Numbers

The form of a complex number a + bi is like the binomial a + bx. Consequently we can add, subtract, and multiply complex numbers using the same methods we used for binomials, remembering that $i^2 = -1$.

Adding and Subtracting Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

In words, this says that you add complex numbers by adding their real parts, adding their imaginary parts, and expression the sum as a complex number.

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

In words, this says that you subtract complex numbers by subtracting their real parts, subtracting their imaginary parts, and expressing the difference as a complex number.

➤ **Example 2.** Perform the indicated operations, writing the result in standard form.

(i) $(5 - 11i) + (7 + 4i)$

(ii) $(-5 + 7i) - (-11 - 6i)$

➤ ***Solution***

(i) $(5 - 11i) + (7 + 4i) = (5 + 7) + (-11 + 4)i$
$= 12 - 7i$

(ii) $(-5 + 7i) - (-11 - 6i) = (-5 + 11) + (7 + 6)i$
$= 6 + 13i$

Multiplying Complex Numbers

Multiplication of complex numbers is performed the same way as multiplication of polynomials, using the distributive property. After completing the multiplication, we replace i^2 with –1. This idea is illustrated in the next example.

➤ **Example 3.** Find the products :

(i) $4i\,(3 - 5i)$

(ii) $(7 - 3i)\,(-2 - 5i)$

(iii) $(4 + 3i)\,(4 - 3i)$

➤ ***Solution***

(i) $4i(3 - 5i) = 4i(3) - 4i(5i) = 12i - 20i^2$
$= 12i - 20\,(-1) = 12i + 20$

(ii) $(7 - 3i)\,(-2 - 5i) = -14 - 35i + 6i + 15i^2$
$= -14 - 35i + 6i + 15(-1) = -14 - 15 - 35i + 6i$
$= -29 - 29i$

(iii) $(4 + 3i)\,(4 - 3i) = 16 - 12i + 12i - 9i^2$
$= 16 - 9(-1) = 25$

Conjugate of a Complex Number

In the previous example, we found that $(4 + 3i)\,(4 - 3i) = 25$.

Thus the product of two complex numbers can be a real number, generalizing from the situation, let us multiply a + bi and a – bi.

$$(a + bi)\,(a - bi) = a^2 - abi + abi - b^2 i^2$$
$$= a^2 - b^2\,(-1) = a^2 + b^2$$

The multiplication of complex conjugates gives a real number.

For the complex number a + bi, we define its complex conjugate to be a –bi.

The complex conjugate of the number a + bi is a – bi, and the complex conjugate of a – bi is a + bi.

We see how conjugates are used when dividing complex numbers.

Using Conjugates to Divide Complex Numbers

When dividing two complex numbers, express the indicated division as a fraction and then multiply the numerator and the denominator by the complex conjugate of the denominator.

www.pdfworld.net

www.pdfworld.net

➤ **Example 4.** Divide $7 + 4i$ by $2 - 5i$

➤ ***Solution*** We first write the problem as $\frac{7+4i}{2-5i}$.

The conjugate of the denominator, $2 - 5i$, is $2 + 5i$, so we multiply the numerator and the denominator by $2 + 5i$.

$$\frac{7+4i}{2-5i} = \frac{(7+4i)}{(2-5i)} \cdot \frac{(2+5i)}{(2+5i)}$$

Multiply the numerator and the denominator by the conjugate of the denominator.

$$= \frac{14+35i+8i+20i^2}{2^2+5^2} = \frac{14+43i+20(-1)}{29}$$

$$= \frac{-6+43i}{29} = \frac{6}{29} + \frac{43}{29}\,i$$

Observe that the quotient is expressed in the form $a + bi$, with $a = -\frac{6}{29}$ and $b = \frac{43}{29}$.

Roots of Negative Numbers

The square of $4i$ and the square of $-4i$ both result in -16,

$(4i)^2 = 16i^2 = 16(-1) = -16$

$(-4i)^2 = 16i^2 = -16$

Consequently, in the complex number system -16 has two square roots, namely, $4i$ and $-4i$.

Principal square root of a negative number :

For any positive number real number b, the principal square root of the negative number $-b$ is defined by $\sqrt{-b} = i\sqrt{b}$.

We call $4i$ the principal square root of -16.

CAUTION

Do not apply the properties $\sqrt{b}\sqrt{c} = \sqrt{bc}$ and $\frac{\sqrt{b}}{\sqrt{c}} = \sqrt{\frac{b}{c}}$ to the pure imaginary numbers because these properties can only be used when b and c are positive. One way to avoid confusion is to represent all imaginary numbers in terms of i before adding, subtracting, multiplying, or dividing.

Correct	**Incorrect**
$\sqrt{-25}\,\sqrt{-4}$	$\sqrt{-25}\,\sqrt{-4}$
$= i\sqrt{25}\; i\sqrt{4}$	$= \sqrt{(-25)(-4)}$
$= (5i)\,(2i)$	$= \sqrt{100}$
$= 10i^2$	$= 10$
$= -10$	

Operations Involving Square Roots of Negative Numbers

➤ **Example 5.** Simplify and write the result in standard form :

(i) $\sqrt{-18} - \sqrt{18}$ (ii) $(-1 + \sqrt{-5})^2$

(iii) $\frac{-25+\sqrt{-50}}{15}$ (iv) $\sqrt{-12}\,(\sqrt{-3} - \sqrt{2})$

STUDY TIP The first thing to do in each part of this example is to express $\sqrt{-b}$ as $i\sqrt{b}$.

➤ ***Solution***

(i) $\sqrt{-18} - \sqrt{8} = i\sqrt{18} - i\sqrt{8} = i\sqrt{9.2} - i\sqrt{4.2}$

$= 3i\sqrt{2} - 2i\sqrt{2} = i\sqrt{2}$

(ii) $(-1 + \sqrt{5})^2 = (-1 + i\sqrt{5})^2$

$= (-1)^2 + 2(-1)\,(i\sqrt{5}) + (i\sqrt{5})^2$

$= 1 - 2i\sqrt{5} + 5i^2 = 1 - 2i\sqrt{5} + 5(-1)$

$= -4 - 2i\sqrt{5}$

(iii) $\frac{-\sqrt{25}+\sqrt{-50}}{15} = \frac{-25+i\sqrt{50}}{15}$

$= \frac{-25}{15} + \frac{5i\sqrt{2}}{15} = -\frac{5}{3} + i\frac{\sqrt{2}}{3}$

(iv) $\sqrt{-12}(\sqrt{-3} - \sqrt{2}) = i\sqrt{12}(i\sqrt{3} - \sqrt{2})$

$= i^2\sqrt{36} - i\sqrt{24}$

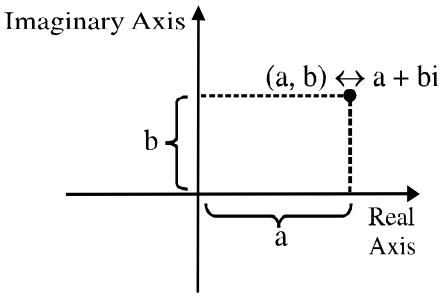
$= (-1)(6) - 2i\sqrt{6} = -6 - 2i\sqrt{6}$

www.pdfworld.net

www.pdfworld.net

Plotting Complex Numbers

The complex plane shown below, consists of horizontal axis, called the real axis, and a vertical axis, called the imaginary axis. Every complex number corresponds to a point in the complex plane and every point in the complex plane corresponds to a complex number.



➤ **Example 6.** Plot the following in the complex plane: (i) 3 + 4i (ii) –1 – 2i (iii) –3 (iv) –4i

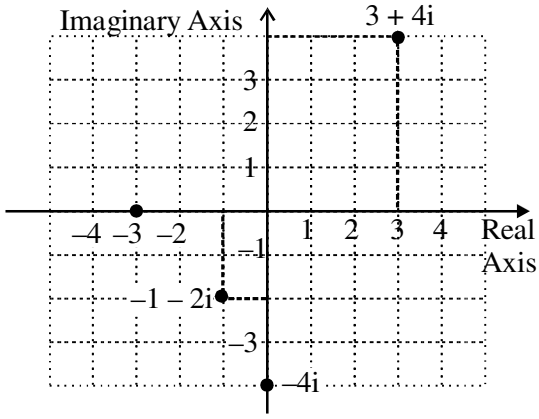
➤ ***Solution***

(i) The complex number 3 + 4i is plotted in the same way we plot (3, 4) in the Cartesian coordinate plane. We move 3 units to the right on the real axis and 4 units up parallel to the imaginary axis.

(ii) The complex number –1 – 2i corresponds to the point (–1, –2) in the Cartesian coordinate plane.

(iii) Since –3 = –3 + 0i, this number corresponds to the point (–3, 0). We plot –3 by moving 3 units to the left on the real axis.

(iv) Since –4i = 0 – 4i, we plot –4i by moving 4 units down on the imaginary axis.



1.3 INTERVALS

Inequalities are used to describe subsets of real numbers, called intervals. For instance, the inequality $x \leq 3$ describes all real numbers less than or equal to 3. Using set-builder notation, we write

$\{x \mid x \leq 3\}$ reading this as "the set of all real numbers x such that x is less than or equal to 3".

Using interval notation, we write $(-\infty, 3]$.

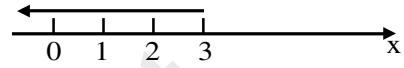


Figure
The interval $(-\infty, 3]$

As shown in the figure the square bracket at 3 shows that 3 is to be included. The negative infinity symbol $-\infty$ does not represent a real number. It indicates that the interval includes all real numbers less than or equal to 3, extending indefinitely to the left.

Let's consider another example of a subset of real numbers that can be expressed in both set-builder and interval notations. The inequality $x > -2$ describes all real numbers greater than –2.

Using set-builder notation, we write $\{x \mid x > -2\}$ reading this as "the set of all real numbers x such that x is greater than –2."

Using interval notation, we write $(-2, \infty)$

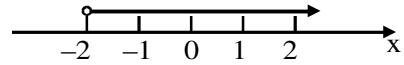


Figure
The interval $(-2, \infty)$

As shown in the figure, the parenthesis at –2 indicates that –2 is not included in the interval. The infinity symbol ∞ does not represent a real number. It indicates that the interval extends indefinitely to the right.

Some inequalities do not extend indefinitely in positive or negative directions. For example, the inequality $-2 < x \leq 4$ indicates that $x > -2$ and $x \leq 4$. This double or compound inequality denotes all real numbers between –2 and 4, excluding –2 but including 4. Using set-builder notation, we write

$\{x \mid -2 < x \leq 4\}$

www.pdfworld.net

www.pdfworld.net

reading this as "the set of all real numbers x such that x is greater than –2 and less than or equal to 4." Using interval notation, we write

(–2, 4].

Figure illustrates that the parenthesis at –2 excludes –2 from the set and the square bracket at 4 includes 4 as a member of the set.

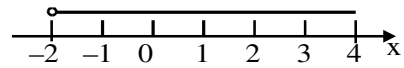
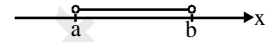
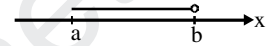
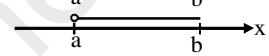
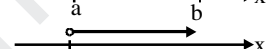
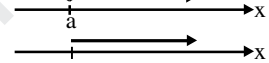
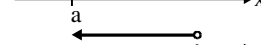
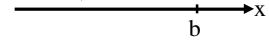
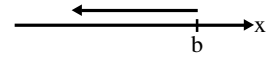


Figure
The interval (–2, 4]

The following table lists the nine possible types of intervals.

Table Intervals on the Real Number Line

Let a and b be real numbers such that a < b.

Interval Notation	*Set-Builder Notation*	*Graph*
(a, b)	{x | a < x < b}	
[a, b]	{x | a ≤ x ≤ b}	
[a, b)	{x | a ≤ x < b}	
(a, b]	{x | a < x ≤ b}	
(a, ∞)	{x | a > a}	
[a, ∞)	{x | a ≥ a}	
(–∞, b)	{x | a < b}	
(–∞, b)	{x | a ≤ b}	
(–∞, ∞)	{x | a ≤ b}	

Intervals and Inequalities

➤ **Example 1.** Express the intervals in terms of inequalities and graph :

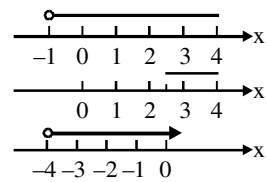
(i) (–1, 4] (ii) [2.5, 4] (iii) (–4, ∞)

➤ ***Solution***

(i) (–1, 4] = { x| – 1 < x ≤ 4}

(ii) [2, 3, 4] = {x | 2.5 ≤ x ≤ 4}

(iii) (–4, ∞) = {x | x > – 4}



➤ **Example 2.** Rewrite each sentence in the left column in both set-builder and interval notations.

➤ ***Solution*** **Set-Builder Interval**	**Notation**	**Notation**
x is less than 5.	{x | x < 5}	(–∞, 5)
x is greater than or equal to 3.	{x | x ≥ 3}	[3, ∞)
x lies between –2 and 5, excluding –2 and 5.	{x | – 2 < x < 5]	(–2, 5)
x is greater than or equal to 2 and less than 7.	{x | 2 ≤ x < 7}	[2, 7)

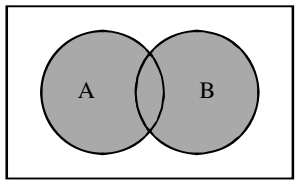
www.pdfworld.net

www.pdfworld.net

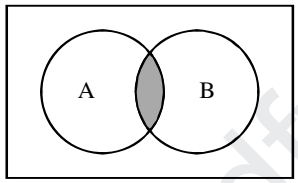
x is non-negative. "Non-negative" indicates a number is greater than or equal to 0.	$\{x \mid x \geq 0\}$	$[0, \infty)$
x is at most 4. "At most" means $\leq$.	$\{x \mid x \leq 4\}$	$(-\infty, 4]$
x is at least 2. "At least means $\geq$.	$\{x \mid x \geq 2\}$	$[2, \infty)$
x is positive but not more than 5. "Not more than" means $\leq$.	$\{x \mid 0 < x \leq 5\}$	$(0, 5]$

Union and Intersection of Sets

If A and B are sets, then their union, $A \cup B$, is the set containing all elements that are in A or in B (or in both A and B). Their intersection, $A \cap B$, is the set containing all elements that are in both A and B



$A \cup B$ is the set of all elements in A or B, or in both A and B.



$A \cap B$ is the set of elements common in both A and B.

➤ **Example 3.** Express each set in interval notation :

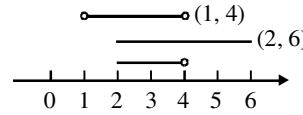
(i) $(1, 4) \cap [2, 6]$ (ii) $(-2, \infty) \cap (3, \infty)$

(iii) $(-2, -1) \cap (2, 3)$ (iv) $(1, 4) \cap [2, 6]$

(v) $(-2, \infty) \cup (3, \infty)$ (vi) $(-2, -1) \cap (2, 3)$

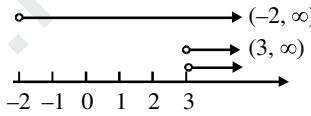
➤ ***Solution***

(i) To find $(1, 4) \cap [2, 6]$, we graph each set above the number line and then locate the interval on the number line common to both. Figure illustrates that numbers common to both intervals begin at 2 and extend up to 4 but not including 4. Thus, $(1, 4) \cap [2, 6] = [2, 4)$.



$(1, 4) \cap [2, 6] = [2, 4)$

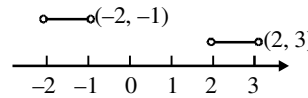
(ii) Figure illustrates those numbers common to both intervals are all real numbers greater than 3. Thus, $(-2, \infty) \cap (3, \infty) = (3, \infty)$



$(-2, \infty) \cap (3, \infty) = (3, \infty)$

(iii) Figure shows that there are no numbers common to the two intervals. Their intersection is a set with no elements, called the empty set or the null set, denoted by ϕ. Thus,

$(-2, -1) \cap (2, 3) = \phi)$

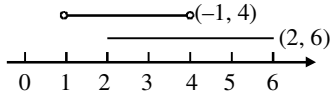


$(-2, -1) \cap (2, 3) = \phi$

(iv) To find $(1, 4) \cup [2,6]$, we graph each set above the number line and then locate the interval on the number line that unites or joins these intervals. As shown in the figure. numbers in either or both intervals are all real numbers greater than 1, extending up to and including 6. Thus, $(1, 4) \cup [2, 6] = (1, 6]$

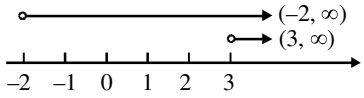
www.pdfworld.net

www.pdfworld.net



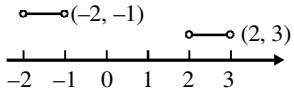
$(1, 4) \cup [2, 6] = (1, 6]$

(v) Figure illustrates that uniting both intervals shown above the number line results in all real numbers greater than –2. Thus, $(-2, \infty) \cup (3, \infty) = (-2, \infty)$.



$(-2, \infty) \cup (3, \infty) = (-2, \infty)$

(vi) Figure indicates that the union of the intervals consists of real number that are either in (–2, –1) or (2, 3). We cannot express the answer as a single interval. Using interval notation, the best we can do is to write $(-2, -1) \cup (2, 3)$



$(-2, -1) \cup (2, 3)$ is not a single interval

Using set-builder notation, we could express this union as $\{x \mid -2 < x < -1 \text{ or } 2 < x < 3\}$.

1. For some particular values of x, we use symbol { }. For example if $x = 1, 2$ we write it as $x \in \{1, 2\}$
2. If there is no value of x, then we write $x \in \phi$.

Practice Problems [B]

1. Express each interval in terms of an inequality and graph the interval on a number line :
   (i) (1, 6] (ii) [–5, 2)
   (iii) [–3, 1] (iv) (2, ∞)
   (v) [–3, ∞) (vi) (–∞, 3)
2. Express each English phrase in both set-builder and interval notations :
   (i) x is less than 6.
   (ii) x is greater than or equal to –1.
   (iii) x lies between 5 and 12, excluding 5 and 12.
   (iv) x lies between 2 and 13, excluding 2 and including 13.
   (v) x is at most 6.
   (vi) x is at least 2 and at most 5.
3. Express each set in problems in interval notation.
   (i) $(1, 5) \cap [2, 7]$ (ii) $(-1, \infty) \cap (5, \infty)$
   (iii) $(-3, -1) \cap (-\infty, 5)$ (iv) $(-1, \infty) \cup (5, \infty)$
   (v) $(-\infty, 2) \cup (-\infty, 4)$ (vi) $(-3, -1) \cup [2, 4]$

1.4 MATHEMATICAL INDUCTION

In many branches of arithmetic, algebra, geometry, analysis it turns out to be necessary to prove the truth of propositions P(n), dependent on a natural variable, for all the values of that variable. This proof is often carried out by a method known as mathematical induction and based on the following principle.

The proposition P(n) is assumed to be true for all the natural values of the variable, provided the following two conditions are fulfilled:

1. The proposition P(n) is true for n = 1.
2. From the supposition that P(n) is true for n = k(where k is any natural number) it follows that it is also true for the value n = k + 1 next in order.

This principle is known as method of mathematical induction.

Mathematical induction is the following method of establishing the truth: to prove the truth of the proposition P(n) for all natural n, we must verify the truth of the statement P(1), and then, assuming that the sentence P(k) is true, try to prove the truth of the statement P(k + 1). If the latter is true and this remains valid for every natural value of k, then, in accordance with the principle of mathematical induction, the proposition P(n) is true for all values of n.

www.pdfworld.net

www.pdfworld.net

➤ **Example 1.** Prove the truth of the proposition $P(n) \equiv \{\text{number } 5.2^{3n-2} + 3^{3n-1} \text{ is multiple of } 19\}$, $n \in N$.

➤ ***Solution*** The statement $P(1) \equiv \{\text{the number } 5.2 + 3^2 \text{ is a multiple of } 19\}$ is true.

Let us assume that for a certain value n = k the proposition

$P(k) \equiv \{\text{number } 5.2^{3k-2} + 3^{3k-1} \text{ is a multiple of } 19\}$ is true. Then, since

$$5.2^{3(k+1)-2} + 3^{3(k+1)-1} = 8.5.2^{3k-2} + 27.3^{3k-1}$$
$$= 8(5.2^{3k-2} + 3^{3k+1}) + 19.3^{3k-1},$$

it is evident that P (k + 1) is true as well. Indeed, the first summand can be divided by 19 on the assumption that P(k) is true; the second summand can also be divided by 19 because it contains 19 as a factor. Both conditions of the principle of mathematical induction are fulfilled and consequently, the proposition P(n) is true for all values of n.

➤ **Example 2.** Prove the formula

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

➤ ***Solution*** For n = 1, the statement is true. Suppose that the formula is true for n = k, i.e.

$$1^3 + 2^3 + 3^3 + ... + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Let us add $(k + 1)^3$ to both sides

$$1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3$$
$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4} + k + 1\right)$$
$$= \left(\frac{k+1}{2}\right)^2 (k^2 + 4k + 4) = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Then from the fact that the formula is true for n = k it follows that it is also true for n = k + 1. This assertion is valid for all natural values of k. Thus the second condition of the principle of mathematical induction is also fulfilled. We have proved the formula.

➤ **Example 3.** Prove that if n is a natural number, then $4^n + 15n - 1$ is divisible by 9.

➤ ***Solution*** For n = 1, the number $4^n + 15n - 1$ is 18, which is divisible by 9.

Assume that $4^k + 15k - 1$ is divisible by 9 and take n = k + 1.

Then $4^{k+1} + 15(k+1) - 1 = 4(4^k + 15k - 1) - 45k + 18$
$= 4(4^k + 15k - 1) - 9(5k - 2)$

But by the induction hypothesis $4^k + 15k - 1$ is divisible by 9, and therefore the right member, and so too the left member, of the equation is divisible by 9, which completes the proof.

➤ **Example 4.** Show by using mathematical induction that $n^5 - n$ is a multiple of 5.

➤ ***Solution*** Let $f(n) = n^5 - n$

Let P(n) : f(n) i.e. $n^5 - n$ is divisible by 5.

Now f(1) = 0, which is divisible by 5

$\therefore$ P(1) is true ...(A)

Let P(m) be true $\Rightarrow$ $f(m) = m^5 - m$ is divisible by 5

Now $f(m + 1) - f(m) = (1 + m)^5 - m^5 - 1$

$\therefore$ $f(m + 1) - f(m) = (1 + m)^5 - m^5 - 1$

$= (1 + 5m + 10m^2 + 10m^3 + 5m^4 + m^5) - m^5 - 1$

$= 5(m + 2m^2 + 2m^3 + m^4)$, which is divisible by 5.

$\therefore$ f(m + 1) is divisible by 5, whenever f(m) is divisible by 5.

$\therefore$ P(m + 1) is true whenever P(m) is true ...(B)

From (A) and (B) it follows that P(n) is true for every natural number n.

➤ **Example 5.** Show by using induction that for all natural numbers n > 2, $2^n > 2n + 1$.

➤ ***Solution*** Let P(n) : $2^n > 2n + 1$, where n > 2

When n = 3, L.H.S. = $2^3 = 8$ and R.H.S. = 2.3 + 1 = 7

Clearly, 8 > 7

$\therefore$ P(3) is true ...(A)

Let P(m) be true $\Rightarrow$ $2m > 2m + 1$, $m \geq 3$ ...(1)

To prove P(m + 1) is true

i.e. $2^{m+1} > 2m + 3 = \alpha$ (say) ...(2)

From (1), $2^{m+1} > 4m + 2 = \beta$ (say) ...(3)

Now $\beta - \alpha = 4m + 2 - 2m - 3 = 2m - 1 > 0$

$\therefore$ $\beta > \alpha$ ...(4)

From (3) and (4), $2^{m+1} > \alpha$ or $2^{m+1} > 2m + 3$

Hence P(m + 1) is true whenever P(m) is true. ...(B)

From (A) and (B) it follows that P(n) is true for all natural numbers n.

www.pdfworld.net

www.pdfworld.net

Practice Problems [C]

Prove by induction the truth of the following equalities for every natural value of n :

1. $1.2 + 2.5 + ... + n(3n-1) = n^2(n+1)$

2. $1^2 + 3^2 + ... + (2n-1)^2 = \frac{n(4n^2-1)}{3}$

3. $1.2^2 + 2.3^2 + ... + (n-1)n^2 = \frac{n(n^2-1)(3n+2)}{12}$

4. $\frac{1}{1.5} + \frac{1}{5.9} + ... + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$

5. $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right),....\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2}$

6. $3 + 33 + ... + 33..... 3 = \frac{10^{n+1} - 9^n - 10}{27}, n \in N$
(the left hand side contains n summands).

Prove by induction the validity of the following inequalities for all natural $n > 1$:

7. $\frac{1}{2}.\frac{3}{4}.\frac{5}{6}..... \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$

8. $\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} > \frac{13}{24}$

9. $\sqrt{n} < 1 + \frac{1}{\sqrt{2}} + ... \frac{1}{\sqrt{n}} < 2\sqrt{n}$

1.5 DIVISIBILITY TEST

(i) A number will be divisible by 2 iff the digit at the unit place is divisible by 2.

(ii) A number will be divisible by 3 iff the sum of its digits of the number is divisible by 3.

(iii) A number will be divisible by 4 iff last two digits of the number together are divisible by 4.

(iv) A number will be divisible by 5 iff digit at the unit place is either 0 or 5.

(v) A number will be divisible by 6 iff the digit at the unit place of the number is divisible by 2 & sum of all digits of the number is divisible by 3.

(vi) A number will be divisible by 8 iff the last 3 digits, all together, is divisible by 8.

(vii) A number will be divisible by 9 iff sum of all it's digits is divisible by 9.

(viii) A number will be divisible by 10 iff it's last digit is 0.

(ix) A number will be divisible by 11 iff the difference between the sum of the digits at even places and sum of the digits at odd places is a multiple of 11.
e.g. 1298, 1221, 123321, 12344321, 1234554321, 123456654321, 795432.

(x) The product of k consecutive integers is divisible by k!.

➤ **Example 1.** Prove that :

(a) the sum $\overline{ab} + \overline{ba}$ is multiple of 11;

(b) a three-digit number written by one and the same digit is entirely divisible by 37.

➤ ***Solution***

(a) $\overline{ab} + \overline{ba} = (10a + b) + (10b + a) = 11(a + b)$;

(b) $\overline{aaa} = 100a + 10a + a = 111a = 37.3a$.

➤ **Example 2.** Prove that the difference $10^{25} - 7$ is divisible by 3.

➤ ***Solution*** Write the given difference in the form $10^{25} - 7 = (10^{25} - 1) - 6$.

The number $10^{25} - 1 = \underbrace{\overline{99..9}}_{25 \text{ digits}}$ is divisible by 3 (and 9). Since the numbers $(10^{25} - 1)$ and 6 are divisible by 3, the number $10^{25} - 7$, being their difference, is also divisible by 3 without a remainder.

➤ **Example 3.** If the number A 3 6 4 0 5 4 8 9 8 1 2 7 0 6 4 4 B is divisible by 99 then find the ordered pair of digits (A, B).

➤ ***Solution*** Sum of digits at odd position $S_O = A + 37$;

Sum of digits at even position $S_E = B + 34$

$\Rightarrow$ $A - B + 3 = 0$ or 11 and $A + B + 71$ is a multiple of 9

$\Rightarrow$ $A - B = -3$ or 8 and $A + B = 1$ or 10

Hence the ordered pair is (9, 1).

www.pdfworld.net

www.pdfworld.net

➤ **Example 4.** Consider a number N = 2 1 P 5 3 Q 4. Find the number of ordered pairs (P, Q) so that the number 'N is divisible by 44.

➤ ***Solution***

$S_O = P + 9$ $S_E = Q + 6$ $S_O - S_E = P - Q + 3$

'N' is divisible is 11 if $P - Q + 3 = 0, 11$

$P - Q = -3$ ...(1)

or $P - Q = 8$ ...(2)

N is divisible by 4 if Q = 0, 2, 4, 6, 8

From equation (1)

$Q = 0$ $P = -3$ (not possible)

$Q = 2$ $P = -1$ (not possible)

$Q = 4$ $P = 1$

$Q = 6$ $P = 3$

$Q = 8$ $P = 5$

$\therefore$ number of ordered pairs is 3

From equation (2)

$Q = 0$ $P = 8$

$Q = 2$ $P = 10$

(not possible) similarly $Q \neq 4, 6, 8$

$\therefore$ No. of ordered pairs is 1

$\therefore$ Total number of ordered pairs, so that number 'N' is divisible by 44, is 4

➤ **Example 5.** Prove that for an arbitrary even n the number $N = n^3 + 20n$ is divisible by 48.

➤ ***Solution*** Let n = 2k, where k is an integer therefore $N = 8k(k^2+5)$.

If we show that for any integer k the number $k(k^2+5)$ is divisible by 6, it will be clear that N is divisible by 48.

Now, $k(k^2+5) = k(k^2-1+6) = (k-1)k(k+1) + 6k$

We see that 6k is divisible by 6. Now $(k-1)k(k+1)$ is a product of three successive integers and for this reason one of them is definitely divisible by 3. Of two successive integers, one must definitely be even. Since 2 and 3 are relatively prime, it follows that $k(k^2+5)$ is indeed divisible by 6 for any integer k.

➤ **Example 6.** Prove that for any natural number n, the number $N = n^2 + 1$ is not divisible by 3.

➤ ***Solution*** The only possible remainders upon the division of a natural number by 3 are 0,1, 2. In solving this problem we therefore find it convenient to partition all the natural numbers into three classes; 3k, where k is a natural number; 3k + 1, where k is a whole number; 3k + 2, where k is a whole number.

If $n = 3k$, we have $n^2 + 1 = 9k^2 + 1$. Since the first term in the right member is divisible by 3 and the second one is not, the number N is not divisible by 3 for these values of n.

If $n = 3k + 1$, then $n^2 + 1 = 9k^2 + 6k + 2$. It is obvious in this case that when N is divided by 3 there is a remainder of 2.

The case of $n = 3k + 2$ is considered in a similar manner.

Practice Problems [D]

1. Prove that the number which is the square of a natural number is either exactly divisible by three or yields unity as the remainder.
2. Prove that the number $n(n^2 + 5)$ is divisible by 6 for any natural n.
3. Prove that the sum of the cubes of three successive natural numbers is divisible by three.
4. Prove that the product of three successive natural numbers, the middle number being the square of a natural number, is divisible by 60.
5. Prove that the product of two successive even numbers is divisible by eight.
6. Prove that the difference between the cube of an odd number and the number itself is divisible by 24.
7. Prove that the square of any odd number, diminished by 1, is divisible by 8.
8. Prove that the sum of two successive odd numbers is divisible by 4.
9. Let $p \leq 5$ be a prime number. Prove that the number $p^2 - 1$ is divisible by 24.
10. Prove that if a is an integer, then $a^3 - a$ is divisible by 6. Also deduce that $5a^3 + 13a - 30$ is divisible by 6.

www.pdfworld.net

www.pdfworld.net

11. Suppose m, n are integers and $m = n^2 - n$. Then show that $m^2 - 2m$ is divisible by 24.

12. Find all five digit numbers of of the form $\overline{34x5y}$ (x and y are digits) divisible by 36.

1.6 RATIO AND PROPORTION

Ratio

1. If A and B are two quantities of the same kind, then their ratio is A : B; which may be denoted by the fraction $\frac{A}{B}$ (This may be an integer or fraction)

2. A ratio may be represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb} = \ldots\ldots$ where m, n,..... are non-zero numbers.

3. To compare two or more ratio, reduce them to common denominator.

4. Ratio between two ratios may be represented as the ratio of two integers e.g. $\frac{a}{b} : \frac{c}{d} : \frac{a/b}{c/d} = \frac{ab}{bc}$ or ad : ac. duplicate, triplicate ratio.

5. Ratios are compounded by multiplying them together i.e. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} \ldots\ldots = \frac{ace}{bdf} \ldots\ldots$

6. If a : b is any ratio then its duplicate ratio is $a^2 : b^2$; triplicate ratio is $a^3 : b^3$..... etc.

7. If a : b is any ratio, then its sub-duplicate ratio is $a^{1/2} : b^{1/2}$; sub-triplicate ratio is $a^{1/3} : b^{1/3}$ etc.

Proportion

When two ratios are equal, then the four quantities compositing them are said to be proportional.

If $\frac{a}{b} = \frac{c}{d}$, then it is written

as a : b = c : d or a : b : : c : d

1. 'a' and 'd' are known as extremes and 'b and c' are known as means.

2. An important property of proportion product of extremes = product of means.

3. If a : b = c : d, then b : a = d : c (Invertando)

4. If a : b = c : d, then a : c = b : d (Alternando)

5. If a : b = c : d,

then $\frac{a+b}{b} = \frac{c+d}{d}$ (Componendo)

6. If a : b = c : d, then $\frac{a-b}{b} = \frac{c-d}{d}$ (Dividendo)

7. If a : b = c : d, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Componendo and dividendo)

8. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$, then each

$$\frac{a+c+e+\ldots\ldots}{b+d+f+\ldots\ldots} = \frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$$

9. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$, then each

$$= \frac{xa + yc + ze + \ldots\ldots}{xb + yb + zf + \ldots\ldots}$$

10. If, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$ then each

$$= \left(\frac{xa^n + ye^n + ze^n}{xb^n + yd^n + zf^n}\right)^{1/n}$$

➤ **Example 1.** If, $\frac{x+y}{2} = \frac{y+z}{3} = \frac{z+x}{4}$ then find x : y : z.

➤ ***Solution*** Each $= \frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$

$$= \frac{2(x+y+z)}{9} = \frac{x+y+z}{9/2}$$

and therefore each

$$= \frac{(x+y+z)-(x+z)}{\frac{9}{2}-4} = \frac{(x+y+z)-(x+z)}{\frac{9}{2}-4}$$

$$= \frac{(x+y+z)-(x+y)}{\frac{9}{2}-2}$$

$$= \frac{x}{3/2} = \frac{y}{1/2} = \frac{z}{5/2} \Rightarrow x : y : z = 3 : 1 : 5$$

www.pdfworld.net

www.pdfworld.net

➤ **Example 2.** If $a(y+z) = b(z+x) = c(x+y)$, then show that $\frac{a-b}{x^2-y^2} = \frac{b-c}{y^2-z^2} = \frac{c-a}{z^2-x^2}$

➤ ***Solution*** Given condition can be written as

$$\frac{y+z}{1/a} = \frac{z+x}{1/b} = \frac{x+y}{1/c} = k \qquad ...(1)$$

Each $\frac{(z+x)-(y-z)}{\frac{1}{b}-\frac{1}{a}} = \frac{(x+y)-(x+z)}{\frac{1}{c}-\frac{1}{b}}$

$$= \frac{(y+z)-(x+y)}{\frac{1}{a}-\frac{1}{c}}$$

$$= \frac{x-y}{\frac{a-b}{ab}} = \frac{y-z}{\frac{b-c}{bc}} = \frac{z-x}{\frac{c-a}{ca}} = k$$

Form (1) and (2), we get by multiplying

$$\frac{x^2-y^2}{\frac{a-b}{abc}} = \frac{y^2-z^2}{\frac{b-c}{abc}} = \frac{z^2-x^2}{\frac{c-a}{abc}}$$

$$\Rightarrow \quad \frac{x^2-y^2}{a-b} = \frac{y^2-z^2}{b-c} = \frac{z^2-x^2}{c-a}$$

$$\Rightarrow \quad \frac{a-b}{x^2-y^2} = \frac{b-c}{y^2-z^2} = \frac{c-a}{z^2-x^2}$$

➤ **Example 3.** If $x = \frac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}+\sqrt{2a-3b}}$, show that $3bx^2 - 4ax + 3b = 0$.

➤ ***Solution*** Taking the left hand side as $\frac{x}{1}$, using componendo and dividendo, $\frac{x+1}{x-1} = \frac{\sqrt{2a+3b}}{\sqrt{2a-3b}}$

Squaring, $\frac{(x+1)^2}{(x-1)^2} = \frac{2a+3b}{2a-3b}$ and again applying componendo and dividendo $\frac{x^2+1}{2x} = \frac{2a}{3b}$ which gives the answer on cross multiplication.

➤ **Example 4.** If $\frac{2y+2z-x}{a} = \frac{2z+2x-y}{b}$

$$= \frac{2x+2y-z}{c}$$, then show that

$$\frac{9x}{2b+2c-a} = \frac{9y}{2c+2a-b} = \frac{9z}{2a+2b-c}$$

➤ ***Solution***

Since $\frac{2y+2z-x}{a} = \frac{2z+2x-y}{b} = \frac{2x+2y-z}{c}$,

each is equal to

$$\frac{2(2z+2x-y)+2(2x+2y-z)-(2y+2z-x)}{2b+2c-a}$$

by a theorem quoted earlier

$$= \frac{9x}{2b+2c-a}$$ on simplification.

Similarly, each $= \frac{9y}{2c+2a-b}$ and $\frac{9z}{2a+2b-c}$ and hence the result.

➤ **Example 5.** Solve : $\frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}} = 2$

➤ ***Solution*** Writing the R.H.S. as $\frac{2}{1}$ and using componendo and dividendo,

$$\frac{(\sqrt{2+x}+\sqrt{2-x})+(\sqrt{2+x}-\sqrt{2-x})}{(\sqrt{2+x}+\sqrt{2-x})-(\sqrt{2+x}-\sqrt{2-x})} = \frac{2+1}{2-1}$$

(i.e.) $\frac{\sqrt{2+x}}{\sqrt{2-x}} = \frac{3}{1}$

Squaring, $\frac{2+x}{2-x} = \frac{9}{1}$ and again applying componendo and dividendo

$\frac{4}{2x} = \frac{10}{8}$ and hence $x = \frac{8}{5}$

www.pdfworld.net

www.pdfworld.net

1.7 POLYNOMIALS

A rational expression, which contains only two operations, multiplication and raising to a natural power, with respect to the letters entering into it, is called a **monomial**.

Examples of monomials : 3a, 2abc, $\frac{23ab}{7}$, abc, $\frac{2ab}{3}$.

A rational expression is called a **polynomial** if it is integral with respect to every letter entering into that expression.

For instance, the rational expression $\sqrt{35}abc - \frac{16\ ad}{7} + 0.3\ dc$ is a polynomial since it is an integral expression with respect to the letters a, b, c and d.

In particular, a rational expression which contains only one letter and is integral with respect to the letter is said to be a polynomial which is integral with respect to one letter.

A polynomial written in the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ is known as an ordered polynomial. If $a_0 \neq 0$, then we say that the polynomial is of degree n.

If it is not known whether the coefficient a_0 is zero or not, then we say that the polynomial is of a degree not higher than n.

From this definition it follows, in particular, that polynomials of degree zero are numbers different from zero. The number zero is also considered to be a polynomial, being the only polynomial which is not assigned a degree. The following notations are used to make the designations more concise :

P(x), Q(x), R(x), p(x), q(x), r(x).

If we want to emphasize that the polynomial P(x) is of degree n, we write $P_n(x)$.

Two polynomials, which are integral with respect to x, are **identical** if and only if their degrees are equal and the coefficients in like powers of x are equal.

The identity transformation of a polynomial to the product of polynomials is known as **factorization** of a polynomial.

Let us consider an example of factoring a polynomial. Grouping and factoring, we get an identity.

$a^2c + 2abc + b^2c + (a + b)^2 d$
$= c(a^2 + 2ab + b^2) + d(a + b)^2$
$= c(a + b)^2 + d(a + b)^2$
$= (a + b)^2 (c + d)$

Division with Remainder

To divide the polynomial P(x) by the non-zero polynomial T(x) with a remainder is to find two polynomials q(x) and r(x) such that P(x) = T(x) q(x) + r(x), the degree of the polynomial r(x) being either strictly lower than that of the polynomial T(x), or r(x) is zero.

If r(x) = 0, i.e. the remainder is zero, then we say that the polynomial P(x) can be divided by the polynomial T(x) with remainder zero, or the polynomial P(x) is exactly divisible by the polynomial T(x)

Remainder theorem : The remainder of the division of the polynomial P(x) by the binomial $(x - \alpha)$ is equal to the value of the polynomial P(x) for $x = \alpha$, i.e. $r = P(\alpha)$

Factor theorem : The polynomial P(x) is exactly divisible by the binomial $(x - \alpha)$ if and only if the value of the polynomial is zero for $x = \alpha$,, i.e. $P(\alpha) = 0$

Proof :

Necessity. Assume that the polynomial P(x) is exactly divisible by the binomial $(x - \alpha)$.

This means that the remainder r is zero. According to the remainder theorem the remainder

$r = P(\alpha)$. Consequently, $P(\alpha) = 0$

Sufficiency. Let $P(\alpha) = 0$. On the other hand, according to the remainder theorem $r = P(\alpha)$.

Hence r = 0, i.e. P(x) is exactly divisible by $x - \alpha$.

Here are some **corollaries of the factor theorem** :

(i) The polynomial $P_n(x) = x^n - \alpha^n$ is exactly divisible by the binomial $(x - \alpha)$ for any natural n. Indeed, $P_n(\alpha) = \alpha^n - \alpha^n = 0$

(ii) The polynomial $P_n(\alpha) = x^n - \alpha^n$ is exactly divisible by the binomial $(x + \alpha)$

www.pdfworld.net

www.pdfworld.net

for any even n(i.e. n = 2m). Indeed,
$P_{2n}(-\alpha) = (-\alpha)^{2m} - \alpha^{2m} = 0$

(iii) The polynomial $P_n(x) = x^n + \alpha^n$ is exactly divisible by the binomial $(x + \alpha)$ for any odd n (i.e. n = 2m + 1). Indeed,
$P_{2m+1}(-\alpha) = (-\alpha)^{2m+1} + \alpha^{2m+1} = 0.$

Roots of a Polynomial

The number α is a root of the polynomial P(x) if $P(\alpha) = 0$.

Some important formulae :

1. $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$
2. $(a - b)^2 = a^2 - 2ab + b^2 = (a - b)^2 - 4ab$
3. $a^2 - b^2 = (a + b)(a - b)$
4. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
5. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
6. $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = (a+b)(a^2+b^2-ab)$
7. $a^3 - b^3 = (a-b)^3 + 3ab(a-b) = (a-b)(a^2+b^2+ab)$
8. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
$$= a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$
9. $a^2 + b^2 + c^2 - ab - bc - ca$
$$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$$
10. $a^3 + b^3 + c^3 - 3abc$
$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
$$= \frac{1}{2}(a + b + c)[(a-b)^2 + (b-c^2 + (c-a)^2]$$
11. $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$
12. $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

➤ **Example 1.** Find the factorization of the polynomial $P(x) = (x^2 + x + 1)(x^2 + x + 2) - 12$.

➤ ***Solution***

$P(x) = (x^2 + x + 1)((x^2 + x + 1) + 1) - 12$
$= (x^2 + x + 1)^2 + (x^2 + x + 1) - 12.$

Let $x^2 + x + 1 = y$. We then have
$y^2 + y - 12 = (y + 4)(y - 3),$
since the roots of the trinomial $y^2 + y - 12$ are equal to (–4) and 3.

Passing from y to x, we get
$P(x) = (x^2 + x + 5)(x^2 + x - 2).$
Since the trinomial $x^2 + x - 2 = (x - 1)(x + 2)$,
$P(x) = (x - 1)(x + 2)(x^2 + x + 5).$

➤ **Example 2.** What is the remainder when $x + x^9 + x^{25} + x^{49} + x^{81}$ is divided by $x^3 - x$.

➤ ***Solution*** We have $x + x^9 + x^{25} + x^{49} + x^{81}$
$= x(1 + x^8 + x^{24} + x^{48} + x^{80})$
$= x[(x^{80} - 1) + (x^{48} - 1) + (x^{24} - 1) + (x^8 - 1) + 5]$
$= x(x^{80} - 1) + x(x^{48} - 1) + x(x^{24} - 1) + x(x^8 - 1) + 5x$
Now $x^3 - x = x(x^2 - 1)$ and all but the last term 5x are divisible by $x(x^2 - 1)$.
Thus the remainder is 5x.

➤ **Example 3.** Using the method of undetermined coefficients, show that the expression $(x + 1)(x + 2)(x + 3)(x + 4) + 1$ is the square of a trinomial.

➤ ***Solution*** If the given expression is the square of a trinomial, then
$(x + 1)(x + 2)(x + 3)(x + 4) + 1 = (x^2 + ax + b)^2$,
where a and b are the desired coefficients.
We compare the coefficients of x^3 and x^2 in both sides.

We obtain the system $\begin{cases} 2a = 10, \\ a^2 + 2b = 35, \end{cases}$

We find a = 5 and b = 5. We make sure that for these values of a and b, the coefficients of x and x^0 (i.e. constant terms) also coincide. Thus, the given expression is equal to $(x^2 + 5x + 5)^2$.

➤ **Example 4.** Factor the following expression $(x + y + z)^3 - x^3 - y^3 - z^3$.

➤ ***Solution*** We have $(x + y + z)^3 = x^3 + y^3 + z^3$
$+ 3x^2(y + z) + (3y^2(x + z) + 3z^2(x + y) + 6xyz.$
Hence $(x + y + z)^3 - x^3 - y^3 - z^3$
$= 3\{x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + 2xyz\}$
$= 3\{z(x^2 + y^2 + 2xy) + z^2(x + y) + xy(x + y)\}$
$= 3(x + y)\{z(x + y) + z^2 + xy\}$
$= 3(x + y)(x + z)(y + z).$
Thus, $(x + y + z)^3 - x^3 - y^3 - z^3 = 3(x + y)(x + z)(y + z)$.

➤ **Example 5.** Using factor theorem, show that a – b, b – c and c – a are the factors of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** By factor theorem, a – b will be a factor of the given expression if it vanishes by substituting a = b in it.

Substituting a = b in the given expression, we have

$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$

$= b(b^2 - c^2) + b(c^2 - b^2) + c(b^2 - b^2)$

$= b^3 - bc^2 + bc^2 - b^3 + c(b^2 - b^2) = 0$

$\therefore$ (a – b) is a factor of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

Similarly, we can show that (b – c) and (c – a) are also factors of the given expression.

Hence, (a – b), (b – c) and (c – a) are factors of the given expression.

➤ **Example 6.** Show that x – 2y is a factor of $3x^3 - 2x^2y - 13xy^2 + 10y^3$.

➤ ***Solution*** Let $f(x) = 3x^3 - 2x^2y - 13xy^2 + 10y^3$

Then $f(2y) = 3(2y)^3 - 2y(2y)^2 - 13y^2(2y) + 10y^3$

$24y^3 - 8y^3 - 26y^3 + 10y^3 = 0$

Hence x – 2y is a factor of f(x).

➤ **Example 7.** Show that $a^n - b^n$ is divisible by a – b if n is any positive integer odd or even.

➤ ***Solution*** Let $a^n - b^n = f(a)$

By Remainder theorem, $f(b) = b^n - b^n = 0$ (replacing a by b)

$\therefore$ a – b is a factor of $a^n - b^n$.

➤ **Example 8.** Show that $a^n - b^n$ is divisible by (a + b) when n is an even positive integer but not if n is odd.

➤ ***Solution*** Let $a^n - b^n = f(a)$

Now $f(-b) = (-b)^n - b^n = b^n - b^n$ if n is even, = 0 and hence a + b is a factor of $a^n - b^n$

If n is odd, $f(-b) = -b^n - b^n = -2b^n \neq 0$.

1.8 FUNCTION

A function f from a set A to a set B is a rule of correspondence that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain which is the set of inputs of the function f, and the set B is the co-domain which contains the outputs. The actual set of outputs is called the range of the function.

The domain of a function may be explicitly described along with the function, or it may be implied by the expression used to define the function. The implied domain is the set of all real numbers for which the expression is defined.

For instance, the function $f(x) = \frac{1}{x^2 - 4}$ has an implied domain that consists of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined.

Another common type of implied domain is used to avoid even roots of negative numbers. For example, the function $f(x) = \sqrt{x}$ is defined only for $x \geq 0$. Hence, its implied domain is the interval $[0, \infty)$.

The range of a function is more difficult to find, and can best be obtained from the graph of the function.

➤ **Example 1.** Find the domain of each of the following functions.

(i) $f : \{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

(ii) Volume of a sphere : $V = \frac{4}{3}\pi r^3$

(iii) $g(x) = \frac{1}{x + 5}$ (iv) $h(x) = \sqrt{4 - x}$

➤ ***Solution***

(i) The domain of f consists of all first coordinates in the set of ordered pairs, and is therefore the set $\{-3, -1, 0, 2, 4\}$.

(ii) For the volume of a sphere your must choose positive values for the radius r. Thus, the domain is the set of all real numbers r such that $r \geq 0$.

(iii) Excluding x-values that yield zero in the denominator, the domain of g is the set of all real numbers $x \neq -5$.

(iv) Choose x-values for which $4 - x \geq 0$. The domain is all real numbers that are less than or equal to 4.

The Graph of a Function

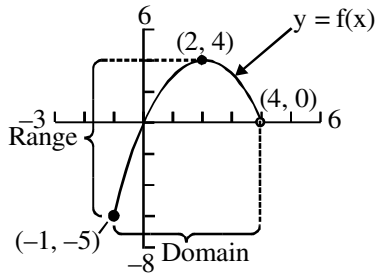
Earlier we studied functions from an algebraic point of view. Here we will study functions from a geometric perspective. The graph of a function f is the collection of ordered pairs (x, f(x)) such that x is in the domain of f. The following is the geometrical interpretation of x and f(x).

www.pdfworld.net

www.pdfworld.net

x = the directed distance from the y-axis.
f(x) = the directed distance from the x-axis.

➤ **Example 2.** Using the graph of the function f, shown in figure (i) find the domain of f (ii) find the function values f(–1) and f(2) (iii) find the range of f.



➤ ***Solution***

(i) The closed dot (on the left) indicates that x = –1 is in the domain of f, whereas the open dot (on the right) indicates x = 4 is not in the domain. Thus, the domain of f is all x in the interval [–1, 4).

(ii) Because (–1, –5) is a point on the graph of f, it follows that f(–1) = –5. Similarly, because (2, 4) is a point on the graph of f, it follows that f(2) = 4.

(iii) Since the graph lies between f(–1) = –5 and f(2) = 4, the range of f is the interval [–5, 4].

Equations

An equation is a statement that two algebraic expressions are equal.
Some examples of equations in x are

$3x - 5 = 7$, $x^2 - x - 6 = 0$, and $\sqrt{2x} = 4$.

To solve an equation in x means to find all values of x for which the equation is true. Such values are solutions. For instance, x = 4 is a solution of the equation 3x – 5 = 7, because 3(4) – 5 = 7 is a true statement.

The solutions of an equation depend upon the kinds of numbers being considered. For instance, in the set of rational numbers, the equation $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers this equation has two solutions $\sqrt{10}$ and $-\sqrt{10}$, because $(\sqrt{10})^2 = 10$ and $(-\sqrt{10})^2 = 10$.

Graphical Solution by Intersection of Graphs

To solve the equation f(x) = g(x) graphically, draw the graphs of $y_1 = f(x)$ and $y_2 = g(x)$. The x-coordinate of any point of intersection of the two graphs is a solution of the equation.

Graphical Solution by x-Intercept

To solve the equation f(x) = g(x) graphically, draw the graphs of y = f(x) – g(x) = h(x). Any x-intercept of the graph of y = h(x) (or zero of the function h) is a solution of the equation.

An equation that is true for every real number in the domain of the variable is an **identity**. For instance, $x^2 - 9 = (x + 3)(x - 3)$ is an identity because it is a true statement for any real value of x. Similarly, $x/(3x^2) = 1/(3x)$, where $x \neq 0$, is an identity because it is true for any non-zero real value of x.

Two equations F = 0 and G = 0 are said to be **equivalent** if each of them is a consequence of the other one, and then the notation is

$$F = 0 \quad \Leftrightarrow \quad G = 0$$

Thus two equations are equivalent if the sets of their roots coincide.

The equation F = 0 is considered to be equivalent to two (or several) equations $F_1 = 0$, $F_2 = 0$ if the set of the roots of equation F = 0 coincides with the union of the sets of the roots of the equations

$F_1 = 0$ and $F_2 = 0$

Some equivalent equations

1. The equation F + G = G is equivalent to the equation F = 0 considered on the set of permissible values of the initial equation.
2. The equation F/G = 0 is equivalent to the equation F = 0 considered on the set of permissible values of the initial equation.
3. The equation F.G = 0 is equivalent to two equations, F = 0 and G = 0, each of which is considered on the set of permissible values of the initial equation.

www.pdfworld.net

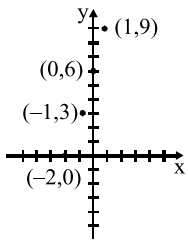
www.pdfworld.net

4. The equation $F^n = 0$ is equivalent to the equation $F = 0$
5. The equation $F^n = G^n$ is equivalent to the equation $F = G$ for odd n and to two equations, $F = G$ and $F = -G$ for even n.

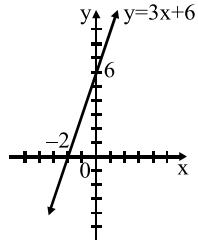
1.9 LINEAR FUNCTION

A function f defined by $f(x) = ax + b$, where a and b are real numbers, is called a linear function.
Similarly, an equation such as $y = 3x + 6$ is called a linear equation. A solution of such an equation is an ordered pair (x, y) that makes the equation true. Verify that (0, 6), (–1, 3), (–2, 0) and (1, 9) are all solutions of $y = 3x + 6$.
The traditional method of graphing linear equations involves plotting points whose coordinates are solutions of the equation, and then joining them with a straight line. Figure shows the ordered pairs just mentioned for the linear equation $y = 3x + 6$. It is accompanied by a table of values. Notice that the points appear to lie in a straight line. Since we may substitute any real number for x, we join these points with a line to obtain the graph of the equation, as shown in the figure.

x	y
–2	0
–1	3
0	6
1	9



(a)



(b)

Using function notation, we can also say that the graph in the figure is the graph of the linear function $f(x) = 3x + 6$.

Simultaneous Equations in Two Unknowns

Simultaneous equations in two unknowns x and y can be solved by the following methods :

1. Method of substitution

Solve $5x – 24y = 16$; $4x – y = 31$.
From the second equation we get $y = 4x – 31$ and substituting for y in the first equation we have

$$5x – 24(4x – 31) = 16$$

which gives $x = 8$ and then $y = 1$.

2. Method of comparison

From each equation find y in terms of x and compare them

$$y = 4x – 31 \text{ and } y = \frac{5x – 16}{24}$$

Hence, $4x – 31 = \frac{5x – 16}{24}$ which gives $x = 8$

and then $y = 1$.

3. Method of elimination

$$\left.\begin{aligned} 5x – 24y = 16 \\ 4x – y = 31 \end{aligned}\right\}$$

Multiply the second by 24 and subtract. y gets eliminated.

$\therefore$ $5x – 96x = –728$ and hence $x = 8$, $y = 1$

4. Method of cross multiplication

If $a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0$, prove that

$$\frac{x}{b_1c_2 – b_2c_1} = \frac{y}{c_1a_2 – c_2a_1} = \frac{z}{a_1b_2 – a_2b_1}$$

Proof :
Multiplying the first equation by c_2 and the second by c_1, we have

www.pdfworld.net

www.pdfworld.net

$$a_1c_2x + b_1c_2y + c_1c_2z = 0$$
$$a_2c_1x + b_2c_1y + c_1c_2z = 0$$

Hence, by subtraction

$$(a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)\,y = 0$$

$$\therefore \quad (c_1a_2 - c_2a_1)x = (b_1c_2 - b_2c_1)\,y$$

$$\therefore \quad \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1}$$

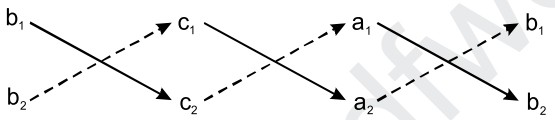
Here we have eliminated z.
Similarly, eliminating x, we get

$$\frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

Hence from (1) and (2),

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

In order to write down the denominators of x, y and z in (3) apply the following rule, "write down the coefficients of x, y and z in order beginning with the coefficients of y and repeat them as in the diagram".



Multiply the coefficients across in the way indicated by the arrows; remembering that informing the products any one obtained by descending is positive and any one obtained by ascending is negative.

In the above equations if we put z = 1, we have

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

which gives the solution of the equations

$$a_1x + b_1y + c_1 = 0,\ a_2x + b_2y + c_2 = 0.$$

➤ **Example 1.** Solve the equation $ax = x + 1$

➤ ***Solution*** $ax = x + 1 \quad \Rightarrow (a - 1)\,x = 1$

If $a - 1 \neq 0$, i.e. $a \neq 1$, then the equation has only one root:

$$x = \frac{1}{a-1}$$

If $a - 1 = 0$, i.e. $a = 1$, then $0.\ x = 1$, the equation has no roots.

Thus, if $a \neq 1$, then $x = \frac{1}{a-1}$; if $a = 1$, then ϕ.

➤ **Example 2.** Solve $\left.\begin{matrix} 2x + 3y - 8 = 0 \\ 3x - 4y + 5 = 0 \end{matrix}\right\}$ by rule of cross multiplication.

➤ ***Solution***

(i) Write all the terms of L.H.S. with R.H.S = 0 (ii) Negative sign is part of the coefficient.

$$\begin{matrix} 3 & (-8) & 2 & 3 \\ (-4) & 5 & 3 & (-4) \end{matrix}$$

$$\frac{x}{(3)(5) - (-4)(-8)} = \frac{y}{(-8)(3) - (5)(2)}$$

$$= \frac{1}{2(-4) - (3)(3)}$$

$$\text{(i.e.)} \ \frac{x}{-17} = \frac{y}{-34} = \frac{1}{-17}$$

or $\frac{x}{1} = \frac{y}{2} = 1$ and hence x = 1, y = 2.

➤ **Example 3.** Solve : $2x - 3y + 4z = 0$;
$7x + 2y - 6z = 0$; $4x + 3y + z = 37$

➤ ***Solution*** From the first two equations we have

$$\begin{matrix} (-3) & 4 & 2 & -3 \\ 2 & (-6) & 7 & 2 \end{matrix}$$

$$\frac{x}{10} = \frac{y}{40} = \frac{z}{25} \text{ or } \frac{x}{2} = \frac{y}{8} = \frac{z}{5} = k \text{ (say)}$$

Then x = 2k, y = 8k, z = 5k
Substituting these values of x, y, z in the third equation,

$$k\,(8 + 24 + 5) = 37 \quad \therefore \quad k = 1$$

Hence x = 2, y = 8, z = 5

www.pdfworld.net

www.pdfworld.net

➤ **Example 4.** Find the condition that the three equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are consistent.

➤ ***Solution*** Consistency means that there must be some set of values of the ordered pair (x, y) which satisfy all the three equations. Solving the first two by rule of cross multiplication, we get

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Substituting these values in the third equation

$$a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

or $a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$

Practice Problems [E]

1. If $\frac{ay - bx}{c} = \frac{cx - az}{b} = \frac{bz - cy}{a}$ show that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

2. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$, then show that $\frac{x^2}{1 - a^2} = \frac{y^2}{1 - b^2} = \frac{x^2}{1 - c^2}$

3. Solve : $12x + 9y - 7z = 2$, $8x - 26y + 9z = 1$, $23x + 21y - 15z = 4$.

4. Factorize the following :

(i) $(a^2 - b^2)(a^2 + ab + b^2) - a^3 + b^3$

(ii) $3x^3 (x^2 - x + 1) + 2x^2(x - x^2 - 1) - x^3 + x^2 - x$

(iii) $a^4 - a^2(b^2 + 1) + b^2$

5. Find a polynomial in x of the third degree which will vanish when $x = 1$ & $x = -2$ and will have the values 4 & 28 when $x = -1$ and $x = 2$ respectively.

6. Solve the following equation, $(a^2 - 9)x = a^3 + 27$.

7. $\frac{x + a}{1 + a} = \frac{x - a}{2 + a}$

8. Determine all non-negative integral pairs (x, y) for which $(xy - 7)^2 = x^2 + y^2$.

9. Solve for integers x, y, z ; $x + y = 1 - z$, $x^3 + y^3 = 1 - z^2$.

1.10 LINEAR INEQUALITIES

Fundamental Concepts and Definitions of Inequalities

We must understand some of the fundamental properties of inequalities.

1. Transitive Property

(i) If $a > b$ and $b > c$, then $a > c$.

(ii) If $a_1 \geq a_2, a_2 \geq a_3, \ldots\ldots, a_{n-1} \geq a_n$, then $a_1 \geq a_n$ The equality sign holds iff all the a_i's are equal.

2. Properties related to addition and subtraction

(i) If $a > b$ and if c be any real number, then $a + c > b + c$.

(ii) If $a > b$ and $c > d$, then $a + c > b + d$.

(iii) If $a_1 \geq b_1, a_2 \geq b_2, \ldots, a_n \geq b_n$, then $a_1 + a_2 + \ldots\ldots a_n \geq b_1 + b_2 + \ldots\ldots b_n$. The equality sign holds iff $a_1 = b_1, \ldots\ldots, a_n = b_n$.

(iv) If $a \geq b$ and $c \geq d$, then $a - d \geq b - c$. The equality sign holds iff $a = b$, $c = d$.

3. Properties related to multiplication

(i) If $a > b$ and $c > 0$, then $ac > bc$. If $a > b$ and $c < 0$, then $ac < bc$.

(ii) If $a \geq b$ and $c > 0$, then $ac \geq bc$. The equality sign holds iff $a = b$.
If $a \geq b$ and $c < 0$, then $ac \leq bc$. The equality sign holds iff $a = b$

(iii) If $a > b > 0$ and $c > d > 0$, then $ac > bd$.

(iv) If $a_1 \geq b_1 > 0, a_2 \geq b_2 > 0, \ldots\ldots, a_n \geq b_n > 0$, then $a_1a_2 \ldots\ldots a_n \geq b_1b_2 \ldots\ldots b_n > 0$. The equality sign holds iff $a_1 = b_1, a_2 = b_2, \ldots\ldots, a_n = b_n$.

www.pdfworld.net

www.pdfworld.net

4. Properties related to division

(i) If $a > b > 0$, then $1/b > 1/a$.
(ii) If $a > b > 0$ and $c > d > 0$, then $a/d > b/c$.
(iii) If $a \geq b > 0$ and $c \geq d > 0$, then $a/d \geq b/c$. The equality sign holds iff $a = b$, $c = d$.

5. Properties related to powers and roots

(i) Let $a > b > 0$, let p and q be positive integers, and let $a^{1/q}$ and $b^{1/q}$ denote positive q^{th} roots of a and b respectively Then
$a^{p/q} > b^{p/q}$ and $b^{-p/q} > a^{-p/q}$

(ii) Let $a \geq b > 0$, let p be a non-negative integer and q a positive integer, and let $a^{1/q}$ and $b^{1/q}$ denote qth roots of a and b respectively. Then
$a^{p/q} \geq b^{p/q}$ and $b^{-p/q} \geq a^{-p/q}$
The equality sign holds iff either $a = b$ or $p = 0$

Polynomial Inequalities in One Variable

Suppose we have to solve an inequality

$$R(x) > Q(x) \quad \text{or} \quad R(x) < Q(x) \qquad ...(1)$$

where R(x) and Q(x) are polynomials in one variable x. Since the domain of permissible values of the polynomial R(x) and Q(x) consists of all real numbers, we can formulate the problem of solving inequality (1) as follows : find all numerical values of the variable x, each of which turns inequality (1) into a true numerical inequality. Each of those numerical values is a solution of inequality (1). Therefore, to solve inequality (1) is to find the set of all its solutions. When the set of all solutions of inequality (1) is empty, we say that inequality (1) has no solution.
Two inequalities $R(x) > Q(x)$ and $T(x) < S(x)$ are **equivalent** if any solution of the first inequality is a solution of the second and conversely, any solution of the second inequality is a solution of the first.

Linear Inequalities

Each of the inequalities

$$2x + 3 > 4, \quad 3x - 4 \leq 2x + 5, \quad \text{and} \quad 3 \geq -5x + 4$$

is a linear inequality in the variable x.
The solution of first degree inequalities can be formulated as follows :
The first degree polynomial $a_0x + a_1$ $(a_0 \neq 0)$

(a) is positive for any $x \in (-\frac{a_1}{a_0}, \infty)$ and negative for any $x \in (-\infty, -\frac{a_1}{a_0})$ when $a_0 > 0$,

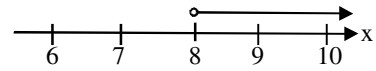
(b) is positive for any $x \in (-\infty, -\frac{a_1}{a_0})$ and negative for any $x \in (-\frac{a_1}{a_0}, \infty)$ when $a_0 < 0$.

CAUTION Remember that when you multiply or divide an inequality by a negative number, you must reverse the inequality symbol.

➤ **Example 1.** Solve the linear inequality $5x - 7 > 3x + 16$.

➤ ***Solution*** $5x - 7 > 3x + 16$
$5x > 3x + 16$
$5x - 3x > 16$
$2x > 16$
$x > 8$

Thus, the solution set consists of all real numbers that are greater than 8. The interval notation for this solution set is $(8, \infty)$. The number line graph of this solution set is shown in the figure.



Solution interval: $(8, \infty)$

➤ **Example 2.** Solve the linear inequality

$1 - \frac{3x}{2} \leq x - 4$ algebraically as well as graphically.

➤ ***Solution*** $1 - \frac{3x}{2} \leq x - 4$
$2 - 3x \leq 2x - 8$
$-3x \leq 2x - 10$

www.pdfworld.net

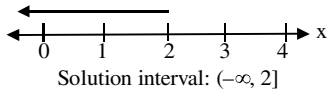
www.pdfworld.net

$$-5x \leq -10$$
$$x \leq 2$$

The solution set consists of all real numbers that are less than or equal to 2. The interval notation for this solution set is $(-\infty, 2]$. The number line graph of this solution set is shown in Figure (a).

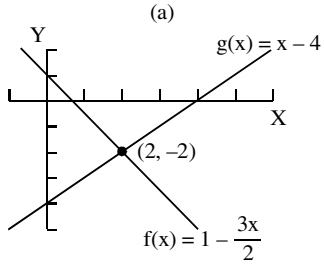
Graphically, the given inequality can be solved by sketching the graphs of the left and right sides.

$$f(x) = 1 - \frac{3x}{2} \quad \text{and} \quad g(x) = x - 4.$$



Solution interval: $(-\infty, 2]$

(a)



(b)

From Figure (b) the graphs appear to intersect at the point (2, –2). This is confirmed by the fact that $f(2) = -2 = g(2)$. Moreover, the graph of f lies above the graph of g to the left of their point of intersection, which implies that $f(x) \geq g(x)$ for $x \leq 2$.

Practice Problems [F]

1. Find all values of $x \in N$ which satisfy $5x - 7 < 2x + 8$
2. Find the greatest integer x satisfying $\frac{2x+1}{3} - \frac{3x-1}{2} > 1$
3. Solve the system of inequalities
$$\begin{cases} 2(3x-1) < 3(4x+1)+16 \\ 4(2+x) < 3x+8 \end{cases}$$
4. Find natural x which satisfy the system of inequalities
$$\begin{cases} x+3 < 4+2x \\ 5x-3 < 4x-1 \end{cases}.$$
5. Solve the following inequality, $ax \leq 1$
6. $a^2 + ax < 1 - x$.

1.11 METHOD OF INTERVALS

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, use the fact that a polynomial can change signs only at its zeros (the values that make the polynomial zero). Between two consecutive zeros a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real line into intervals in which the polynomial has no sign changes. These zeros are the critical numbers of the inequality, and the resulting intervals are the test intervals for the inequality.

Finding Test Intervals for a Polynomial

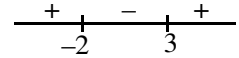
To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order (from smallest to largest). The zeros of a polynomial are its critical numbers.
2. Use the critical numbers of the polynomial to determine its test intervals.
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative, then the polynomial will have negative values for every x-value in the interval. If the value of the polynomial is positive, then the polynomial will have positive values for every x-value in the interval.

➤ **Example 1.** Determine the intervals on which $x^2 - x - 6$ is entirely negative and those on which it is entirely positive.

➤ ***Solution*** By factoring the quadratic as

$$x^2 - x - 6 = (x + 2)(x - 3),$$



www.pdfworld.net

www.pdfworld.net

you can see that the critical numbers occur at x = –2 and x = 3. Therefore, the test intervals for the quadratic are (–∞, –2), (–2, 3), and (3, ∞). In each test interval, choose a representative x-value and evaluate the polynomial, as shown in Table.

Test interval	Representative x – value	Value of polynomial	Conclusion
$(-\infty,-2)$	$x=-3$	$(-3)^2-(-3)-6=6$	Polynomial is positive
$(-2,3)$	$x=0$	$(0)^2-(0)-6=-6$	Polynomial is negative
$(3,\infty)$	$x=4$	$(4)^2-(4)-6=6$	Polynomial is positive

The polynomial has positive values for every x in the intervals (–∞, –2) and (3, ∞), and negative values for every x in the interval (–2, 3).

➤ **Example 2.** Solve $2x^2+5x>12$

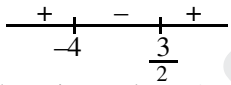
➤ ***Solution*** $2x^2+5x-12>0$ Write in standard form

$(x+4)(2x-3)>0$ Factor

Critical numbers : $x=-4,\ x=\frac{3}{2}$

Test intervals : $(-\infty,-4),\ \left(-4,\frac{3}{2}\right),\ \left(\frac{3}{2},\infty\right)$

Test : Is $(x+4)(2x-3)>0$?



After testing these intervals, you can see that the polynomial $2x^2+5x-12$ is positive in the open intervals $(-\infty,-4)\cup\left(\frac{3}{2},\infty\right)$.

➤ **Example 3.** Solve $x^2+4x+1\le 0$.

➤ ***Solution*** By the quadratic formula, the roots of x^2+4x+1 are

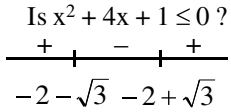
$$x=\frac{-4\pm\sqrt{16-4}}{2}=\frac{-4\pm 2\sqrt{3}}{2}=-2\pm\sqrt{3}.$$

Critical numbers : $x=-2-\sqrt{3},\ x=-2+\sqrt{3}$

Test intervals :

$(-\infty,-2,-\sqrt{3}),(-2-\sqrt{3},-2+\sqrt{3}),(-2+\sqrt{3},\infty)$

Test : Is $x^2+4x+1\le 0$?



After testing these intervals, you can see that the polynomial x^2+4x+1 is less than or equal to zero in the closed interval $[-2-\sqrt{3},\ -2+\sqrt{3}]$. Therefore, the solution set of the inequality is $[-2-\sqrt{3},\ -2+\sqrt{3}]$.

Quadratic Inequalities

We consider a quadratic inequality

$$ax^2+bx+c>0 \quad (a\ne 0) \qquad ...(1)$$

Using the identity transformation known as an isolation of a perfect square, we obtain

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right] \qquad ...(2)$$

where $D=b^2-4ac$. Therefore inequality (1) is equivalent to the inequality

$$a\left[\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right]>0 \quad (a>0). \qquad ...(3)$$

(a) If $D<0$, then for any numerical value of the unknown $x=x_0$ the left hand side of inequality (3) contains the sum of the nonnegative number $\left(x_0+\frac{b}{2a}\right)^2$ and the positive number $-\frac{D}{4a^2}$, i.e. inequality (3) turns into a true numerical inequality. Consequently, inequality (3) holds true for any x. To put it otherwise, in this case the set of all real numbers is the set of all solutions of inequality (3).

www.pdfworld.net

www.pdfworld.net

(b) If D = 0, then inequality (3) evidently turns into a true numerical inequality for any number x, except for $x_0 = -\frac{b}{2a}$. consequently, in this case the set $(-\infty, -\frac{b}{2a}) \cup (-\frac{b}{2a}, \infty)$ is the set of all solutions of inequality (3).

(c) if D > 0, then inequality (1) is equivalent to the inequality

$(x - x_1)(x - x_2) > 0 \quad (a > 0)$ ...(4)

where $x_1 = \frac{-b-\sqrt{D}}{2a}$, $x_2 = \frac{-b+\sqrt{D}}{2a}$. It is evident that $x_1 < x_2$. and therefore, applying the method of intervals, we find that the set $(-\infty, x_1) \cup (x_2, \infty)$ is the set of all solutions of inequality (4).

Let a < 0. Then inequality (2) is equivalent to the inequality

$$\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2} < 0 \quad (a < 0). \qquad ...(5)$$

(a) If D < 0, then, evidently, for any number x this inequality turns into a non-valid numerical inequality and therefore inequality (5) has no solution.

(b) If D = 0, then inequality (5) is equivalent to the inequality

$(x - x_1)(x - x_2) < 0 \; (a < 0)$,

where $x_1 = \frac{-b-\sqrt{D}}{2a}$, $x_2 = \frac{-b+\sqrt{D}}{2a}$.

If is it evident that $x_1 > x_2$, and therefore, applying the method of intervals, we find that the interval (x_2, x_1) is the set of all solutions of inequality (5).

We can solve the inequality $ax^2 + bx + c < 0$ ($a \neq 0$) by analogy. We can collect and tabulate the arguments presented above.

Table

a	*D*	*Inequality*	*Solution of inequality*
a > 0	D > 0	$ax^2 + bx + c < 0$	$(-\infty, \frac{-b-\sqrt{D}}{2a}) \cup (\frac{-b+\sqrt{D}}{2a}, \infty)$
a > 0	D > 0	$ax^2 + bx + c < 0$	$(\frac{-b-\sqrt{D}}{2a}, \frac{-b+\sqrt{D}}{2a})$
a > 0	D = 0	$ax^2 + bx + c > 0$	$(-\infty, \frac{-b}{2a}) \cup (\frac{-b}{2a}, \infty)$
a > 0	D = 0	$ax^2 + bx + c < 0$	no solution
a > 0	D < 0	$ax^2 + bx + c > 0$	$(-\infty, \infty)$
a > 0	D < 0	$ax^2 + bx + c < 0$	no solution
a < 0	D > 0	$ax^2 + bx + c > 0$	$(\frac{-b+\sqrt{D}}{2a}, \frac{-b-\sqrt{D}}{2a})$
a < 0	D > 0	$ax^2 + bx + c < 0$	$(-\infty, \frac{-b+\sqrt{D}}{2a}) \cup (\frac{-b-\sqrt{D}}{2a}, \infty)$
a < 0	D = 0	$ax^2 + bx + c > 0$	no solution
a < 0	D = 0	$ax^2 + bx + c < 0$	$(-\infty, \frac{-b}{2a}) \cup (\frac{-b}{2a}, \infty)$
a < 0	D < 0	$ax^2 + bx + c > 0$	no solution
a < 0	D < 0	$ax^2 + bx + c < 0$	$(-\infty, \infty)$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

Consider some examples of solving quadratic inequalities :

(i) The solution set of $x^2 + 2x + 4 > 0$ consists of the entire set of real numbers, $(-\infty, \infty)$ since $a > 0$ and $D < 0$. In other words, the quadratic $x^2 + 2x + 4$ is positive for every real value of x. (Note that this quadratic inequality has no critical numbers. In such a case, there is only one test interval – the entire real line.)

(ii) The solution set of $x^2 + 2x + 1 \le 0$ consists of the single real number, –1, because $a > 0$ and $D = 0$, in which case the quadratic is zero at $x = \frac{-b}{2a}$.

(iii) The solution set of $x^2 + 3x + 5 < 0$ is empty since $a > 0$ and $D < 0$. In other words, the quadratic $x^2 + 3x + 5$ is not less than zero for any value of x.

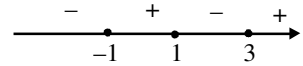
(iv) The solution set of $x^2 - 4x + 4 > 0$ consists of all real numbers except the number 2. In interval notation, this solution set can be written as $(-\infty, 2) \cup (2, \infty)$.

(v) The solution set of $x^2 - x - 6 < 0$ is the interval $(-2, 3)$. The roots of the quadratic $P(x) = x^2 - x - 6$ are $x_1 = 3$ and $x_2 = -2$. This means that the inequality is equivalent to $(x - 3)(x + 2) < 0$.
Applying the sign scheme as discussed above, we find that the set of all solutions of the inequality is the interval $(-2, 3)$

Polynomial Inequalities

➤ **Example 4.** Solve the inequality
$$x^3 - 3x^2 - x + 3 > 0$$

➤ ***Solution*** The roots of the polynomial can be easily found.
In fact, $x^3 - 3x^2 - x + 3 = x^2(x - 3) - (x - 3) = (x - 3)(x - 1)(x + 1)$.
The inequality can now be written as follows :
$$(x - 3)(x - 1)(x + 1) > 0.$$
The critical points of the polynomial are
$$x = -1, x = 1, x = 3$$
These points partition the number line into 4 sub-intervals. Let us mark the points on the number line. To determine the sign of the polynomial on each sub-interval, we can do the following: note that for $x > 3$ all the linear factors in the polynomial are positive and, consequently, on the interval $(3, \infty)$ the polynomial assumes only positive values. We mark the interval $(3, \infty)$ with plus sign.



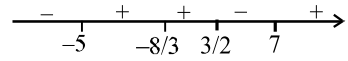
When passing from the interval $(3, \infty)$ to the interval $(1, 3)$, through the point $x = 3$, only one of the linear factors, $x - 3$, changes sign and, consequently, the polynomial becomes negative. So we mark the interval $(1, 3)$ with negative sign. Proceeding to the left, to the interval $(-1, 1)$, we find that only the factor $x - 1$ changes sign. This means that on passing through the point $x = 1$ the left-hand side of the inequality changes sign. Finally, the passage to the last interval, $(-\infty, -1)$ is again followed by a change of sign of the function.
The inequality being strict, the critical points themselves are not the solutions.
Hence the solution set is $(-1, 1) \cup (3, \infty)$.

➤ **Example 5.** Solve the inequality
$$(x + 5)(2x - 3)^5(-x + 7)^3(3x + 8)^2 < 0 \quad ...(1)$$

➤ ***Solution*** First of all we multiply this inequality by $-\left(\frac{1}{2}\right)^5\left(\frac{1}{3}\right)^2$ and get an equivalent inequality

$$[x - (-5)]\left[x - \left(-\frac{8}{3}\right)\right]^2\left(x - \frac{3}{2}\right)^5(x - 7)^3 > 0 \quad ...(2)$$

To solve inequality (2), we use the method of intervals. We mark the numbers $-5, -\frac{8}{3}, \frac{3}{2}, 7$ on the number line. We put the plus sign to the right of the greatest number, ie. the number 7. When passing through the point 7 the polynomial



$$P(x) = [x - (-5)]\left[x - \left(-\frac{8}{3}\right)\right]^2\left(x - \frac{3}{2}\right)^5(x - 7)^3 \quad ...(3)$$

www.pdfworld.net

www.pdfworld.net

changes sign since the binomial x – 7 is of the odd degree in the product (3), and therefore we put the minus sign in the interval $\left(\frac{3}{2},7\right)$, When passing through the point $\frac{3}{2}$, the polynomial P(x) changes sign since the binomial $x-\frac{3}{2}$ is of the odd degree in the product (3), and therefore in the interval $\left(-\frac{8}{3},\frac{3}{2}\right)$ we put the plus sign. When passing through the point $-\frac{8}{3}$, the polynomial P(x) does not change sign since the binomial $x-\left(-\frac{8}{3}\right)$ is of the even degree in the product (3), and therefore we put the plus sign in the interval $\left(-5,-\frac{8}{3}\right)$. Finally, when passing through the point –5, the polynomial P(x) changes sign since the binomial x – (–5) is of the first degree in the product (3), and therefore we put the minus sign in the interval (– ∞, –5). Thus the solution of inequality (2) and of the equivalent inequality (1) is the set of all intervals where the plus sign has been put, i.e. the set of all solutions is the set

$$\left(-5,-\frac{8}{3}\right)\cup\left(\frac{-8}{3},\frac{3}{2}\right)\cup(7,\infty)$$

➤ **Example 6.** Solve the inequality

$$(x^2-3x+2)(x^3-3x^2)(4-x^2)\geq 0 \qquad ...(1)$$

➤ ***Solution*** Since the identities

$x^2-3x+2=(x-2)(x-1)$,

$x^3-3x^2=x^2(x-3)$

$4-x^2=-(x-2)(x+2)$

hold true, it follows that inequality (1) is equivalent to the inequality

$$[x-(-2)]x^2(x-1)\,x-2)^2(x-3)\leq 0. \qquad ...(2)$$

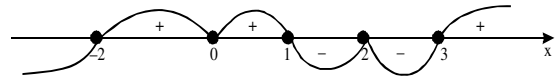
We first solve the equation

$$[x-(-2)]x^2(x-1)(x-2)^2(x-3)=0 \qquad ...(3)$$

It has only five roots : $x_1=-2$, $x_2=0$, $x_3=1$, $x_4=2$, and $x_5=3$. Then we solve the strict inequality

$$[x-(-2)]x^2(x-1)(x-2)^2(x-3)<0 \qquad ...(4)$$

using the method of intervals. The set of all its solutions is the set $(-\infty,-2)\cup(2,3)$.



Uniting the set of solutions of equation (3) and that of the strict inequality (4), we get the set of all solutions of inequality (1) : $(-\infty,-2)\cup\{0\}\cup[1,3]$

Practice Problems [G]

Solve the following inequalities:

1. $x^2-3x-4>0$
2. $x^2-3x-4\leq 0$
3. $x^2-6x+9>0$
4. $4x^2+4x+1\leq 0$
5. $2x^2-x+5>0$
6. $\begin{cases}2x^2-5x+2=0\\ x-2<0\end{cases}$
7. $\begin{cases}x^2-2x-3>0\\ x+4\geq 0\end{cases}$
8. $\begin{cases}x^2-6x+9\leq 0\\ 2x-5>0\end{cases}$
9. $\begin{cases}x^2-4\geq 0\\ x^2-2x-8\geq 0\\ -x^2+5x-4\geq 0\end{cases}$
10. $\begin{cases}x^2+x-6>0\\ x+2\leq 0\\ x^2-x+2>0\end{cases}$
11. $(x+4)^5(x+3)^6(x+2)^7(x-1)^8\leq 0$
12. $x^4-3x^2+x^2+3x-2\geq 0$
13. $x^4+6x^3+6x^2+6x+5<0.$
14. $(x+3)^4+(x-1)^4\geq 82.$
15. Find integral x's which satisfy the inequality $x^4-3x^3-x+3<0.$

www.pdfworld.net

www.pdfworld.net

1.12 RATIONAL FUNCTION

A rational expression is an algebraic fraction which is the quotient of the division of one polynomial by another.

A rational expression which is the quotient of the division of a polynomial A by a polynomial B is usually written as $\frac{A}{B}$, where A is the numerator of the algebraic fraction and B is its denominator.

Examples of rational expressions:

$$\frac{3a+b}{a^3+1}, \frac{ab-b}{d+a}, \frac{a^2+b^2}{a-b}, \frac{xy+6y}{7x+8y}$$

The domain of permissible values of the rational expression $\frac{A}{B}$, which includes n letters, is the set of all number sequences corresponding to the letter sequence of the fraction $\frac{A}{B}$, except those for each of which the respective numerical value of B is zero.

For instance, the domain of permissible values of the algebraic fraction $\frac{a^2+b^2}{a-b}$ is the set $\{(a,b) \mid a \in R, b \in R, a \neq b\}$

A rational function is a function of the form. $y = f(x) = \frac{g(x)}{h(x)}$, where g (x) & h (x) are polynomials. It is defined for all real values of x excluding the roots of h(x) = 0.

➤ **Example 1.** Solve the equation

$$\frac{3x}{x-1} - \frac{2x}{x+2} = \frac{3x-6}{(x-1)(x+2)},$$

➤ ***Solution*** Any solution of this equation is also a solution of the equation

$$3x(x+2) - 2x(x-1) = 3x - 6,$$

which results from multiplying the given equation by (x – 1) (x + 2). This equation is, evidently, equivalent to the following quadratic equation:

$$x^2 + 5x + 6 = 0.$$

We can find its roots by formula (4): x = –3 and x = –2. Consequently, only the numbers – 3 and –2 can be solution of this equation. Verification confirms that the number –3 is its root while the number –2 is not since for x = –2 the sides of the given equation are not defined. The solution is {–3}.

Rational Inequalities

The concepts of critical numbers and test intervals can be extended to inequalities involving rational expressions. To do this use the fact that the value of a rational expression can change sign only at its zeros (the x-values for which its numerator is zero) and undefined values (the x-values for which its denominator is zero). These two types of numbers make up the critical numbers of a rational inequality. For instance, the critical numbers of the inequality $\frac{x-1}{(x-2)(x+3)} < 0$ are x = 1 (the numerator is zero), and x = 2 and x = –3 (the denominator is zero).

From these three critical numbers we get four test intervals : $(-\infty, -3)$, $(-3, 1)$, $(1, 2)$, and $(2, \infty)$.

➤ **Example 2.** Solve : $\frac{x+1}{x+3} < 2$

➤ ***Solution*** It is incorrect to begin the solution of $\frac{x+1}{x+3} < 2$ by multiplying both sides by x + 3 to clear fractions. The problem with this is that because x + 3 contains a variable and can be positive or negative (depending on the value of x), we do not know whether or not to reverse the sense of the inequality. Instead, as in solving quadratic inequalities, begin the solution process by obtaining 0 on one side of the inequality.

Our first step is to obtain 0 on one side of the inequality

$$\frac{x+1}{x+3} < 2$$

$$\Rightarrow \quad \frac{x+1}{x+3} - 2 < 0$$

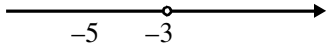
$$\Rightarrow \quad \frac{x+1-2(x+3)}{x+3} < 0$$

www.pdfworld.net

www.pdfworld.net

$$\Rightarrow \quad \frac{-x-5}{x+3} < 0 \quad -x-5=0 \qquad x+3=0$$

$x = -5$ $\qquad$ $x = -3$



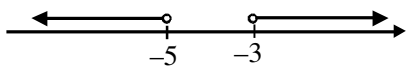
The intervals are $(-\infty, -5)$, $(-5, -3)$, and $(-3, \infty)$. Now we will take one representative number from each test interval and substitute that number into the original inequality.

Test	Substitute into Representative Number	$\frac{x+1}{x+3} < 2$	Conclusion
$(-\infty, -5)$	-6	$\frac{-6+1}{-6+3} \overset{?}{<} 2$	$(-\infty, -5)$
		$\frac{5}{3} < 2$ True	belongs to the solution set.
$(-5, -3)$	-4	$\frac{-4+1}{-4+3} \overset{?}{<} 2$	$(-5, -3)$ does
		$3 < 2$ False	not belong to the solution set.
$(-3, \infty)$	0	$\frac{0+1}{0+3} \overset{?}{<} 2$	$(-3, \infty)$
		$\frac{1}{3} < 2$ True	belongs to the solution set.

Our analysis indicates that the solution set is

$$(-\infty, -5) \cup (-3, \infty),$$

shown on a number line in Figure.



Before considering another example, let's summarize the process involved in the solution of rational inequalities.

Procedure for solving rational inequalities

1. Write the inequality so that one side is 0.
2. In necessary, combine terms on the other side into a single quotient.
3. Locate points on the number line for which the quotient is zero (numbers making the numerator zero) or undefined (numbers making the denominator zero). These critical numbers will divide the number line into test intervals.
4. Choose one representative number for each test interval. If substituting that value into the original inequality produces a true statement, then all real numbers in the test interval belong to the solution set. The solution set is the union of all such test intervals.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

➤ **Example 3.** Solve $\frac{2x-7}{x-5} \le 3$

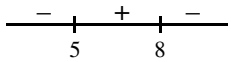
➤ ***Solution*** $\frac{2x-7}{x-5} \le 3 \Rightarrow \frac{2x-7}{x-5} - 3 \le 0$

$\Rightarrow \frac{2x-7-3x+15}{x-5} \le 0 \Rightarrow \frac{-x+8}{x-5} \le 0$

Critical numbers : $x = 5, x = 8$

Test intervals : $(-\infty, 5), (5, 8), (8, \infty)$

Test : Is $\frac{-x+8}{x-5} \le 0$?



By testing these intervals, you can determine that the rational expression $(-x + 8)/(x - 5)$ is negative in the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because $(-x + 8)/(x - 5) = 0$ when $x = 8$, you can conclude that the solution set of the inequality is $(-\infty, 5) \cup [8, \infty)$.

➤ **Example 4.** Solve $\frac{x}{x-2} > \frac{1}{x+3}$

➤ ***Solution*** $\frac{x}{x-2} > \frac{1}{x+3} \Rightarrow \frac{x}{x-2} > \frac{1}{x+3} > 0$

$\Rightarrow \frac{x^2+3x-x+2}{(x-2)(x+3)} > 0 \Rightarrow \frac{x^2+2x+2}{(x-2)(x+3)} > 0$

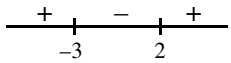
The discriminant of the numerator is

$b^2 - 4ac = 4 - 8 = -4 < 0$ which indicates that the numerator has no zeros. Therefore, $x = 2$ and $x = -3$ are the only critical numbers

Critical numbers : $x = -3, x = 2$

Test intervals : $(-\infty -3), (-3, 2) (2, \infty)$

Test : Is $\frac{x^2+2x+2}{(x-2)(x+3)} > 0$?



By testing these intervals, you can determine that the rational expression $(x^2 + 2x + 2)/[(x - 2)(x + 3)]$ is positive in the open intervals $(-\infty, -3)$ and $(2, \infty)$. Therefore, the solution set of the inequality is $(-\infty, -3) \cup (2, \infty)$.

➤ **Example 5.** Solve $\frac{3}{x-1} - \frac{2}{x+1} \ge 1$

➤ ***Solution*** $\frac{3}{x-1} - \frac{2}{x+1} \ge 1$

$\frac{3}{x-1} - \frac{2}{x+1} - 1 \ge 0$

$\frac{3(x+1)}{(x-1)(x+1)} - \frac{3(x-1)}{(x-1)(x+1)} - \frac{(x-1)(x+1)}{(x-1)(x+1)} \ge 0$

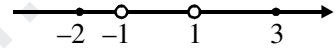
$\frac{3(x+1)-2(x-1)-(x-1)(x+1)}{(x-1)(x+1)} \ge 0$

$\frac{3x+3-2x+2-x^2+1}{(x-1)(x+1)} \ge 0$

$\frac{-x^2+x+6}{(x-1)(x+1)} \ge 0$

$-x^2 + x + 6 = 0; \quad (x - 1)(x + 1) = 0$

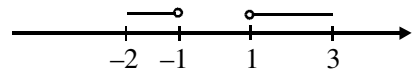
$x = 3$ or $x = -2; \quad x = 1$ or $x = -1$



The open circles at –1 and 1 indicate that they cannot be solutions. The intervals are $(-\infty, -2)$, $(-2, -1)$, $(-1, 1)$, $(1, 3)$, and $(3, \infty)$. By testing a representative number from each interval, you can determine that the original inequality is satisfied in $(-2, -1)$ and $(1, 3)$. Furthermore, since the simplified form of the inequality is

$$\frac{-x^2+x+6}{(x-1)(x+1)} \ge 0$$

we must include values of x that cause the expression on the left to equal 0. These were the values that result in zero in the numerator $(-x^2 + x + 6 = 0)$, which we found to be –2 and 3. We can conclude that the solution set of the inequality is $[-2, -1) \cup (1, 3]$, shown on a number line in Figure.



➤ **Example 6.** Solve the inequality

$$\frac{(x^2-3x+2)(x^2+2x+2)}{(x-3)(x-1-2x^2)} \ge 0$$

➤ ***Solution*** Note that the discriminants of the quadratic trinomials $x^2 + 2x + 2$ and $-2x^2 + x - 1$ are negative. Therefore, the trinomials attain values of the same sign coinciding with the sign of the coefficient of

www.pdfworld.net

www.pdfworld.net

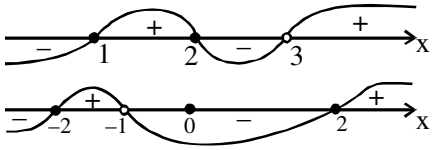
$x^2 : x^2 + 2x + 2 > 0$ and $-2x^2 + x - 1 < 0$ for any x. We then rewrite the given inequality as follows:

$$\frac{(x^2 - 3x + 2)}{(x - 3)} \le 0 \text{ or } \frac{(x-1)(x-2)}{x-3} \le 0$$

Mark the points x = 1, x = 2 and x = 3 on the number line.

In the interval x < 1 we take an arbitrary point, say x = 0. At this point $\frac{(x-1)(x-2)}{x-3} = -\frac{3}{2} < 0$.

The **wavy curve** in the figure indicates alternation in sign



At the points x = 1 and x = 2 the fraction vanishes; x = 1 and x = 2 are solutions of the given inequality. For x = 3 the fraction loses meaning.

Thus, the solution of the inequality will be :

$x \le 1, 2 \le x < 3$.

Practice Problems [H]

1. Solve the following equations :

(i) $3x + 1 = (4x - 3) - (x - 4)$

(ii) $ax = a^2$

(iii) $(a - 2)x = a^2 - 4$

(iv) $(a^2 - 9)x = a^3 + 27$

(v) $\frac{3}{x-1} = \frac{2}{x}$

(vi) $\frac{4}{x+2} = a$

2. Solve the following inequations :

(i) $\frac{x^2 - x - 6}{x^2 + 6x} \ge 0$

(ii) $\frac{x^2 - 6x + 9}{5 - 4x - x^2} \ge 0$

(iii) $\frac{2x+3}{x^2 + x - 12} \le \frac{1}{2}$

(iv) $\frac{(2 - x^2)(x-3)^3}{(x+1)(x^2 - 3x - 4)} \ge 0$

(v) $\frac{x^4 - 3x^3 + 2x^2}{x^2 - x - 30} > 0$

(vi) $\frac{2x}{x^2 - 9} \le \frac{1}{x+2}$

3. Solve the following inequations :

(i) $\frac{x^2 - 5x + 4}{x - 4} \ge 0$

(ii) $\frac{x+3}{x^2 - 6x + 8} < 0$

(iii) $\frac{1}{2x-3} \le \frac{1}{2x+5}$

(iv) $\frac{1}{x^2 + x} \le \frac{1}{2x^2 + 2x + 3}$

(v) $\frac{1}{x-1} - \frac{4}{x-2} + \frac{4}{x-3} - \frac{1}{x-4} < \frac{1}{30}$

(vi) $\frac{5 - 2x}{3x^2 - 2x - 16} < 0$

(vii) $\frac{(x+2)(x^2 - 2x + 1)}{4 + 3x + x^2} \ge 0$

(viii) $\frac{1 - 2x - 3x^2}{3x - x^2 - 5} > 0$

4. Find all values of $n \in N - \{10\}$ for which the equation $\frac{x-8}{n-10} = \frac{n}{x}$ has no solutions.

5. For what values of a is the inequality $\frac{x - 2a - 1}{x - a} < 0$ satisfied for all x lying in the interval [1, 2] ?

1.13 PARTIAL FRACTIONS

Introduction

The sums of rational expressions are found by combining two or more rational expressions into one rational expression. Here, the reverse process is considered : given one rational expression, express it as the sum of two or more rational expressions. A special type of sum of rational expressions is called the partial fraction decomposition; each term

www.pdfworld.net

www.pdfworld.net

in the sum is a partial fraction. The technique of decomposing a rational expression into partial fractions is useful in calculus and other areas of mathematics.

Partial Fraction Decomposition of $\frac{f(x)}{g(x)}$

To form a partial fraction decomposition of a rational expression, we use the following steps:

Step-1 : If f(x)/g(x) is not a proper fraction (a fraction with the numerator of lower degree than the denominator), divide f(x) by g(x). For example,

$$\frac{x^4 - 3x^3 + x^2 + 5x}{x^3 + 3} = x^2 - 3x - 2 + \frac{14x + 6}{x^2 + 3}.$$

Then, apply the following steps to the remainder, which is a proper fraction.

Step-2 : Factor g(x) completely into factors of the form $(ax + b)^m$ or $(cx^2 + dx + e)^n$, where $cx^2 + dx + e$ is irreducible and m and n are integers.

Step-3 :

(a) For each distinct linear factor (ax + b), the decomposition must include the term $\frac{A}{ax + b}$.

(b) For each repeated linear factor $(ax + b)^m$, the decomposition must include the terms

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + ... + \frac{A_m}{(ax + b)^m}$$

Step-4 :

(a) For each distinct quadratic factor $(cx^2 + dx + e)$, the decomposition must include the term $\frac{Bx + C}{cx^2 + dx + e}$.

(b) For each repeated quadratic factor $(cx^2 + dx + e)^n$, the decomposition must include the terms

$$\frac{B_1x + C_1}{cx^2 + dx + e} + \frac{B_2x + C_2}{(cx^2 + dx + e)^2} + ... + \frac{B_nx + C_n}{(cx^2 + dx + e)^n}$$

Step-5 : Use algebraic techniques to solve for the constants in the numerators of the decomposition.

To find the constants in Step-5, the goal is to get a system of equations with as many equations as there are unknowns in the numerators. One method for getting these equations is to substitute values for x on both sides of the rational equation formed from Steps 3 or 4.

Distinct Linear Factors

➤ **Example 1.** Find the partial fraction decomposition of $\frac{2x^4 - 8x^2 + 5x - 2}{x^3 - 4x}$.

➤ ***Solution*** The given fraction is not a proper fraction; the numerator has higher degree than the denominator. Perform the division.

$$\begin{array}{r} 2x \\ x^3 - 4x \overline{)\ 2x^4 - 8x^2 + 5x - 2} \\ \underline{2x^4 - 8x^2} \\ 5x - 2 \end{array}$$

The quotient is $\frac{2x^4 - 8x^2 + 5x - 2}{x^2 - 4x} = 2x + \frac{5x - 2}{x^3 - 4x}$. Now, work with the remainder fraction.

Factor the denominator as $x^3 - 4x = x(x + 2)(x - 2)$. Since the factors are distinct linear factors, use step 3(a) to write the decomposition as

$$\frac{5x - 2}{x^3 - 4x} = \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{x - 2}, \qquad ...(1)$$

where A, B, and C are constants that need to be found. Multiply both sides of equation (1) by x(x + 2) (x – 2), getting

$$5x - 2 = A(x + 2)(x - 2) + Bx(x - 2) + Cx(x + 2). \quad ...(2)$$

Equation (1) is an identity, since both sides represent the same rational expression. Thus, equation (2) is also an identity. Equation (1) holds for all values of x except 0, –2, and 2. However, equation (2) holds for all values of x.

In particular, substituting 0 for x in equation (2) gives –2 = –4A, so that $A = \frac{1}{2}$.

www.pdfworld.net

www.pdfworld.net

Similarly, choosing $x = -2$ gives $-12 = 8B$, so that $B = -\frac{3}{2}$.

Finally, choosing $x = 2$, $8 = 8C$, so that $C = 1$.

The remainder rational expression can be written as the following sum of partial fractions :

$$\frac{5x-2}{x^3-4x} = \frac{1}{2x} + \frac{-3}{2(x+2)} + \frac{1}{x-2},$$

and the given rational expression can be written as

$$\frac{2x^4-8x^2+5x-2}{x^3-4x} = 2x + \frac{1}{2x} + \frac{-3}{2(x+2)} + \frac{1}{x-2}$$

Verification can be done by combining the terms on the right.

Repeated linear factors

➤ **Example 2.** Find the partial fraction decomposition of $\frac{2x}{(x-1)^3}$.

➤ ***Solution*** This is a proper fraction. The denominator is already factored with repeated linear factors. Write the decomposition as shown, by using step 3(b) above.

$$\frac{2x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Clear the denominators by multiplying both sides of this equation by $(x-1)^3$.

$$2x = A(x-1)^2 + B(x-1) + C$$

Substituting 1 for x leads to $C = 2$, so that

$$2x = A(x-1)^2 + B(x-1) + 2 \quad ...(1)$$

The only root has been substituted, and values for A and B still need to be found. However, any number can be substituted for x. For example, when we choose $x = -1$ (because it is easy to substitute), equation (1) becomes

$$-2 = 4A - 2B + 2$$
$$-4 = 4A - 2B$$
$$-2 = 2A - B \quad ...(2)$$

Substituting 0 for x in equation (1) gives

$$0 = A - B + 2$$
$$2 = -A + B \quad ...(3)$$

Now, solve the system of equation (2) and (3) to get $A = 0$ and $B = 2$.

The partial fraction decomposition is

$$\frac{2x}{(x-1)^3} = \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3}$$

Three substitutions were needed because there were three constants to evaluate, A, B, and C.

To check this result, combine the terms on the right.

Distinct Linear and Quadratic Factors

➤ **Example 3.** Find the partial fraction decomposition of $\frac{x^2+3x-1}{(x+1)(x^2+2)}$.

➤ ***Solution*** This denominator has distinct linear and quadratic factors, where neither is repeated. Since $x^2 + 2$ cannot be factored, it is irreducible. The partial fraction decomposition is

$$\frac{x^2+3x-1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}.$$

Multiply both sides by $(x + 1)(x^2 + 2)$ to get

$$x^2 + 3x - 1 = A(x^2 + 2) + (Bx + C)(x + 1).$$

First, substitute -1 for x to get

$$(-1)^2 + 3(-1) - 1 = A[(-1)^2 + 2] + 0$$
$$-3 = 3A$$
$$A = -1$$

Replace A with -1 in equation (1) and substitute any value for x. For instance, if $x = 0$.

$$0^2 + 3(0) - 1 = -1(0^2 + 2) + (B \cdot 0 + C)(0 + 1)$$
$$-1 = -2 + C$$
$$C = 1$$

Now, letting $A = -1$ and $C = 1$, substitute again in equation (1), using another value for x. For $x = 1$,

$$3 = -3 + (B + 1)(2)$$
$$6 = 2B + 2$$
$$B = 2.$$

Using $A = -1$, $B = 2$, and $C = 1$, the partial fraction decomposition is

$$\frac{x^2+3x-1}{(x+1)(x^2+2)} = \frac{-1}{x+1} + \frac{2x+1}{x^2+2}.$$

Again, this work can be checked by combining terms on the right.

For fractions with denominators that have quadratic factors, another method is often more convenient.

www.pdfworld.net

www.pdfworld.net

The system of equations is formed by equating coefficients of like terms on both sides of the partial fraction decomposition. For instance, in example 3, after both sides were multiplied by the common denominator, the equation was

$x^2 + 3x - 1 = A(x^2 + 2) + (Bx + C)(x + 1)$.

Multiplying on the right and collecting like terms, we have

$x^2 + 3x - 1 = Ax^2 + 2A + Bx^2 + Bx + Cx + C$

$x^2 + 3x - 1 = (A + B)x^2 + (B + C)x + (C + 2A)$.

Now, equating the coefficient of like powers of x gives the three equations

$1 = A + B$

$3 = B + C$

$-1 = C + 2A$

Solving this system of equations for A, B and C would give the partial fraction decomposition. The next example uses a combination of the two methods.

Repeated Quadratic Factors

➤ **Example 4.** Find the partial fraction decomposition of $\frac{2x}{(x^2+1)^2(x-1)}$.

➤ ***Solution*** The expression has both a linear factor and a repeated quadratic factor. By steps 3(a) and 4(b) from the beginning of this section,

$$\frac{2x}{(x^2+1)^2(x-1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1}$$

Multiplication of both sides by $(x^2 + 1)^2 (x - 1)$ leads to

$2x = (Ax + B)(x^2 + 1)(x - 1)$

$+ (Cx + D)(x - 1) + E(x^2 + 1)^2$. ...(1)

If x = 1, equation (1) reduces to 2 = 4E, or $E = \frac{1}{2}$.

Substituting 1/2 for E in equation (1) and combining terms on the right gives

$2x = (A + 1/2)x^4 + (-A + B)x^3$

$+ (A - B + C + 1)x^2 + (-A + B + D - C)x$

$+ (-B - D + 1/2)$.

To get additional equations involving the unknowns, equate the coefficients of like powers of x on the two sides of equation (2). Setting corresponding coefficients of x^4 equal,

$0 = A + \frac{1}{2}$ or $A = -\frac{1}{2}$. From the corresponding coefficients of x^3, $0 = -A + B$. Since

$A = -\frac{1}{2}$, $B = -\frac{1}{2}$. Using the coefficients of x^2, $0 = A - B + C + 1$. Since

$A = -\frac{1}{2}$ and $B = -\frac{1}{2}$, $C = -1$.

Finally, from the coefficient of x, $2 = -A + B + D - C$. Substituting for A, B and C gives D = 1. With

$A = -\frac{1}{2}$, $B = -\frac{1}{2}$, $C = -1$, $D = 1$, and $E = \frac{1}{2}$, the given fraction has the partial fraction decomposition

$$\frac{2x}{(x^2+1)^2(x-1)} = \frac{-\frac{1}{2}x - \frac{1}{2}}{x^2+1} + \frac{-x+1}{(x^2+1)^2} + \frac{\frac{1}{2}}{x-1}$$

or

$$\frac{2x}{(x^2+1)^2(x-1)} = \frac{-(x+1)}{2(x^2+1)} + \frac{-x+1}{(x^2+1)^2} + \frac{1}{2(x-1)}$$

Practice Problems [I]

Find the partial fraction decomposition for the following rational expressions :

1. $\frac{7x-10}{(x-2)(x-1)}$

2. $\frac{4x^2+13x-9}{x^3+2x^2-3x}$

3. $\frac{6x-11}{(x-1)^2}$

4. $\frac{x^2+2x+7}{x(x-1)^2}$

5. $\frac{5x^2-6x+7}{(x-1)(x^2+1)}$

6. $\frac{6x^2-x+1}{x^3+x^2+x+1}$

www.pdfworld.net

www.pdfworld.net

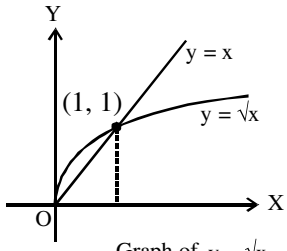
7. $\dfrac{x^4+2x^2-x-1}{x(x^2+1)^2}$

8. $\dfrac{3x^4-9x^3+14x^2-9x+2}{(x-1)(x^2-2x+2)^2}$

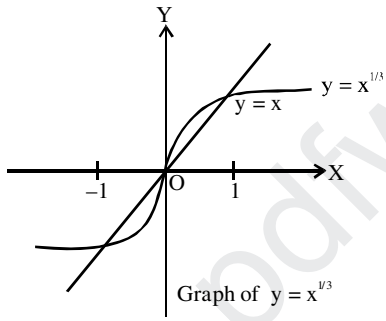
1.14 IRRATIONAL FUNCTION

Irrational functions are those which contain atleast one fractional power of x.

For example f(x) = $\sqrt{x}$ is square root function which is irrational.



Graph of $y = \sqrt{x}$



Graph of $y = x^{1/3}$

1. If roots are all even (i.e. $\sqrt{x}, \sqrt[4]{x}, \sqrt[6]{x}$, ... etc.), then it is defined for non-negative real values of the radicand. In other words, if the radicand is negative (i.e. $x < 0$), then the root is imaginary, if the radicand is zero, then the root is also zero, and if the radicand is positive, then the value of the root is also positive.
2. If roots are all odd (i.e. $\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}$, etc.) of an equation then it is defined for all real values of the radicand. If the radicand is negative, then the root is negative, if the radicand is zero then the root is zero and if the radicand is positive then the root is positive.

Irrational Equations

The irrational equation is an equation in which the unknown quantities are under the radical sign. The domain of permissible values of an irrational equation consists of the values of the unknown for which all the expressions under the radical signs of an even degree are non-negative.

One of the ways of solving an irrational equation is to raise both sides of the equation successively to a power which is the least common multiple of the exponents of all the radicals entering into the given equation. If the power to which the equation is raised is even, the resulting consequence of the initial equation can have extraneous roots. In that case the roots must be verified.

When solving irrational equations, we frequently use the formula $\left(\sqrt[n]{f(x)}\right)^n = f(x)$. In the case of an even n, its application may lead to extending the domain of the given equation (for $\left(\sqrt[n]{f(x)}\right)^n$ the constraint $f(x) \geq 0$ is naturally used for an even n, whereas with $\left(\sqrt[n]{f(x)}\right)^n$ replaced by f(x) this constraint is removed). For this reason, when solving irrational equations, we must check solutions in most cases.

The Method of Raising Both Sides of an Equation to the same Power.

➤ **Example 1.** Solve the equation

$$\sqrt{(6-4x-x^2)} = x+4$$

➤ ***Solution*** The equation $\sqrt{(6-4x-x^2)} = x+4$ is equivalent to the system

www.pdfworld.net

www.pdfworld.net

$$\begin{cases} x + 4 \geq 0 \\ 6 - 4x - x^2 = (x + 4)^2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \geq -4 \\ x^2 + 6x + 5 = 0 \end{cases}$$

Solving the equation $x^2 + 6x + 5 = 0$

We find that $x_1 = -1$, and $x_2 = -5$ only $x_1 = -1$ satisfies the condition $x \geq -4$.

Consequently, the number -1 is the only solution of the given equation.

➤ **Example 2.** Prove that the following equations have no solution :

(i) $\sqrt{(2x + 7)} + \sqrt{(x + 4)} = 0$

(ii) $\sqrt{(x - 4)} = -5$

(iii) $\sqrt{(6 - x)} - \sqrt{(x - 8)} = 2$

(iv) $\sqrt{-2 - x} = \sqrt[5]{(x - 7)}$

(v) $\sqrt{x} + \sqrt{(x + 16)} = 3$

(vi) $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 8$

(vii) $\sqrt{(x - 3)} - \sqrt{x + 9} = \sqrt{(x - 1)}$

➤ ***Solution***

(i) We have $\sqrt{(2x + 7)} + \sqrt{(x + 4)} = 0$

The left hand side of the equation is positive but right hand side is zero. Therefore the equation has no roots.

(ii) We have $\sqrt{(x - 4)} = -5$

The left hand side of the equation is positive but right hand side is negative. Therefore the equation has no roots.

(iii) We have $\sqrt{(6 - x)} - \sqrt{x - 8} = 2$

The equation is defined for

$6 - x \geq 0$

and $x - 8 \geq 0$

$$\therefore \quad \begin{cases} x \leq 6 \\ x \geq 8 \end{cases}$$

Consequently, there is no x for which both expressions would have meaning. therefore the equation has no roots.

(iv) We have $\sqrt{(-2 - x)} = \sqrt[5]{(x - 7)}$

This equation is defined for

$-2 - x \geq 0$

$\Rightarrow \quad x \leq -2$

For $x \leq -2$ the left hand side is positive but right hand side is negative. Therefore the equation has no roots.

(v) We have $\sqrt{x} = \sqrt{(x + 16)} = 3$

The equation is defined for

$x \geq 0$

and $x + 16 \geq 0$

$$\Rightarrow \quad \begin{cases} x \geq 0 \\ x \geq -16 \end{cases}$$

Hence $x \geq 0$

For $x \geq 0$ the left hand side ≥ 4 but right hand side is 3. Therefore the equation has no roots.

(vi) We have $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 98$

For $x < 0$, the expression $7\sqrt{x}$ is meaningless,

For $x > 0$, the expression $8\sqrt{-x}$ is meaningless and

For $x = 0$, the expression $\dfrac{15}{x^3}$ is meaningless.

Consequently, the left hand side of the original equation is meaningless for any $x \in R$. Therefore the equation has no roots.

(vii) We have $\sqrt{(x - 3)} - \sqrt{(x - 9)} = \sqrt{x - 1}$

This equation is defined for

$$\begin{cases} x - 3 \geq 0 \\ x + 9 \geq 0 \\ x - 1 \geq 0 \end{cases} \Rightarrow \quad \begin{cases} x \geq 3 \\ x \geq -9 \\ x \geq 1 \end{cases}$$

Hence $x \geq 3$

www.pdfworld.net

www.pdfworld.net

For $x \geq 3, \sqrt{x-3} < \sqrt{x+9}$

i.e., $\sqrt{(x-3)} - \sqrt{(x+9)} < 0$

Hence for $x \geq 3$, the left hand side of the original equation is negative and right hand side is positive. Therefore the equation has no roots.

➤ **Example 3.** Solve

$$\sqrt{3x^2+6x+7} + \sqrt{5x^2+10x+14} = 4 - 2x - x^2.$$

➤ ***Solution*** $\sqrt{3x^2+6x+7} = \sqrt{3\left(x^2+2x\right)+7}$

$$= \sqrt{3\left(x+1\right)^2+4} \geq 2$$

Similarly $\sqrt{5x^2+10x+14} \geq 3$

Hence $4 - 2x - x^2 \geq 5$,

$x^2 + 2x + 1 \leq 0 \Rightarrow (x+1)^2 \leq 0$

This is possible if $x = -1$

➤ **Example 4.** Solve the equation

$$\sqrt{x-1} + \sqrt{2x+6} = 6$$

➤ ***Solution*** $\sqrt{x-1} + \sqrt{2x+6} = 6$ ...(1)

Squaring both sides of the equation, we get :

$$x - 1 + 2\sqrt{(x-1)(2x+6)} + 2x + 6 = 36,$$

and further $2\sqrt{2x^2+4x-6} = -3x + 31$.

Squaring the last equation, we get

$8x^2 + 16x - 24 = 9x^2 - 186x + 961$,

$\Rightarrow \quad x^2 - 202x + 985 = 0 \quad \Rightarrow x_1 = 5, x_2 = 197$

The found roots are readily checked directly by substituting them into equation (1).

(i) $\sqrt{x_1 - 1} + \sqrt{2x_1 + 6} = \sqrt{5-1} + \sqrt{2\times5+6} = 6$

Thus $x_1 = 5$ is a root of the given equation.

(ii) $\sqrt{x_2 - 1} + \sqrt{2x_2 + 6}$

$= \sqrt{197-1} + \sqrt{2\times197+6} \neq 6$

that is, $x_2 = 197$ is an extraneous root. Thus, $x = 5$ is the only root of the given equation.

➤ **Example 5.** Solve the equation

$$\sqrt{3x+4} + \sqrt{x-4} = 2\sqrt{x}$$

➤ ***Solution*** $\sqrt{3x+4} + \sqrt{x-4} = 2\sqrt{x}$ ...(1)

We square both sides of the equation

$$3x + 4 + 2\sqrt{(3x+4)(x-4)} + x - 4 = 4x \quad ...(2)$$

$$\Rightarrow \quad 2\sqrt{(3x+4)(x-4)} = 0$$

whose roots are $x = -4/3$ and $x = 4$. One of the roots obtained, namely, $x = -4/3$, does not satisfy the initial equation since it does not belong to the set of its permissible values. Verification shows that for $x = 4$ satisfies the equation.

➤ **Example 6.** Solve the equation

$$\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)} = 1$$

➤ ***Solution*** We have $\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)} = 1$ ...(1)

Cubing both sides of (1), we obtain

$$2x - 1 + x - 1 + 3\sqrt[3]{(2x-1)(x-1)}\left(\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)}\right) = 1$$

$$\Rightarrow \quad 3x - 2 + 3 \cdot \sqrt[3]{(2x^2-3x+1)}\,(1) = 1$$

from (1)

$$\Rightarrow \quad 3 \cdot \sqrt[3]{(2x^2-3x+1)} = 3 - 3x$$

$$\Rightarrow \quad \sqrt[3]{(2x^2-3x+1)} = (1-x)$$

Again cubing both sides, we obtain

$2x^2 - 3x + 1 = (1-x)^3$

$\Rightarrow \quad (2x-1)(x-1) = (1-x)^3$

$\Rightarrow \quad (2x-1)(x-1) = -(x-1)^3$

$\Rightarrow \quad (x-1)\{2x - 1 + (x-1)^2\} = 0$

$\Rightarrow \quad (x-1)(x^2) = 0$

$\therefore \quad x_1 = 0$ and $x_2 = 1$

$\because \quad x_1 = 0$ doesnot satisfy the equation (1).

$x_2 = 1$ is the only root of the original equation.

www.pdfworld.net

www.pdfworld.net

The Method of Introducing New Variables

➤ **Example 7.** Solve the equation

$$x^2 + 3 - \sqrt{2x^2 - 3x + 2} = 1.5\,(x + 4)$$

➤ ***Solution*** $x^2 + 3 - \sqrt{2x^2 - 3x + 2} = 1.5\,(x + 4)$ ...(1)

Isolating the radical and squaring both sides of equation (1) would lead to an awkward equation. Multiplying both sides by 2, we get :

$$2x^2 + 6 - 2\sqrt{2x^2 - 3x + 2} = 3x + 12,$$

and further $2x^2 - 3x + 2 - 2\sqrt{2x^2 - 3x + 2} - 8 = 0$

Setting $y = \sqrt{2x^2 - 3x + 2}$, we get :

$$y^2 - 2y - 8 = 0 \Rightarrow y_1 = 4,\ y_2 = -2.$$

Hence, $\sqrt{2x^2 - 3x + 2} = 4$ or $\sqrt{2x^2 - 3x + 2} = -2$

From the first equation we find : $x_1 = \frac{7}{2}$, $x_2 = -2$. the second equation has no real roots.

Since equation (1) is equivalent to the equation $\sqrt{2x^2 - 3x + 2} = 4$ (because the second equation has no solution), the found values can be checked by substituting them into the equation $\sqrt{2x^2 - 3x + 2} = 4$. This substitution shows that both values of x are roots of the indicated equation, and hence of equation (1).

➤ **Example 8.** Solve the equation

$$\sqrt[4]{1 - x} + \sqrt[4]{15 + x} = 2$$

➤ ***Solution*** $\sqrt[4]{1 - x} + \sqrt[4]{15 + x} = 2$ ...(1)

Let us assume $\begin{cases} u = \sqrt[4]{1 - x} \\ v = \sqrt[4]{15 + x} \end{cases}$

Then equation (1) takes the form : $u + v = 2$. But to find the values of the new variables, one equation is not sufficient. Raising both sides of each equation to the fourth power, we get

$$\begin{cases} u^4 = 1 - x \\ v^4 = 15 + x \end{cases}$$

We then add together the equations of the last system:

$$u^4 + v^4 = 16.$$

Thus, for finding u, v we have the following symmetric system of equations :

$$\begin{cases} u + v = 2 \\ u^4 + v^4 = 16 \end{cases}$$

solving this system we find (confining ourselves to real solutions):

$$\begin{cases} u_1 = 0 \\ v_1 = 2 \end{cases} ; \begin{cases} u_2 = 2 \\ v_2 = 0 \end{cases}$$

The problem has been reduced to solving the collection of the systems :

$$\begin{cases} \sqrt{1 - x} = 0 \\ \sqrt{15 + x} = 2 \end{cases} ; \qquad \begin{cases} \sqrt[4]{1 - x} = 2 \\ \sqrt{15 + x} = 0 \end{cases}$$

Solving this collection, we find : $x_1 = 1$, $x_2 = -15$.

➤ **Example 9.** Solve the equation

$$\sqrt[3]{x + 1} = \sqrt{x - 3}$$

➤ ***Solution*** We introduce the designation

$$\sqrt[3]{x + 1} = u,\ \sqrt{x - 3} = v.$$

Eliminating x in the equations

$u^3 = x + 1$ and $v^2 = x - 3$, we arrive at a system of equations

$$u = v,\ u^3 - v^2 = 4.$$

Its solution reduces to that of the equation $v^3 - v^2 - 4 = 0$, whose only real root is $v = 2$. Returning to the initial unknown, we get a linear equation $4 = x - 3$, whose root is the only root of the initial equation. Hence, $x = 7$.

➤ **Example 10.** Prove that the trinomial $x^3 + px + q$ vanishes at

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

➤ ***Solution*** Let us denote

www.pdfworld.net

www.pdfworld.net

$$\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}=u,\ \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}=v.$$

Then $x = u + v$.

Consequently $x^3 = (u+v)^3 = u^3 + v^3 + 3uv\,(u+v)$.

But $u^3 + v^3 = -q,\ uv = -\frac{p}{3}$.

Therefore $x^3 = -q - px$

or $x^3 + px + q = 0$

which is the required result.

Practice Problems [J]

1. Find a factor which will rationalise $x^{1/6} + a^{1/4}$
2. Express with rational denominator

$$\frac{2\sqrt{x+1}}{\sqrt{x-1}-\sqrt{2x}+\sqrt{x+1}}$$

3. Find the square root of
   (i) $3/2\,(x-1)+\sqrt{2x^2-7x-4}$
   (ii) $2a-\sqrt{3a^2-2ax-x^2}$
4. Find the cube root of
   (i) $72-32\sqrt{5}$ (ii) $99-70\sqrt{2}$
5. Solve the following equations :
   (i) $\sqrt{x+1}=a$
   (ii) $\sqrt{x+3}=\sqrt{a-x}$
   (iii) $\sqrt{25-x}=2-\sqrt{9+x}$
   (iv) $(x+4)(x+1)-3\sqrt{x^2+5x+2}=6$
   (v) $\sqrt{x+\sqrt{x+11}}+\sqrt{x-\sqrt{x+11}}=4$
   (vi) $\sqrt{3x^2-2x+15}+\sqrt{3x^2-2x+8}=7$
6. Solve the following equations :
   (i) $\frac{8}{\sqrt{10-2x}}-\sqrt{10-2x}=2$
   (ii) $\sqrt{6x-x^2-5}=2x-6$
   (iii) $\frac{2+\sqrt{19-2x}}{x}=1.$
   (iv) $\sqrt[3]{16-x^3}=4-x.$
   (v) $\sqrt{2x-4}-\sqrt{x+5}=1.$
   (vi) $\sqrt[3]{12-x}+\sqrt[3]{14+x}=2$
   (vii) $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1.$
   (viii) $\sqrt{x+3}-1=\sqrt{x-\sqrt{x-2}}$
   (ix) $\sqrt{x^2+2x+1}-\sqrt{x^2-4x+4}=3$
   (x) $x^3+1=2\sqrt[3]{2x-1}$

1.15 IRRATIONAL INEQUALITIES

1. An inequation of the form $\sqrt[2n]{f(x)} < g(x)$, $n \in N$ is equivalent to the system

$$\begin{cases} f(x) \geq 0 \\ g(x) > 0 \\ f(x) < g^{2n}(x) \end{cases},$$

and inequation of the form $\sqrt[2n+1]{f(x)} < g(x),\ n \in N$ is equivalent to the inequation $f(x) < g^{2n+1}(x)$

➤ **Example 1.** Solve the inequation

$$\sqrt{(x+14)} < (x+2)$$

➤ ***Solution*** This inequation is equivalent to the system

$$\begin{cases} x+14 \geq 0 \\ x+2 > 0 \\ x+14 < (x+2)^2 \end{cases} \Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ x^2+3x-10 > 0 \end{cases}$$

www.pdfworld.net

www.pdfworld.net

$$\Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ (x+5)(x-2) > 0 \end{cases} \Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ x < -5 \text{ and } x > 2 \end{cases}$$

Taking intersection, we get $x > 2$ i.e. $x \in (2, \infty)$

2. An inequation of the form $\sqrt[2n]{f(x)} > g(x)$, $n \in N$ is equivalent to the collection of two systems of inequations

$$\begin{cases} g(x) \geq 0 \\ f(x) > g^{2n}(x) \end{cases} \text{ and } \begin{cases} g(x) < 0 \\ f(x) \geq 0 \end{cases}$$

and inequation of the form

$\sqrt[2n+1]{f(x)} > g(x)$, $n \in N$ is equivalent to the inequation $f(x) > g^{2n+1}(x)$

➤ **Example 2.** Solve the inequality

$$\sqrt{x^2+3x+2} < 1 + \sqrt{x^2-x+1}$$

➤ ***Solution*** The domain of the variable here consists of two intervals: $x \leq -2$ and $x \geq -1$.

In the domain, both members of our inequality are non-negative and so squaring yields the equivalent inequality $2x < \sqrt{x^2-x+1}$

(a) For $x \leq -2$ and $-1 \leq x < 0$, this inequality is true since for each of these values of x there is a negative number on the left and a positive number on the right. Thus, all these values of x are solutions to the original inequality.

(b) For $x \geq 0$, both members of the inequality $2x < \sqrt{x^2-x+1}$ are non-negative and so squaring yields the equivalent inequality $3x^2 + x - 1 < 0$. $\Rightarrow (-1-\sqrt{13})/6 < x < (-1+\sqrt{13})/6$.

Taking condition (b) into account, we find that in the latter case the solution of the original inequality will consist of all values of x in the interval $0 \leq x < (-1+\sqrt{13})/6$.

Combining both cases we get the answer: $x \leq -2$ and $-1 \leq x < (1+\sqrt{13})/6$

CAUTION It should be noted that if we do not consider the cases (a) and (b) and square the inequality $2x < \sqrt{x^2-x+1}$ from the start, we will naturally loose some of the solutions. This is loss of "vigilance" while squaring.

➤ **Example 3.** Solve the inequality

$$\sqrt{x-5} - \sqrt{9-x} > 1. \qquad ...(1)$$

➤ ***Solution*** The set of permissible value is $x \in [5, 9]$. Inequality (1) is equivalent to the inequality

$$\sqrt{x-5} > \sqrt{9-x} + 1 \qquad ...(2)$$

whose both sides are nonnegative. Squaring both sides of inequality (2), we get an equivalent inequality $2x - 15 > 2\sqrt{9-x}$. ...(3)

(1) If $2x - 15 \leq 0$, i.e. $x \leq 15/2$, then the left-hand side of the inequality is negative or equal to zero and the right-hand side is positive. Therefore, for any value of x on the interval [5, 15/2] inequality (3) is not satisfied.

(2) If $2x - 15 > 0$, i.e. $x > 15/2$, then both sides of the inequality are nonnegative, and after squaring we get an inequality which is equivalent to inequality (3):
$(2x-15)^2 > 4(9-x)$.
Thus the set of solutions of inequality (1) is obtained as the set of solutions of the system of inequalities

$5 \leq x \leq 9$, $2x - 15 > 0$, $(2x-15)^2 > 4(9-x)$,

hence we get $x \in \left(\frac{14+\sqrt{7}}{2}, 9\right]$.

➤ **Example 4.** Solve the inequality

$$\sqrt{3x} - \sqrt{2x+1} \geq 1. \qquad ...(1)$$

➤ ***Solution*** Inequality (1) is equivalent to the following system :

$$\begin{cases} 3x \geq 0 \\ 2x+1 \geq 0 \\ \sqrt{3x} - \sqrt{2x+1} \geq 1. \end{cases} \qquad ...(2)$$

It is advisable to rewrite the last inequality of system (2) in the form $\sqrt{3x} \geq 1 + \sqrt{2x+1}$, where

www.pdfworld.net

www.pdfworld.net

both sides are nonnegative, and therefore the squaring of both sides of this inequality is an equivalent transformation. Thus, from system (2) we pass to the following system which is equivalent to (2):

$$\begin{cases} x \ge 0 \\ (\sqrt{3x})^2 \ge (1+\sqrt{2x+1})^2 \end{cases} \text{ or } \begin{cases} x \ge 0 \\ \sqrt{2x+1} \le \dfrac{x}{2} - 1 \end{cases}$$

Further, we have : $\begin{cases} x \ge 0 \\ \dfrac{x}{2} - 1 \ge 0 \\ 2x+1 \le \left(\dfrac{x}{2}-1\right)^2 \end{cases}$,

whence we get $[12, \infty)$ which is the solution of the last system and, at the same time, of inequality (1).

➤ **Example 5.** Solve the inequality

$$\sqrt{2x+5} + \sqrt{x-1} > 8. \qquad ...(1)$$

➤ ***Solution*** Inequality (1) is equivalent to the system

$$\begin{cases} 2x+5 \ge 0 \\ x-1 \ge 0 \\ \sqrt{2x+5} + \sqrt{x-1} > 8 \end{cases} \qquad ...(2)$$

Since both sides of the last inequality of system (2) take on only nonnegative values, system (2) is equivalent to the following system :

$$\begin{cases} 2x+5 \ge 0 \\ x-1 \ge 0 \\ (\sqrt{2x+5} + \sqrt{x-1})^2 > 64 \end{cases}$$

or $\begin{cases} x \ge 1 \\ 2\sqrt{2x^2+3x-5} > 60-3x \end{cases}$ ...(3)

System (3) is equivalent to the following collection of systems :

$$\begin{cases} x \ge 1 \\ 60-3x \ge 0 \\ x^2 - 372x + 3620 < 0 \end{cases} ; \quad \begin{cases} x \ge 1 \\ 60-3x < 0 \end{cases}.$$

Note that for $x \ge 1$ the inequality $2x^2 + 3x - 5 \ge 0$ is true (since $2x^2 + 3x - 5 = (2x + 5)(x - 1)$), therefore the last collection of systems of inequalities is equivalent to the collection

$$\begin{cases} x \ge 1 \\ x \le 20 \\ (x-10)(x-362) < 0 \end{cases} ; \quad \begin{cases} x \ge 1 \\ x > 20 \end{cases}.$$

Solving this collection, we get: $10 < x \le 20$; $x > 20$. Combining these solutions, we get : $(10, \infty)$ which is the solution of Inequality (1).

➤ **Example 6.** Solve the inequality

$$\sqrt{x-2} + \sqrt{3-x} > \sqrt{x-1} - \sqrt{6-x}$$

➤ ***Solution*** This inequality

$$\sqrt{x-2} + \sqrt{3-x} > \sqrt{x-1} - \sqrt{6-x} \qquad ...(1)$$

is equivalent to the following system of inequalities

$$\begin{cases} x-2 \ge 0 \\ 3-x \ge 0 \\ x-1 \ge 0 \\ 6-x \ge 0 \\ \sqrt{x-2} + \sqrt{3-x} > \sqrt{x-1} - \sqrt{6-x} \end{cases}$$

or $\begin{cases} 2 \le x \le 3 \\ \sqrt{x-2} + \sqrt{3-x} > \sqrt{x-1} - \sqrt{6-x} \end{cases}$ ...(2)

Since $2 \le x \le 3$, we have $x - 1 \le 2$, and therefore

$$\sqrt{x-1} \le \sqrt{2}.$$

Further $6 - x \ge 3$, therefore $\sqrt{6-x} \ge \sqrt{3}$

Hence, $\sqrt{x-1} - \sqrt{6-x} \le \sqrt{2} - \sqrt{3}$, and the more so

$$\sqrt{x-1} - \sqrt{6-x} < 0.$$

But $\sqrt{x-2} + \sqrt{3-x} > 0$, consequently, the second inequality of system (2) is fulfilled for any permissible values of x from the domain of inequality (1), that is, system (2) and hence also inequality(1) have the following solution : [2 , 3] .

www.pdfworld.net

www.pdfworld.net

Practice Problems [K]

1. Solve the following inequations :

(i) $(x^2-1)\sqrt{x^2-x-2} \geq 0$

(ii) $\sqrt{\frac{x-2}{1-2x}} > -1$

(iii) $\frac{\sqrt{x-3}}{x-2} > 0$

2. Solve the following inequations :

(i) $\sqrt{1-\frac{x+2}{x^2}} < \frac{2}{3}$

(ii) $\frac{3}{\sqrt{2-x}} - \sqrt{2-x} < 2$

(iii) $\frac{x^2-13x+40}{\sqrt{19x-x^2-78}} \leq 0$

3. Solve the following inequations :

(i) $\sqrt{2x-x^2} < 5-x.$

(ii) $\sqrt{x^2-3x-10} < 8-x.$

(iii) $3-x > 3\sqrt{1-x^2}$

4. Solve the following inequations :

(i) $\sqrt{8+2x-x^2} > 6-3x.$

(ii) $\sqrt{-x^2+6x-5} > 8-2x.$

(iii) $x-3\sqrt{x-3}-1 > 0$

5. Solve the following inequations :

(i) $\frac{\sqrt{2x-1}}{x-2} < 1$

(ii) $\frac{1-\sqrt{21-4x-x^2}}{x+1} \geq 0$

(iii) $\frac{4-\sqrt{x+1}}{1-\sqrt{x+3}} \leq 3$

6. Solve the following inequations :

(i) $\sqrt{x+3} + \sqrt{x+15} < 6.$

(ii) $\sqrt{x-6} - \sqrt{10-x} \geq 1$

7. Solve the inequality $\sqrt{\frac{3x-1}{a-2}} < 1$ for all real values of a.

1.16 MODULUS FUNCTION

The modulus of a real number x is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Modulus is also known as **absolute value**.

The case $x \geq 0$ tells us that the modulus of non-negative number is the number itself. Thus,

$|5| = 5$ $\quad |\pi| = \pi$ $\quad \left|\frac{1}{3}\right| = \frac{1}{3}$

$|0| = 0$

The case $x < 0$ tells us that the modulus of a real number is never negative. For example,

$|-3| = -(-3) = 3$ $\quad |-\pi| = -(-\pi) = \pi$

$$\left|-\frac{1}{3}\right| = -\left(\frac{1}{3}\right) = \frac{1}{3}$$

➤ **Example 1.** Evaluate by rewriting as an equivalent expression without modulus sign :

(i) $|\sqrt{3}-1|$ $\quad$ (ii) $|2-\pi|$

(iii) $\frac{|x|}{x}$ if $x > 0$ $\quad$ (iv) $\frac{|x|}{x}$ if $x < 0$

➤ ***Solution***

(i) Since $\sqrt{3} \approx 1.7$, the expression inside the modulus sign is positive. The modulus of a positive number is the number itself. Thus, $|\sqrt{3}-1| = \sqrt{3}-1$.

(ii) Since $\pi \approx 3.14$, the number inside the modulus sign is negative. The modulus of x when $x < 0$ is $-x$. Thus, $|2-\pi| = -(2-\pi) = \pi - 2$.

(iii) If $x > 0$, then $|x| = x$. Thus, $\frac{|x|}{x} = \frac{x}{x} = 1$.

(iv) If $x < 0$, then $|x| = -x$. Thus, $\frac{|x|}{x} = \frac{-x}{x} = -1$.

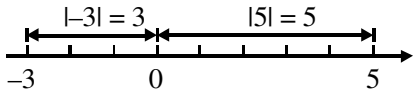
pdfworld.net

www.pdfworld.net

www.pdfworld.net

Absolute Value and Distance

Geometrically, the absolute value of x is the distance between the real number x and the origin on the real number line (Figure).

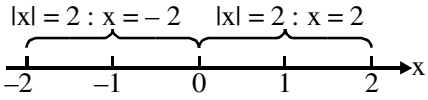


Absolute value as the distance from 0

➤ **Example 2.** Rewrite using absolute value : The distance between x and the origin is 2. What are the possible values for x ?

➤ ***Solution*** Since the distance between x and the origin is 2 we have $|x - 0| = 2$ i.e. $|x| = 2$.

There are two numbers whose distance from the origin on the number line is 2, namely, –2 and 2, as shown in the figure.



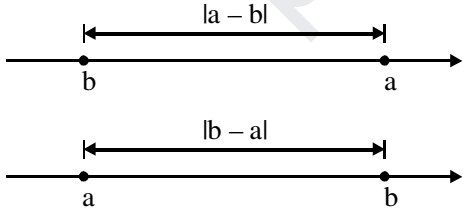
$|x| = 2$ means $x = 2$ or $x = -2$

Generalizing from this example, if c is any positive number and **$|x| = c$, then $x = c$ or $x = -c$.**

Distance between two points on the real number line

The distance between real numbers a and b is the absolute value of their difference.

$|a - b| = |b - a|$



➤ **Example 3.** Express the distance between the numbers –5 and 3 using absolute value. Then evaluate the absolute value.

➤ ***Solution*** The distance between –5 and 3 can be expressed in two equivalent ways, namely,

$|-5 - 3| = |-8|$

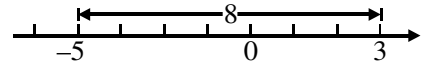
$|3 - (-5)| = |8|$

We evaluate these absolute values as

$|-8| = 8$

$|8| = 8$

Figure verifies that the calculation in correct.

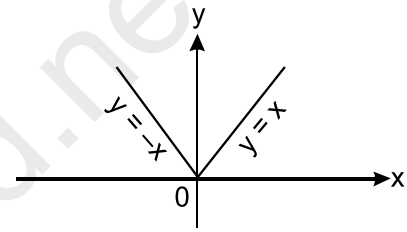


The distance between –5 and 3 is 8.

Modulus Function

The function $y = |x|$ is called the modulus function.

It is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



CAUTION

When in place of x, we have f (x) under the modulus sign, then in place of the proper equation

$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$ we often find the

students writing $|f(x)| = \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(x) & \text{if } x < 0 \end{cases}$

which is obviously quite incorrect.

➤ **Example 4.** Remove a factor from the radicand in the expression $\sqrt{16a^4b^6c^3}$, where $b < 0$ and $c > 0$.

➤ ***Solution*** We have

$$\sqrt{16a^4b^6c^3} = \sqrt{16} \cdot \sqrt{b^6} \cdot \sqrt{a^4} \cdot \sqrt{c^3}$$

$$= 4|a^2|\,|b^3|\,|c|.\sqrt{c}\,.$$

The number a^2 is always non-negative; $|a^2| = a^2$. Since $b < 0$ and $c > 0$, $|b^3| = -b^3$ and $|c| = c$. Therefore

$$\sqrt{16a^4b^6c^3} = -4a^2b^3c\sqrt{c}\,.$$

www.pdfworld.net

www.pdfworld.net

➤ **Example 5.** Bring a factor in the radicand in the expression $\frac{\sqrt{x}}{y}$, where $x \geq 0$ and $y < 0$.

➤ ***Solution*** Since $y < 0$, $\sqrt{y^2} = |y| = -y$.

Hence, $y = -\sqrt{y^2}$.

Therefore, $\frac{\sqrt{x}}{y} = \frac{\sqrt{x}}{-\sqrt{y^2}} = -\sqrt{\frac{x}{y^2}}$.

➤ **Example 6.** Let $f(x) = |x-2| + |x-4| - |2x-6|$. Find the sum of the largest and smallest values of $f(x)$ if $x \in [2, 8]$.

➤ ***Solution*** If $2 \leq x \leq 3$, $f(x) = x - 2 + 4 - x - (6 - 2x)$

$\Rightarrow \quad f(x) = 2x - 4$

hence minimum = 0, maximum = 2

If $3 \leq x \leq 4$, $f(x) = x - 2 + 4 - x - (2x - 6)$

$\Rightarrow \quad f(x) = 8 - 2x$,

hence minimum = 0, maximum = 2

If $4 \leq x \leq 8$, $f(x) = x - 2 + x - 4 - (2x - 6) = 0$

$\therefore$ minimum = 0, maximum = 0

Finally, minimum = 0, maximum = 2

$\therefore$ Their sum is equal to 2.

➤ **Example 7** Solve the equation

$$\left|\frac{x^2 - 8x + 12}{x^2 - 10x + 21}\right| = -\frac{x^2 - 8x + 12}{x^2 - 10x + 21}$$

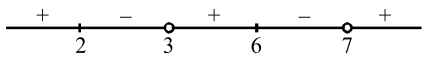
➤ ***Solution*** This equation has the form $|f(x)| = -f(x)$ where $f(x) = \frac{x^2 - 8x + 12}{x^2 - 10x + 21}$

It is equivalent to $f(x) \leq 0$

$\therefore$

$$\frac{x^2 - 8x + 12}{x^2 - 10x + 21} \leq 0 \quad \Rightarrow \quad \frac{(x-2)(x-6)}{(x-3)(x-7)} \leq 0$$

By method of intervals



$\therefore \quad x \in [2, 3) \cup [6, 7)$

➤ **Example 8.** Simplify the function

$f(x) = 2|x-1| - 3|2-x| + 5|x-7|$

➤ ***Solution*** This problem requires considering a total of 8 combinations of signs, but we can manage things so as to consider only four. This is achieved by a special technique called the "method of intervals".

Mark on the number line those values of x for which each of the expressions under the absolute value sign vanishes: the points 1, 2 and 7. Thus, the entire number line is divided into four intervals:

$x < 1, \; 1 \leq x < 2, \; 2 \leq x < 7, \; x \geq 7.$

$$f(x) = \begin{cases} -2(x-1) - 3(2-x) - 5(x-7) & x < 1 \\ 2(x-1) - 3(2-x) - 5(x-7) & 1 \leq x < 2 \\ 2(x-1) + 3(2-x) - 5(x-7) & 2 \leq x < 7 \\ 2(x-1) + 3(2-x) + 5(x-7) & x \geq 7 \end{cases}$$

$$\text{Hence } f(x) = \begin{cases} -4x + 31, & x < 1 \\ 27, & 1 \leq x < 2 \\ -6x + 39, & 2 \leq x < 7 \\ 4x - 31, & x \geq 7 \end{cases}$$

Modulus Equation

➤ **Example 9.** Solve $|3x - 1| = |x + 5|$

➤ ***Solution*** $|3x - 1| = |x + 5|$

$3x - 1 = x + 5$	or	$3x - 1 = -(x + 5)$
$3x - 1 = x + 5$	or	$3x - 1 = -x - 5$
$2x - 1 = 5$		$4x - 1 = -5$
$2x = 6$		$4x = -4$
$x = 3$		$x = -1$

The solution set is $\{-1, 3\}$.

➤ **Example 10.** Solve the equation $|2x - 5| = x - 1$

➤ ***Solution*** The expression $2x - 5$ under the sign of an absolute value is non-negative for $x \geq 5/2$ and negative for $x < 5/2$. Let us consider the initial equation separately on each of the intervals.

Assume $2x - 5 \geq 0$, i.e. $x \geq 5/2$. Then. by the definition of an absolute value, $|2x - 5| = 2x - 5$, and the given equation reduces to the form

$2x - 5 = x - 1$

Solving this equation, we find that $x = 4$. Since the

www.pdfworld.net

www.pdfworld.net

number 4 satisfies the assumption we have made $(2\times4-5\geq 0)$, $x=4$ as a solution of the initial equation.

Let us now assume that $2x-5<0$. Then, by the definition of an absolute value, $|2x-5|=-(2x-5)$, and the given equation assumes the form

$$-(2x-5)=x-1.$$

Solving this equation, we find that $x=2$, since the number 2 satisfies the assumption $(2\times2-5<0)$, $x=2$ is a solution of the initial equation.

The solutions are $x=2, 4$

➤ **Example 11.** Solve the equation $|x^2-x-6|=x+2$

➤ ***Solution***

Case A: Let $x^2-x-6<0 \Rightarrow -2<x<3$

We have the equation $-x^2+x+6=x+2$ with roots $x_1=2, x_2=-2$.

Now check to see whether x_1 and x_2 satisfy the above condition. The first is valid, the second is not; and so only 2 is a root of the original equation.

Case B: Let $x^2-x-6\geq 0$

$\Rightarrow \quad x\leq -2$ or $x\geq 3$

We have the equation $x^2-x-6=x+2$ whose roots are $x_1=4, x_2=-2$.

Since both of these values of x satisfy the above condition, both 4 and -2 are roots of the original equation. Thus the original equation has three roots $-2, 2, 4$.

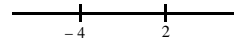
STUDY TIP Let us examine this solution more carefully; first we rejected the value $x=-2$ and then found it again, so that in the end this value was a root of the original equation. How is this to be explained? The point is that in the first case we rejected $x=-2$ but we did not assert that is not a root of the original equation. The only thing we maintained was that this value is discarded due to the restrictions imposed on x by the condition of Case A. Quite naturally, there is nothing to stop this value from satisfying the condition of another case and thus to become a root of the original equation.

➤ **Example 12.** Solve the equation

$$2|x-2|-3|x+4|=1$$

➤ ***Solution*** We first equate each quantity in the modulus signs to zero and find the roots. These roots are used to partition the number line into several parts so that it becomes convenient to decide the sign of each quantity in the modulus sign.

Here the number line should be partitioned at -4 and 2.



Let $-\infty<x<-4$

The given equation becomes $2(2-x)-(3)\{-(x+4)\}=1$

$\Rightarrow \quad x=-15$ and this agrees with the initial condition

$$-\infty<x<-4$$

$\therefore \quad x=-15$ is a solution.

Let $-4\leq x<2$

The given equation becomes $2(2-x)-3(x\,4)=1$

$\Rightarrow \quad x=\dfrac{9}{5}$ which agrees with the initial condition.

Let $2\leq x<\infty$

The given equation becomes

$$2(x-2)-3(x+4)=1$$

$\Rightarrow \quad x=-17$ and this does not belong to the interval $[2, \infty)$.

Hence $x=-17$ is not acceptable.

Hence the solution set is $\left\{-15,-\dfrac{9}{5}\right\}$

➤ **Example 13.** Solve the equation

$$|x^2-9|+|x^2-4|=5$$

➤ ***Solution*** We consider three cases:

(a) $x^2<4$, (b) $4\leq x^2\leq 9$ (c) $9<x^2$

In the first case, $|x^2-9|=9-x^2$, $|x^2-4|=4-x^2$ or

$$9-x^2+4-x^2=5, \quad x^2=4,$$

$x=\pm 2$ are unsuitable and the given equation has no roots in this case.

In the second case, $|x^2-9|=9-x^2$, $|x^2-4|=5$ or $5=5$.

At this point some students think the equation has disappeared. Actually, the original equation is equivalent to the identity $5=5$ when $4\leq x^2\leq 9$, which is to say it is satisfied for all values of x.

www.pdfworld.net

www.pdfworld.net

This means that any value of x which satisfied the condition $4 \le x^2 \le 9$ is a solution of the equation. It now remains to solve this double inequality. We then get $-3 \le x \le -2, 2 \le x \le 3$.
In the third case, $x^2 - 9 + x^2 - 4 = 5, x^2 = 9$,
$x = \pm 3$ are unsuitable and the given equation has no roots in this case.
Finally, we have the solution $-3 \le x \le -2, 2 \le x \le 3$.

Sometimes $ax^2 + bx + c$ in $|ax^2 + bx + c|$ cannot be factorized in real factors. It happens when $ax^2 + bx + c = 0$ has imaginary roots, i.e., $D < 0$. In this case, if $a > 0$, $ax^2 + bx + c$ will be always positive and so $|ax^2 + bx + c|$ can be replaced by $ax^2 + bx + c$, If $a < 0$, $ax^2 + bx + c$ will be always negative and so $|ax^2 + bx + c|$ will be replaced by $-(ax^2 + bx + c)$.

The equation of the form $h(|f(x)|) = g(x)$
(when h is the function of $|f(x)|$, is equivalent to the collection of systems

$$\begin{cases} h(f(x)) = g(x), \text{ if } f(x) \ge 0 \\ h(-f(x)) = g(x), \text{ if } f(x) < 0 \end{cases}$$

➤ **Example 14.** Solve the equation
$|x - |4 - x|| - 2x = 4$

➤ ***Solution***. This equation is equivalent to the collection of systems

$$\begin{cases} |x - (4 - x)| - 2x = 4, \text{ if } 4 - x \ge 0 \\ |x + (4 - x)| - 2x = 4, \text{ if } 4 - x < 0 \end{cases}$$

$$\Rightarrow \begin{cases} |2x - 4| - 2x = 4, \text{ if } x \le 0 \\ 4 - 2x = 4, \text{ if } x > 4 \end{cases}$$

The second system of this collection gives $x = 4$ but $x > 4$
Hence second system has no solution.
The first system of collection (1) is equivalent to the system

$$\begin{cases} 2x - 4 - 2x = 4, \text{ if } 2x \ge 4 \\ -2x + 4 - 2x = 4, \text{ if } 2x < 4 \end{cases}$$

$$\Rightarrow \begin{cases} -4 = 4, \text{ if } x \ge 2 \\ -4x = 0, \text{ if } x < 2 \end{cases}$$

The first equation has no solution
and second one gives $x = 0$
Hence $x = 0$ is unique solution of the given equation.

➤ **Example 15.** Prove that $\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$ is equal to 2, if $1 \le x \le 2$, and to $2\sqrt{x-1}$, if $x > 2$.

➤ ***Solution*** $\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$

$$= \sqrt{x - 1 + 2\sqrt{x-1} + 1} + \sqrt{x - 1 - 2\sqrt{x-1} + 1}$$

$$= \left|\sqrt{x-1} + 1\right| + \left|\sqrt{x-1} - 1\right|$$

$$= \sqrt{x-1} + 1 + \left|\sqrt{x-1} - 1\right|$$

$$= \begin{cases} 2\sqrt{x-1} & \text{if } \sqrt{x-1} > 1, \\ 2 & \text{if } \sqrt{x-1} \le 1. \end{cases}$$

$$= \begin{cases} 2\sqrt{x-1} & \text{if } x > 2, \\ 2 & \text{if } 1 \le x \le 2. \end{cases}$$

➤ **Example 16.** Solve the equation

$$\sqrt{x - 1 + 2\sqrt{x-2}} - \sqrt{x - 1 - 2\sqrt{x-2}} = 1$$

➤ ***Solution*** We assume $\sqrt{x-2} = t$; then the initial equation assumes the form

$$\sqrt{t^2 + 2t + 1} - \sqrt{t^2 - 2t + 1} = 1$$

Since the radicands on the left hand side of equation are perfect squares, it can be represented in the following equivalent form : $|t + 1| - |t - 1| = 1$.
The only root of this equation is $t = 0.5$.
Returning to the initial unknown, we get an equation

$$\sqrt{x-2} = 0.5$$

whose root is $x = 2.25$.

➤ **Example 17.** Solve the equation

$$2^{|x+1|} - 2^x = |2^x - 1| + 1$$

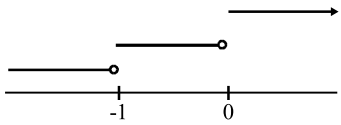
➤ ***Solution*** Let us find the partition points :
$x + 1 = 0, 2^x - 1 = 0$
$\therefore \quad x = -1, x = 0$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

Consider the following cases :



$x < -1$:

$2^{-(x+1)} - 2^x = -(2^x - 1) + 1$

$\Rightarrow \quad 2^{-(x+1)} = 2$

$\therefore \quad -(x+1) = 1$

$\therefore \quad x = -2$

$-1 \le x < 0$:

$2^{x+1} - 2^x = -(2^x - 1) + 1$

$\Rightarrow \quad 2^{x+1} = 2$

$\therefore \quad x + 1 = 1$

$\therefore \quad x = 0$

$x \neq 0 \qquad (\because -1 \le x < 0)$

$x \ge 0$:

$2^{x+1} - 2^x = 2x - 1 + 1$

$\Rightarrow \quad 2^{x+1} = 2 \cdot 2^x$

$\Rightarrow \quad 2^{x+1} = 2^{x+1}$

which is true for all $x \ge 0$

Now combining all cases, we have the final solution as :

$x \in \{-2\} \cup [0, \infty)$

Practice Problems [L]

1. Solve
(i) $|x^2 + x - 20| = x^2 + x - 20$
(ii) $|x^2 + 6x + 8| = -x^2 - 6x - 8.$

2. Solve
(i) $|x - 3| = x$
(ii) $|x^2 - 1| = x + 3$
(iii) $2x - 1 = |x + 3|$

3. Solve
(i) $3|x^2 - 4x + 2| = 5x - 4$
(ii) $|x^2 + 4x + 3| + 2x + 5 = 0$
(iii) $(x-1)|x^2 - 4x + 3| + 2x^2 + 3x - 5 = 0$
(iv) $|x^3 + 1| + x^2 - x - 2 = 0$

4. Solve
(i) $(x+3).|x+2| + |2x+3| + 1 = 0$
(ii) $|x| - 2|x + 1| + 3|x + 2| = 0$
(iii) $|x - 4| + |x + 4| = 8$
(iv) $|x - 3| + |x + 2| - |x - 4| = 3$
(v) $|(x+3)|.(x+1) + |2x+5| = 0$

5. Solve
(i) $|x - |4 - x|| = 2x + 4$
(ii) $|x^2 - 3|x| + 2| = x^2 - 2x$

6. Solve
(i) $|2^{x+1} - 1| + |2^{x+1} + 1| = 2^{|x+1|}$
(ii) $2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$
(iii) $(x + 4)\, 3^{1-|x-1|} - x$
$= (x + 1)\, |3^x - 1| + 3^{x+1} + 1$

7. For $a \le 0$, determine all real roots of the equation $x^2 - 2a|x - a| - 3a^2 = 0$.

8. Solve for x and y:

(i) $\begin{cases} |x| + y = 4 \\ x + 3|y| = 6 \end{cases}$ (ii) $\begin{cases} \sqrt{(x+y)^2} = 5 \\ \sqrt{(x-y)^2} = 1 \end{cases}$

9. Solve for x : $(4a - 15)x^2 + 2a|x| + 4 = 0$

10. Simplify the expression

$\sqrt{9 - 6a + a^2} + \sqrt{9 + 6a + a^2}$ if $a < -3$.

11. Find all solutions of the equation $(3|x| - 3)^2 = |x| + 7$ belonging to the domain of $y = \sqrt{x(x-3)}$

12. Find the least integral value of x which satisfies the equation $|x - 3| + 2|x + 1| = 4$.

1.17 MODULUS INEQUALITIES

Inequations Containing Absolute Values

Let x be a variable or an algebraic expression and let a be a positive real number i.e. $a > 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a. That is,
$|x| < a$ if and only if $-a < x < a$.

2. The solution of $|x| > a$ are all values of x that are less than $-a$ or greater than a. That is,

www.pdfworld.net

www.pdfworld.net

$|x| > a$ if and only if $x < -a$ or $x < a$.

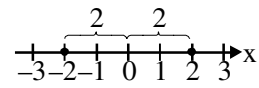
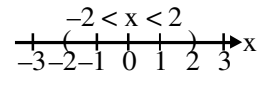
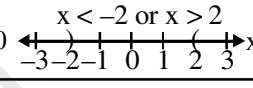
In short, assuming $a > 0$

$|x| < a \Rightarrow -a < x < a$

$|x| \le a \Rightarrow -a \le x \le a$

$|x| > a \Rightarrow x < -a$ and $x > a$

and $|x| \ge a \Rightarrow x \le -a$ and $x \ge a$

Eqn./Ineqn.	*Solution*	*Verbal Description*	*Graph*
$\|x\| = 2$	$x = -2$ and $x = 2$	Values of x that lie 2 units from 0	
$\|x\| < 2$	$-2 < x < 2$	Values of x that lie less than 2 units from 0	
$\|x\| > 2$	$x < -2$ or $x > 2$	Values of x that lie more than 2 units from 0	

➤ **Example 1.** Solve $|x - 5| < 2$

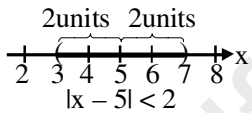
➤ ***Solution*** $|x - 5| < 2$

$-2 < x - 5 < 2$

$-2 + 5 < x - 5 + 5 < 2 + 5$

$3 < x < 7$

The solution set consists of all real numbers that are greater than 3 and less than 7. The interval notation, for this solution set is (3, 7). The graph of this solution set is shown in the figure.



$|x - 5| < 2$

➤ **Example 2.** Solve $|x + 3| \ge 7$

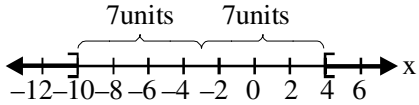
➤ ***Solution*** $|x + 3| \ge 7$

$x + 3 \le -7$ or $x + 3 \ge 7$

$x + 3 - 3 \le -7 - 3$ $x + 3 - 3 \ge 7 - 3$

$x \le -10$ $x \ge 4$

The solution set consists of all real numbers that are less than or equal to –10 or greater than or equal to 4. The interval notation for this solution set is $(-\infty, -10] \cup [4, \infty)$. The graphs of this solution set is shown in the figure.



$|x + 3| \ge 7$

Forms of Inequations Containing Absolute Values

1. The inequation of the form $f(|x|) < g(x)$ is equivalent to the collection of systems

$$\begin{cases} f(x) < g(x), \text{ if } x \ge 0 \\ f(-x) < g(x), \text{ if } x < 0 \end{cases}$$

2. The inequation of the form $|f(x)| < g(x)$ is equivalent to $\begin{cases} -g(x) < f(x) < g(x) \\ g(x) > 0 \end{cases}$

In particular $|f(x)| < a$ has no solution for $a \le 0$ and for $a > 0$ it is equivalent to $-a < f(x) < a$

3. The inequation of the form $|f(x)| > g(x)$ is equivalent to the

$$\begin{cases} f(x) > g(x), \text{ if } g(x) < 0 \\ -f(x) > g(x), \text{ if } g(x) < 0 \end{cases}$$

$$\Rightarrow \begin{cases} f(x) < -g(x) \text{ or } f(x) > g(x), & \text{if } g(x) > 0 \\ \text{always true}, & \text{if } g(x) < 0 \end{cases}$$

In particular $|f(x)| > a$ has no solution for $a \le 0$ and for $a > 0$ is equivalent to $f(x) < -a$, $f(x) > a$

4. The inequation of the form $|f(x)| \ge |g(x)|$ is equivalent to the system $f^2(x) \ge g^2(x)$

➤ **Example 3.** Solve : $\left|\frac{3x+3}{5}\right| < 3$

➤ ***Solution*** $\left|\frac{3x+3}{5}\right| < 3$

$$-3 < \frac{3x+3}{5} < 3$$

www.pdfworld.net

www.pdfworld.net

$-15 < 3x + 3 < 15$
$-18 < 3x < 12$
$-6 < x < 4$

➤ **Example 4.** Solve : $|-2x + 5| \geq 11$

➤ ***Solution*** $|-2x + 5| \geq 11$

$-2x + 5 \geq 11$ or $-2x + 5 \leq -11$

$-2x \geq 6$ $\quad\quad$ $-2x \leq -16$

$x \leq -3$ $\quad\quad$ $x \geq 8$

Using interval notation, the solution set is $(-\infty, -3]$ or $[8, \infty)$. This can be expressed as $(-\infty, -3] \cup [8, \infty)$.

➤ **Example 5.** Solve the inequality $x^2 - |5x + 6| > 0$.

➤ ***Solution*** Let us divide the entire number axis into two subintervals

$(-\infty, -6/5)$ and $[-6/5, \infty)$

on each of which the inequality can be written without the modulus sign.

The equality $|5x + 6| = -5x - 6$ is true for the subinterval $(-\infty, -6/5)$ and consequently, the inequality assumes the form

$x^2 + 5x + 6 > 0$ or $(x + 3)(x + 2) > 0$

$\Rightarrow \quad x \in (-\infty, -3) \cup (-2, \infty)$.

Taking intersection with the subinterval we get

$(-\infty, -3) \cup (-2, -6/5)$.

$|5x + 6| = 5x + 6$ on the second subinterval,

$[-6/5, \infty)$

and consequently, we can write the inequality as follows

$x^2 - 5x - 6 > 0$ or $(x + 1)x - 6) > 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup (6, +\infty)$.

Taking into account that the variable belongs to the subinterval $[-6/5, -\infty)$, we get the set of solutions of the inequality on that subinterval

$[-6/5, -1] \cup (6, \infty)$

The final answer is $(-\infty, -3) \cup (-2, -1) \cup (6, \infty)$

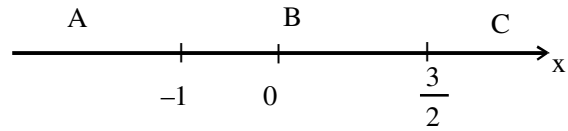
➤ **Example 6.** Solve the inequality

$$x + |3 - 2x| > |x + 1| - 1.$$

➤ ***Solution*** Let us write the given inequality in the form $x + 2\left|x - \frac{3}{2}\right| > |x + 1| - 1$ and mark on the number axis the points $x = -1$ and $x = \frac{3}{2}$ at which the expressions standing inside the modulus sign vanish. These points break the number axis into three intervals;

$x < -1$ (A), $\quad -1 \leq x < \frac{3}{2}$ (B), $\quad x \geq \frac{3}{2}$ (C).

Considering x consecutively on each of these intervals, we obtain that the original inequality is equivalent to the set of the following three systems of inequalities :



$$\begin{cases} x < -1 \\ x - 2\left(x - \frac{3}{2}\right) > -(x + 1) - 1; \end{cases}$$

$$\begin{cases} -1 \leq x < \frac{3}{2} \\ x - 2\left(x - \frac{3}{2}\right) > x + 1 - 1; \end{cases}$$

$$\begin{cases} x \geq \frac{3}{2}, \\ x + 2\left(x - \frac{3}{2}\right) > x + 1 - 1; \end{cases}$$

Solving each of these systems, we find that the solution of the first system is the interval $x < -1$, the solution of the second system is the interval $-1 \leq x < \frac{3}{2}$ and the solution of the third system is the interval $x > \frac{3}{2}$. Consequently, the given inequality is satisfied by all real numbers x, except for $x = \frac{3}{2}$.

Hence, the solution is $x \neq \frac{3}{2}$.

➤ **Example 7.** Solve the inequality

$$|x^2 + 3x| + x^2 - 2 \geq 0$$

➤ ***Solution***

Case A : $\quad x^2 + 3x \geq 0$

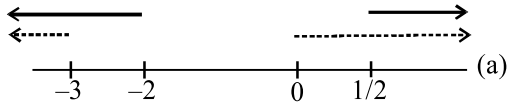
$\Rightarrow \quad x \leq -3$ and $x \geq 0$ $\quad\quad$ ...(1)

We get the inequality $2x^2 + 3x - 2 \geq 0$ the solutions of which are $x \leq -2$ and $x \geq 1/2$.

www.pdfworld.net

www.pdfworld.net

We now have to choose from these solutions those which satisfy condition (1). This is shown in Fig. (a)

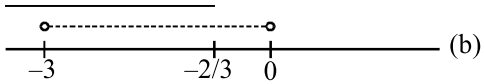


We get the solution for case A as $x \leq -3$ and $x \geq 1/2$.

Case B: $x^2 + 3x < 0 \Rightarrow -3 < x < 0$ ...(2)

We get the inequality $-3x - 2 \geq 0 \Rightarrow x \leq -2/3$.

We now have to choose from these solutions those which satisfy condition (2). This is shown in Fig. (b)



We get the solution for case B as $-3 < x \leq -2/3$.

Combining the solutions found in A and B , we get the answer : $x \leq -2/3$ and $x \geq 1/2$

➤ **Example 8.** Solve the inequality

$|x - 1| - |x| + |2x + 3| > 3x + 4$

➤ ***Solution*** Mark on the number line those values of x for which each of the expressions under the absolute value sign vanishes: the points $-3/2$, 0 and 1. Thus, the entire number line is divided into four intervals:

$$x < -\frac{3}{2}, -\frac{3}{2} \leq x < 0, 0 \leq x < 1, x \geq 1$$

Let us consider each of these intervals.

(a) Let $x < -\frac{3}{2}$

In this case, $3x + 3 < 0$, $x < 0$ and $x - 1 < 0$,

i.e. the initial inequality takes the form

$-x + 1 + x - 2x - 3 < 3x + 4$.

It is satisfied for $x < -3/2$;

Taking intersection with condition (a) we find that $x < -3/2$ is a solution of the original inequality.

(b) Let $-3/2 \leq x < 0$

In this case, $2x + 3 \geq 0$, $x < 0$ and $x - 1 < 0$,

and therefore the original inequality takes the form $-x + 1 + x + 2x + 3 > 2x + 4$, i.e. $0 > 0$.

It is simple that for every x in the interval $-3/2 \leq x < 0$, the original inequality turns into the invalid inequality $0 > 0$ and therefore has no solution for Case (b).

(c) Let $0 \leq x < 1$

In this case, $2x + 3 \geq 0$ and $x - 1 < 0$;

consequently the original inequality reduces to the inequality $-x + 1 - x + 2x + 3 > 2x + 4$.

It is satisfied for $x < 0$. But this relation is inconsistent with condition (c); there are no solutions.

(d) Let $1 \leq x$

In this case the inequality takes the form

$x - 1 - x + 2x + 3 > 3x + 4$,

or $2 > 4$; in other words, there are no values among $x \geq 1$ that satisfy the original inequality.

Hence the proposed inequality holds true for union of solutions in the four cases i.e. $x < -3/2$.

STUDY TIP

It is quite clear from above that the function

$y = |x - 1| - |x| + |2x + 3|$

may be written in the following form without absolute value sign:

$$y = \begin{cases} -2x - 2 & \text{if} \quad x < -\frac{3}{2}, \\ 2x + 4 & \text{if} \quad -\frac{3}{2} \leq x < 0, \\ 4 & \text{if} \quad 0 \leq x < 1, \\ 2x + 2 & \text{if} \quad 1 \leq x, \end{cases}$$

This form of representing a function can be very useful in problem solving.

CAUTION

Note that the intervals may also be written as $x \leq -\frac{3}{2}, -\frac{3}{2} < x < 0, 0 \leq x \leq 1, 1 < x$. We can verify that this does not change anything in the solution.

➤ **Example 9.** Solve the inequality

$|x^2 - 3x - 3| > |x^2 + 7x - 13|$

➤ ***Solution*** Since both sides are non-negative we can square both sides of the inequality.

$|x^2 - 3x - 3|^2 > |x^2 + 7x - 13|^2$

www.pdfworld.net

www.pdfworld.net

But $|a|^2 = a^2$ so that this inequality can be rewritten as

$$(x^2 - 3x - 3) > (x^2 + 7x - 13)^2$$

Now, transposing all terms to the right side and using the formula for the difference of squares, we get

$$2(x^2 + 2x - 8) \cdot 10 (x - 1) < 0$$

$$\Rightarrow \quad (x + 4)(x - 2)(x - 1) < 0$$

This inequality is readily solved by the method of intervals. Its solutions, and consequently the solutions of the original inequality are $x < -4$ and $1 < x < 2$.

➤ **Example 10.** Solve the inequality

$$|x - 1| + |x + 1| < 4.$$

➤ ***Solution*** By the definition of modulus, we have $|x - 1| = -x + 1$, $|x + 1| = -x - 1$ on the interval $(-\infty, -1)$ and consequently, on that interval the inequality is equivalent to the linear inequality $-2x < 4$ which is valid for a $x > -2$. Thus, the set of solution includes the interval $(-2, -1)$.

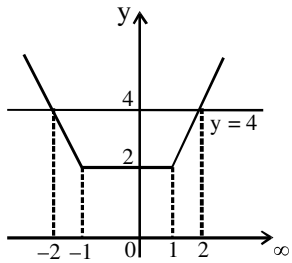
On the interval $[-1, 1)$ the original inequality is equivalent to the true numerical inequality $2 < 4$. Therefore, all the values of the variable belonging to that interval belong to the set of solutions.

On the interval $(1, \infty)$ inequality $2x < 4$ which is true for $x < 2$. Therefore, the interval $(1, 2)$ also belongs to the set of solutions. Collecting all the results obtained we come to the conclusion that the inequality is satisfied by all the values of the variable from the interval $(-2, 2)$.

We can obtain the same result from geometrical considerations. The figure shows the graphs of the functions

$$y = f(x) = |x - 1| + |x + 1| \text{ and } y = 4.$$

On the interval $(-2, 2)$ the graph of the function $y = f(x)$ is located below the graph of $y = 4$ and this means that the inequality $f(x) < 4$ is valid.



Practice Problems [M]

1. Solve :

(i) $|x - 1| > 2$ (ii) $\left|\frac{2x - 1}{x - 1}\right| > 2$

(iii) $\left|\frac{x^2 - 3x - 1}{x^2 + x + 1}\right| < 3$ (iv) $\frac{2}{|x + 2|} \leq 1$

2. Solve :

(i) $|x^2 + x| - 5 < 0$ (ii) $|x - 3| > |x^2 - 3|$

(iii) $\frac{2}{|x - 2|} > \left|\frac{-3}{2x - 1}\right|$

(iv) $|x - 2| \leq 2x^2 - 9x + 9$.

3. Solve :

(i) $|2x + 1| > x$ (ii) $|x + 2| + |x - 3| > 5$

(iii) $|x + 1| + |x - 3| > 2|x - 1|$

4. Solve :

(i) $|x + 2| - |x - 1| < x - 3/2$

(ii) $(|x - 1| - 3)(|x + 2| - 5) < 0$.

(iii) $|2x + 1| - |5x - 2| \geq 1$

5. Solve :

(i) $x^2 + 2|x + 3| - 10 \leq 0$

(ii) $x^2 - |3x + 2| + x \geq 0$

(iii) $|x^2 - 6x + 8| \leq 4 - x$

6. Solve :

(i) $|x^3 - x| \leq x$ (ii) $|x| \geq \frac{2x}{|x - 3|}$

(iii) $|x^2 - 2|x|| \geq x - 2$

(iv) $|34 + 2|x - x^2|| \leq -1$

1.18 PROPERTIES OF MODULUS

Next, we list several basic properties of absolute value. Each of these properties can be derived from the definition of absolute value.

For all real numbers a and b, and natural number n,

(i) $|a| \geq 0$

(ii) $|-a| = |a|$

www.pdfworld.net

www.pdfworld.net

(iii) $-|a| \le a \le |a|$
(iv) $|ab| = |a|\,|b|$
(v) $|a_1a_2.....a_n| = |a_1|\,|a_2|.....|a_n|$
(vi) $|a^n| = |a|^n$
(vii) $|a^{2n}| = a^{2n}$
$|a^2| = |a|^2 = a^2$
(viii) $\sqrt[2n]{a^{2n}} = |a|$, $\sqrt[2n+1]{a^{2n+1}} = a$
(ix) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$
(x) $|a + b| \le |a| + |b|$
(xi) $|a + b| = |a| + |b|$ only if $ab \ge 0$
(xii) $|a - b| \ge ||a| - |b||$
(xiii) $|a - b| = ||a| - |b||$ only if $ab \ge 0$
(xiv) $|a + b + c| \le |a| + |b| + |c|$
(xv) $|a + b + c| = |a| + |b| + |c|$ only if $ab \ge 0$, $bc \ge 0$ and $ca \ge 0$
(xvi) $|a \pm b \pm c| \le |a| + |b| + |c|$

To prove property $|a + b| \le |a| + |b|$
note that $|a + b|^2 = (a + b)^2 = a^2 + 2ab + b^2$
and $(|a| + |b|)^2 = |a|^2 + 2|a|.|b| + |b|^2 = a^2 + 2|ab| + b^2$
but $ab \le |ab|$, so that $|a + b|^2 \le (|a| + |b|)^2$
Here equality holds when $ab = |ab|$, which happens when $ab \ge 0$

STUDY TIP

The equation $|f(x) + g(x)| = |f(x)| + |g(x)|$ is equivalent to $f(x).g(x) \ge 0$
The equation $|f(x)| + |g(x)| = f(x) - g(x)$ is equivalent to $f(x) \ge 0$ and $g(x) \le 0$

➤ **Example 1.** For what values of x are the following inequalities true:
(i) $|-x| \ge x$; (ii) $x\,|x| \ge x^2$;
(iii) $\sqrt{x^2} \le -x$; (iv) $x\sqrt{2} > \sqrt{2x^2}$?

➤ ***Solution***
(i) $|-x| = |x|$; $|x| \le x$ for $x \ge 0$.
(ii) The inequality is not fulfilled for $x < 0$. For $x \ge 0$, $|x| = x$, and the slack inequality $x^2 \ge x^2$ is correct.
(iii) $\sqrt{x^2} = |x|$; $|x| \le -x$ for $x \le 0$.
(iv) $\sqrt{2x^2} = |x|\sqrt{2}$; $x\sqrt{2} > |x|\sqrt{2}$ or $x > |x|$ is incorrect for any x.

➤ **Example 2** Find the real roots of the equation,
$$\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$$

➤ ***Solution*** $\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$

$$\Rightarrow \sqrt{(\sqrt{x-1})^2 + 4 - 4\sqrt{x-1}} + \sqrt{(\sqrt{x-1})^2 + 9 - 6\sqrt{x-1}} = 1$$

$$\Rightarrow \left|\sqrt{x-1} - 2\right| + \left|\sqrt{x-1} - 3\right| = 1$$

$$\Rightarrow |y - 2| + |y - 3| = 1 \text{ where } y = \sqrt{x-1}$$

Solving this we get $2 \le y \le 3$
Hence, $x \in [5, 10]$

➤ **Example 3.** Solve the equation
$|x^2 - 1| + |2 - x^2| = 1$

➤ ***Solution*** Since $x^2 - 1 + 2 - x^2 = 1$, we can write the given equation as
$|x^2 - 1| + |2 - x^2| = |(x^2 - 1) + (2 - x^2)|$
This is compared with $|a| + |b| = |a + b|$; which is true only when $ab \ge 0$
(i.e.) $(x^2 - 1)(2 - x^2) \ge 0$.
This is true when $1 \le x^2 \le 2$
$\Rightarrow x \in [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$.

➤ **Example 4.** Solve $|(x^2 + 2x + 2) + (3x + 7)| < |x^2 + 2x + 2| + |3x + 7|$.

➤ ***Solution*** We have $|a + b| < |a| + |b|$ if a, b are of opposite signs.
Now, $x^2 + 2x + 2 = (x + 1)^2 + 1 > 0 \;\forall\; x \in R$.
Thus, the equation is satisfied for those values of x at which $3x + 7 < 0$ i.e. $x < \frac{-7}{3}$.

➤ **Example 5.** Solve $|(x^4 - 9) - (x^2 + 3)| = |x^4 - 9| - |x^2 + 3|$.

➤ ***Solution*** We have $|a - b| = |a| - |b|$ if a, b have the same sign and $|a| \ge |b|$
i.e. $x^4 - 9$ and $x^2 + 3$ must be of same sign

www.pdfworld.net

www.pdfworld.net

i.e. $(x^4 - 9)(x^2 + 3) > 0$
i.e. $(x^2 - 3)(x^2 + 3)^2 > 0$
i.e. $x^2 - 3 > 0$ ...(1)
and $|x^4 - 9| \geq |x^2 + 3|$
i.e. $(x^2 - 3)(x^2 + 3) \geq x^2 + 3$
[using result (1)]
i.e. $(x^2 + 3)(x^2 - 4) \geq 0$ i.e. $x^2 \geq 4$
gives $x \in (-\infty, -2] \cup [2, \infty)$.

Practice Problems [N]

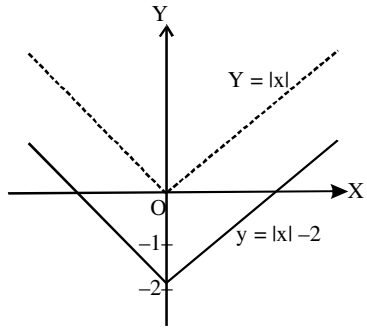
1. If $|a - b| < \ell$ and $|b - c| < m$ prove that $|a - c| < \ell + m$.
2. Prove that $|x + y + z| \geq |x| - |y| - |z|$, $x, y, z \in R$
3. Prove that if the numbers x, y are of one sign, then
$$\left|\frac{x+y}{2} - \sqrt{xy}\right| + \left|\frac{x+y}{2} + \sqrt{xy}\right| = |x| + |y|$$
4. Solve $|x^2 - x| + 2|x - 1| = |x^2 + x - 2|$
5. Solve $\left|\frac{x}{x-1}\right| + |x| = \frac{x^2}{|x-1|}$
6. Solve $|x^2 + x - 1| = \big||x^2 + 2x| - |x + 1|\big|$
7. Solve $|x + 2| + |x^2 - 5x + 1| \leq |x^2 - 4x + 3|$

1.19 GRAPHS OF MODULUS FUNCTIONS

Let us first draw the graphs of simple functions involving modulus.

➤ **Example 1.** Draw the graph of $y = |x| - 2$

➤ ***Solution***



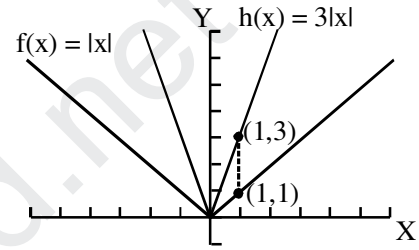
➤ **Example 2.** Compare the graph of the following functions with the graph of $f(x) = |x|$:

(i) $h(x) = 3|x|$ (ii) $g(x) = \frac{1}{3}|x|$

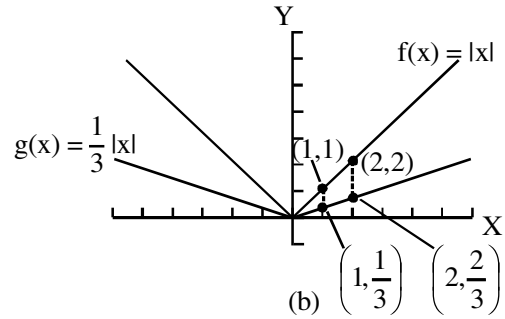
➤ ***Solution***

(i) Relative to the graph of $f(x) = |x|$, the graph of $h(x) = 3|x| = 3f(x)$ is a vertical stretch (multiply each y-value by 3) of the graph of f.

(ii) Similarly, the equation $g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$ indicates that the graph of h is a vertical shrink of the graph of f.



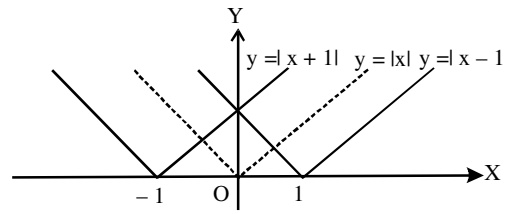
(a)



(b)

➤ **Example 3.** Draw the graphs of $y = |x - 1|$ and $y = |x + 1|$.

➤ ***Solution.***



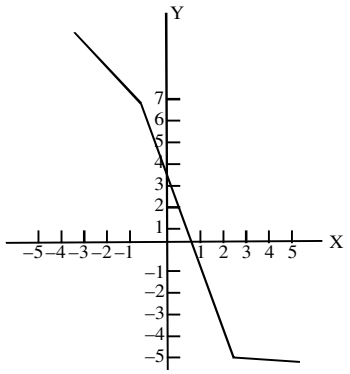
www.pdfworld.net

www.pdfworld.net

➤ **Example 4.** Sketch the graph of the function $f(x) = 2|x-2| - |x+1| - x$

➤ ***Solution***

$$f(x) = \begin{cases} -2(x-2)+(x+1)-x & x < -1 \\ -2(x-2)-(x+1)-x & -1 \le x < 2 \\ 2(x-2)-(x+1)-x & x \ge 2 \end{cases}$$

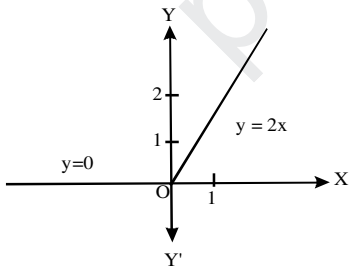


$$\Rightarrow \quad f(x) = \begin{cases} 5-2x\ , & x < -1 \\ 3-4x\ , & -1 \le x < 2 \\ -5\ , & x \ge 2 \end{cases}$$

➤ **Example 5.** Sketch the graph of the function

$f(x) = x + \sqrt{x^2}$

➤ ***Solution*** $f(x) = x + |x|$

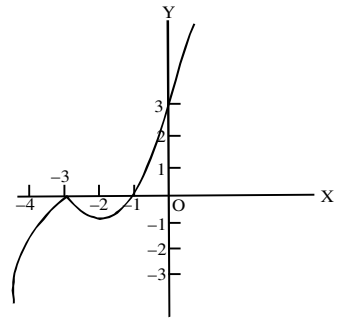


➤ **Example 6.** Sketch the graph of the function

$f(x) = |x+3|(x+1)$

➤ ***Solution*** The given function can be written as

$$f(x) = \begin{cases} -(x+3)(x+1), & x < -3 \\ (x+3)(x+1), & x \ge -3 \end{cases}$$

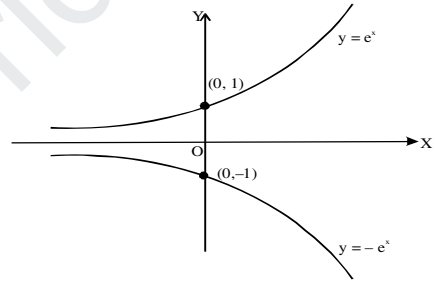


1. Drawing the graph of y = –f(x) from the known graph of y = f(x) :

The graph of function $y = -f(x)$ is obtained from the graph of the function $y = f(x)$ by reflection of the graph of $y = f(x)$ about the x–axis.

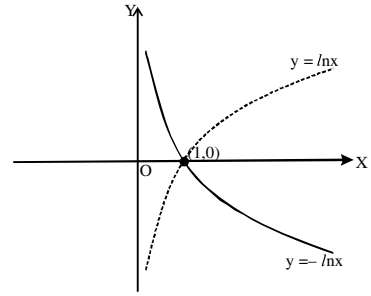
➤ **Example 7.** Draw the graph of $y = -e^x$

➤ ***Solution*** Take reflection of $y = e^x$ considering x–axis as plane mirror.



➤ **Example 8.** Draw the graph of $y = -\ln x$

➤ ***Solution***



2. Drawing the graph of y = f(–x) from the known graph of y = f(x) :

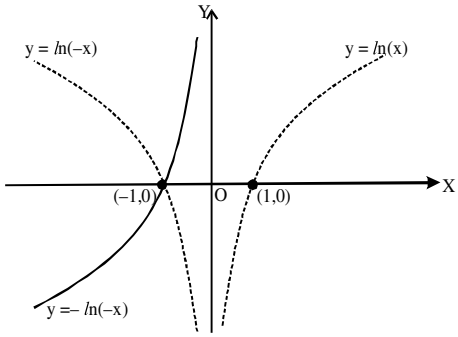
The graph of function $y = f(-x)$ is obtained from the graph of the function $y = f(x)$ by reflection of the graph of $y = f(x)$ about the y–axis.

www.pdfworld.net

www.pdfworld.net

➤ **Example 9.** Draw the graph of $y = -\ln(-x)$

➤ ***Solution***

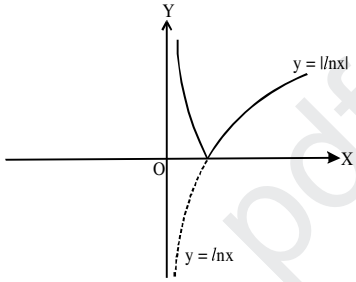


3. Drawing the graph of y = | f(x)|, from the graph of y = f(x) :

The graph of function y = |f(x)| is obtained from the graph of the function y = f(x) as follows : the portion of the graph of the function y = f(x) lying above the x–axis remains unchanged and its portion located below the x–axis, is reflected above the x-axis.

➤ **Example 10.** Draw the graph of $y = |\ln x|$

➤ ***Solution*** Take the image of the negative part of f(x) considering the x–axis as plane mirror.



4. Drawing the graph of y = f(|x|) , from the graph of y = f(x) :

The graph of function y = f(|x|) is obtained from the graph of the function y = f(x) as follows :

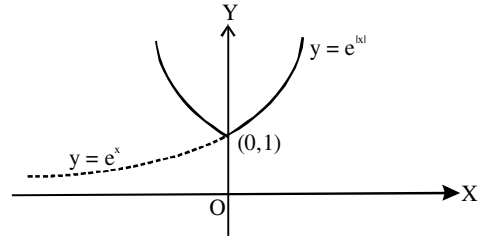
For $x \geq 0$, the graph of the function is retained, and this retained portion of the graph is reflected symmetrically about the y–axis, thus determining the graph of the function for $x \leq 0$.

(i) The part of f(x) lying on the left of the y–axis (if it exits) is omitted.

(ii) f(| x |) is an even function.

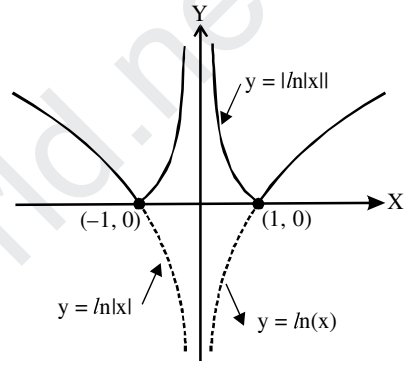
➤ **Example 11.** Draw the graph of the function $y = e^{|x|}$

➤ ***Solution***



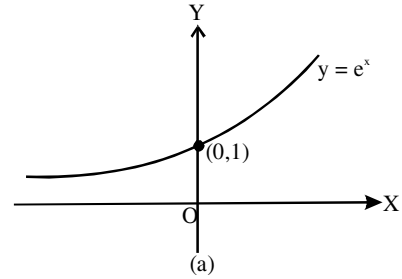
➤ **Example 12.** Draw the graph of the function $y = |\ln|x||$

➤ ***Solution.***

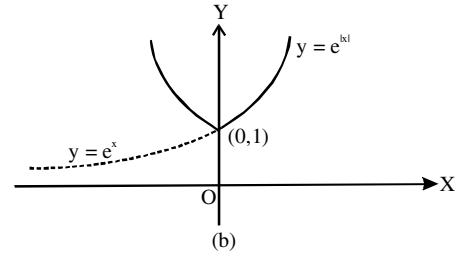


➤ **Example 13** Draw the graph of the function $y = |e^{|x|} - 3|$

➤ ***Solution***

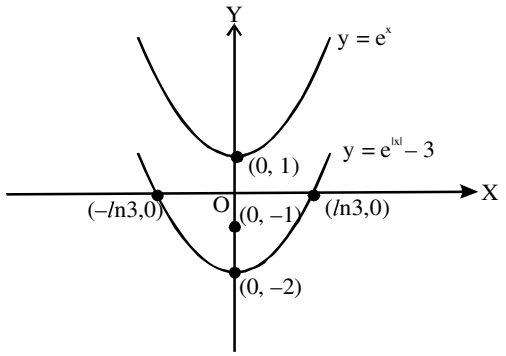


⇒



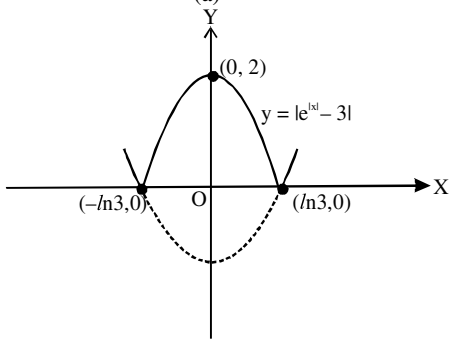
www.pdfworld.net

www.pdfworld.net



(a)

⇒



(b)

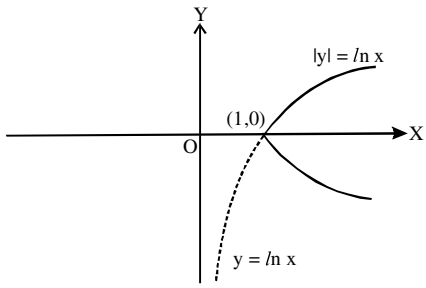
5. Drawing the graph of |y| = f(x), from the graph of y = f(x) :

The graph of function |y| = f(x) is obtained from the graph of the function y = f(x) as follows :

First delete its portion located below the x–axis. The portion of the graph of the function y = f(x) lying above the x–axis remains unchanged. Then take the image of the positive part of f(x) about the x–axis considering it as a plane mirror.

➤ **Example 14.** Draw the graph of the function
| y | = ln x.

➤ ***Solution.***

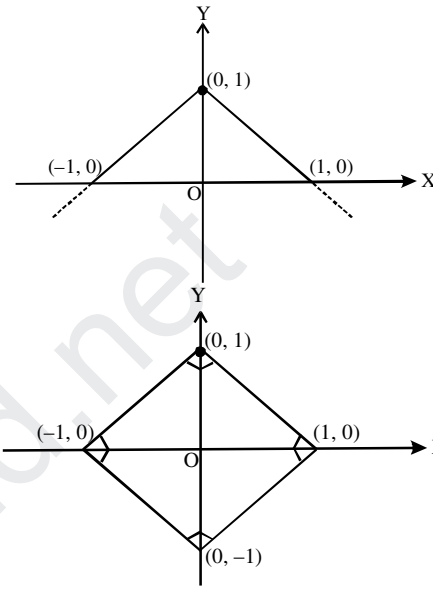


➤ **Example 15.** Draw the graph of the function
| x | + | y | = 1

➤ ***Solution*** The given curve can be rewritten as
| y | = 1 – | x |

Graph of y = 1 – | x |

Now graph of | y | = 1 – | x |



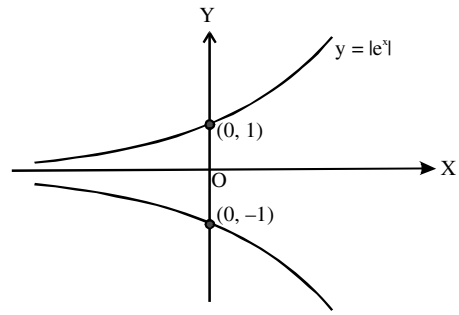
6. Drawing the graph of | y | = | f(x) |, from the graph of y = f(x) :

The graph of function |y| = |f(x)| is obtained from the graph of the function y = f(x) as follows :

First draw the graph of y = f(x). Then take the image of the graph of f(x) about the x–axis considering it as a plane mirror. The union of the graphs is the required graph.

➤ **Example 16.** Draw the graph of the function
| y | = |e^x|

➤ ***Solution***

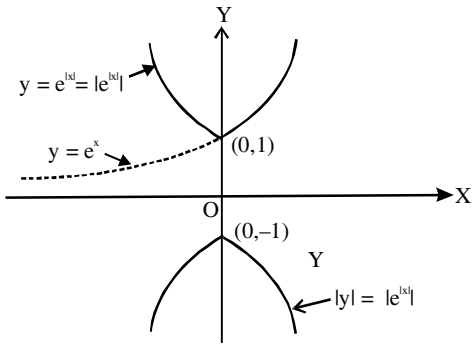


www.pdfworld.net

www.pdfworld.net

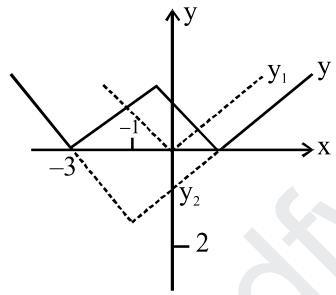
➤ **Example 17.** Draw the graph of the function
$|y| = |e^{|x|}|$

➤ ***Solution***



➤ **Example 18.** Construct the graph of the function
$y = ||x + 1| - 2|$.

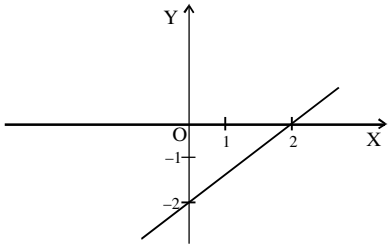
➤ ***Solution*** Taking the graph of the function $y_1 = |x|$, translate it one unit leftwards along the x– axis and two units.



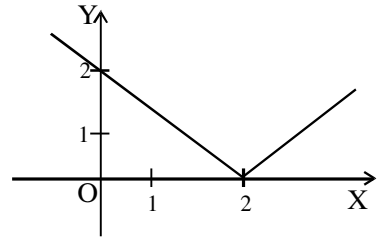
downwards along the y-axis. This yields the graph of the function $y_2 = |x + 1| - 2$. then replace the portion of the graph below the x–axis corresponding to $-3 \le x \le 1$, by the portion symmetric to it about the x–axis. The resulting polygonal line is the graph of the function y.

➤ **Example 19.** Draw the graph of the function
$y = |2 - |x - 2||$

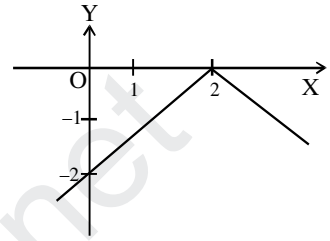
➤ ***Solution.*** Graph of $y = x - 2$



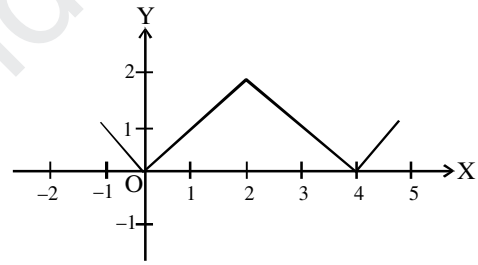
Graph of $y = |x - 2|$:



Graph of $y = -|x - 2|$



Graph of $y = 2 - |x - 2|$



7. To draw the graphs of functions of the form y = max {f(x), g(x)} or y = min {f(x), g(x)} :

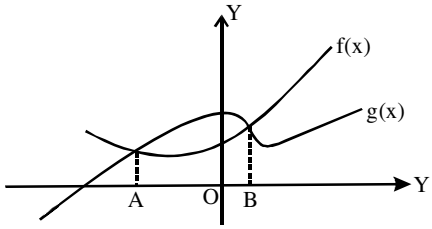
First draw the graphs of f(x) and g(x). Locate their points of intersection.

Between any two consecutive points of intersection, if the graph of f(x) is above the graph of g(x) i.e., $f(x) > g(x)$.

then maximum $\{f(x), g(x)\} = \max\{f(x), g(x)\} = f(x)$
and minimum $\{f(x), g(x)\} = \min\{f(x), g(x)\} = g(x)$
and if graph of f(x) is below the graph of g(x) i.e.,
$f(x) < g(x)$
then $\max\{f(x), g(x)\} = g(x)$ and
$\min\{f(x), g(x)\} = f(x)$

www.pdfworld.net

www.pdfworld.net



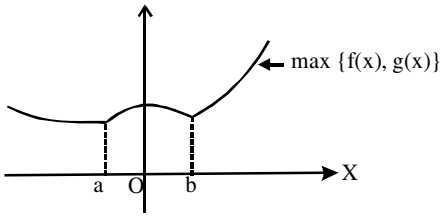
It is clear from the graph.

For $x < a$, $f(x) > g(x) \Rightarrow \max\{f(x), g(x)\} = f(x)$
and $\min\{f(x), g(x)\} = g(x)$

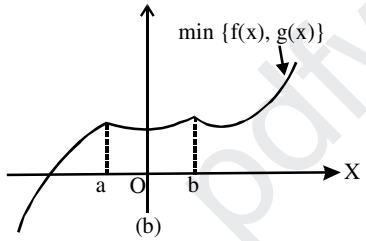
For $a < x < b$, $g(x) > f(x) \Rightarrow \max\{f(x), g(x)\} = g(x)$
and $\min\{f(x), g(x)\} = g(x)$

and For $x < b$, $f(x) > g(x) \Rightarrow \max\{f(x), g(x)\} = f(x)$
and $\min\{f(x), g(x)\} = g(x)$

Graphs of $\max\{f(x), g(x)\}$ and $\min\{(f(x), g(x)\}$ for the above graph.



(a)



(b)

Note

(i) $\max(f(x), g(x)) = \dfrac{f(x)+g(x)}{2} + \left|\dfrac{f(x)-g(x)}{2}\right|$

(ii) $\min(f(x), g(x)) = \dfrac{f(x)+g(x)}{2} - \left|\dfrac{f(x)-g(x)}{2}\right|$

Proof (analytically) :

(i) If $f(x) > g(x)$, then $|f(x) - g(x)| = f(x) - g(x)$ ...(1)

$\max\{f(x), g(x)\} = f(x)$

$= \dfrac{f(x)+g(x)}{2} + \dfrac{f(x)-g(x)}{2}$

$= \dfrac{f(x)+g(x)}{2} + \left|\dfrac{f(x)-g(x)}{2}\right|$ {from (1) }

and if $f(x) < g(x)$,

then $|f(x) - g(x)| = -(f(x) - g(x))$ ...(2)

$\therefore$ $\max\{f(x), g(x)\} = g(x)$

$= \dfrac{f(x)+g(x)}{2} + \dfrac{g(x)-f(x)}{2}$

$= \dfrac{f(x)+g(x)}{2} + \left|\dfrac{f(x)-g(x)}{2}\right|$ {from (2)}

(ii) If $f(x) > g(x)$, then $|f(x) - g(x)| = f(x) - g(x)$ ...(3)

$\therefore$ $\min\{f(x), g(x)\} = g(x)$

$= \dfrac{f(x)+g(x)}{2} - \left(\dfrac{f(x)-g(x)}{2}\right)$

$= \dfrac{f(x)+g(x)}{2} - \left|\dfrac{f(x)-g(x)}{2}\right|$ {from (3)}

and if $f(x) < g(x)$, then

$|f(x) - g(x)| = -(f(x) - g(x)) = g(x) - f(x)$ ...(4)

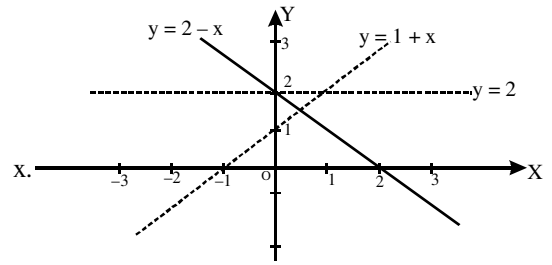
$\therefore$ $\min\{f(x), g(x)\} = f(x)$

$= \dfrac{f(x)+g(x)}{2} - \left(\dfrac{g(x)-f(x)}{2}\right)$

$= \dfrac{f(x)+g(x)}{2} - \left|\dfrac{f(x)-g(x)}{2}\right|$ {from (4)}

➤ **Example 20.** Draw the graph of the function $y = \max\{2 - x, 2, 1 + x\}$

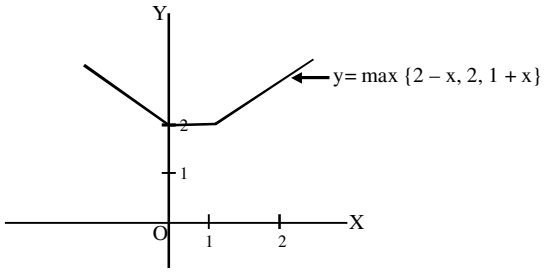
➤ ***Solution*** First draw the graphs of $y = 2 - x$, $y = 2$ and $y = 1 +$



Graph of $y = \max\{2 - x, 2, 1 + x\}$ from the above graph:

www.pdfworld.net

www.pdfworld.net



Graphs of Linear Inequalities

The inequalities of the form $ax + by + c > 0$, $ax + by + c < 0$, $ax + by + c \geq 0$ and $ax + by + c \leq 0$ are called linear inequations in two variables x and y.

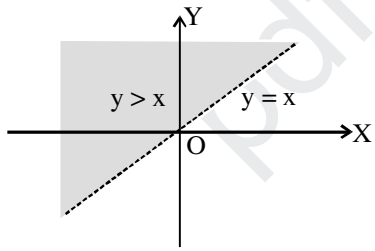
First draw the line $ax + by + c = 0$. For strict inequality i.e., $ax + by + c > 0$ or $ax + by + c < 0$ draw the line dotted (.....) otherwise draw it thick(—). The shaded portion represents the solution set of the given inequality. The dotted line is not a part of the solution, but thick line is a part of the solution.

➤ **Example 21.** Draw the graphs of the solution set of the following inequalities :

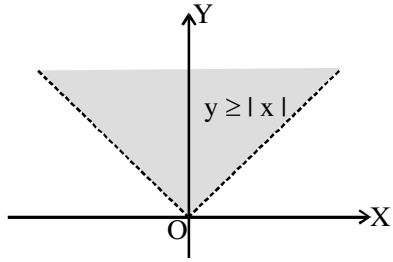
(a) $y > x$ (b) $y \geq |x|$ (c) $2x - y \geq 1$

➤ ***Solution.***

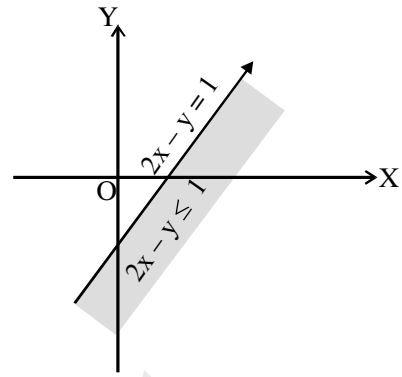
(a) $y > x$



(b) $y \geq |x|$



(c) $2x - y \geq 1$



Practice Problems [O]

1. Draw the graphs :
(a) $y = |x^2 + x|$
(b) $y = -|x^2 - 2x|$

2. Draw the graphs :
(a) $y = x^2 - 4|x| + 3$
(b) $y = x^2 + 4|x| + 3$
(c) $y = 2 - |x| - x^2$

3. Draw the graphs :
(a) $y = |x^2 - 3|x| + 2|$
(b) $y = -|x^2 - |x| - 6|$.

4. Draw the graphs :
(a) $y = |x|(x - 2)$;
(b) $y = (3 - x)|x + 1|$.

5. Draw $y = |x^2 - 4| - |x^2 - 9|$

6. Indicate the points on the plane X-Y which satisfy the following equations :
(i) $y + |y| - x - |x| = 0$
(ii) $|x + y| + |x - y| = 4$
(iii) $|y| x = x$
(iv) $|x - y| + y = 0$

7. On the plane X-Y, indicate the points satisfying the inequality
(a) $|x - y| \leq 1$; (b) $|x + y| \geq 2$
(c) $|x| - |y| \geq 1$; (d) $|x| + |y| \leq 3$;

www.pdfworld.net

www.pdfworld.net

(e) $|x-1|+|y+1| \geq 2$

(f) $|x+y|+|x-y| \leq 2$

8. How many solutions does the system of equations

$$\begin{cases} |x|+|y|=1 \\ x^2+y^2=a^2 \end{cases}$$, possess ?

9. Solve for x, y : $|y| = 1 - x^2$ and $|x^2 - 2x| + y = 1$.

10. Solve $x|x-4| + a = 0$.

1.20 EXPONENTIAL AND LOGARITHMIC FUNCTION

Exponential Function

Exponential functions are used in describing economic and physical phenomena such as compound interest, population growth, memory retention, and decay of radioactive material. Exponential functions involve a constant base and a variable exponent such as $f(x) = 2^x$ or $g(x) = 3^{-x}$.

Definition

The exponential function f with base a is denoted by

$$f(x) = a^x,$$

where $a > 0$, $a \neq 1$, and x is any real number.

The exponential function is considered only for $a > 0$, since for $a < 0$ and $a = 0$ the expression a^x looses sense for some values of the variable x.

For instance, for $a = -4$ the expression $(-4)^x$ has no sense for $x = \frac{1}{2}$. If $a = 1$, then $a^x = 1$ for any x. The case $a = 1$ is not considered, since it is not interesting.

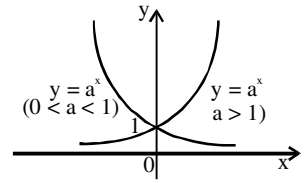
The properties of the exponential function are as follows:

(i) The exponential function $y = a^x$ is defined for all real values of the argument x, that is, its domain is the entire number line $(-\infty, \infty)$.

(ii) $a^0 = 1$ for any base $a \neq 0$.

(iii) The exponential function $y = a^x$ is positive in the entire domain and attains all positive values. The latter means that for any $y > 0$ there exists a value of x such that $a^x = y$.

(iv) The exponential function $y = a^x$ is monotonous, it increases for $a > 1$ and decreases for $0 < a < 1$.



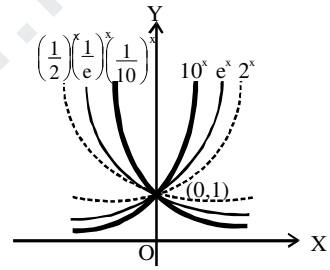
(v) For $a > 1$ we have $a^x > 1$ for $x > 0$ and $a^x < 1$ for $x < 0$; for $0 < a < 1$ we have $0 < a^x < 1$ for $x > 0$ and $a^x > 1$ for $x < 0$.

The Natural Base e

For many applications, the convenient choice for a base is the irrational number

$$e \approx 2.71828.....$$

which is called the natural base. The function $f(x) = e^x$ is the natural exponential function.

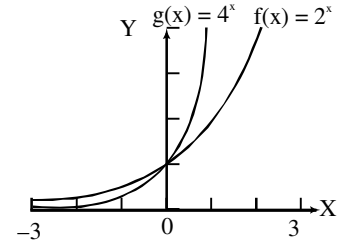


➤ **Example 1.** On the same coordinate plane, graph the following functions:

(i) $f(x) = 2^x$ (ii) $g(x) = 4^x$

➤ ***Solution*** The table lists some values for each function, and figure shows their graph. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	−2	−1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x) = 4^x$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64



(a)

www.pdfworld.net

www.pdfworld.net

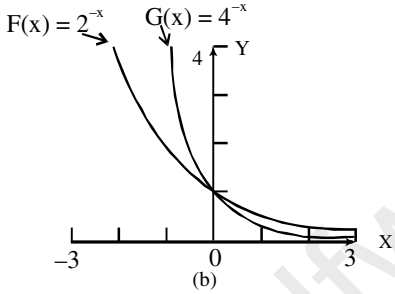
➤ **Example 2.** On the same coordinate plane, graph the following functions.

(i) $F(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$

(ii) $G(x) = 4^{-x} = \left(\frac{1}{4}\right)^x$

➤ ***Solution*** The table lists some values for each function, and figure shows their graphs. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

x	−3	−2	−1	0	1	2
$F(x) = 2^{-x}$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$G(x) = 4^{-x}$	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$



(b)

The graphs in Figures (a) and (b) are typical of the exponential functions a^x and a^{-x} $(a > 1)$.

The basic characteristics of these exponential functions are summarized here.

Graph of $y = a^x$	Graph of $y = a^{-x}$
Domain : $(-\infty, \infty)$	Domain : $(-\infty, \infty)$
Range : $(0, \infty)$	Range : $(0, \infty)$
Intercept (0, 1)	Intercept (0, 1)
Increasing	Decreasing

➤ **Example 3.** Compare the graph of each of the following with the graph of $f(x) = 3^x$. Identify the domain and range of each function :

(i) $g(x) = 3^{x+1}$

(ii) $h(x) = 3^x$

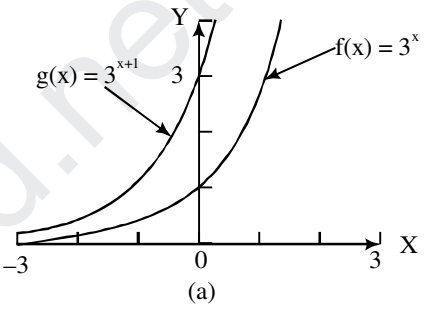
(iii) $k(x) = -3^x$

➤ ***Solution***

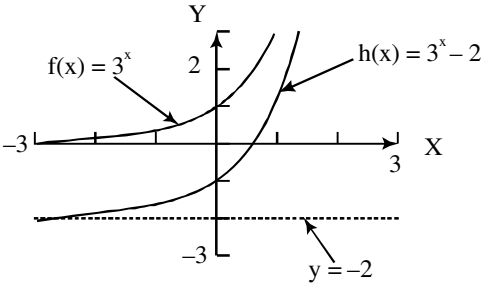
(i) Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the left, as shown in Figure (a). The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

(ii) Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f down two units, as shown in Figure (b). The domain is $(-\infty, \infty)$ and the range is $(-2, \infty)$.

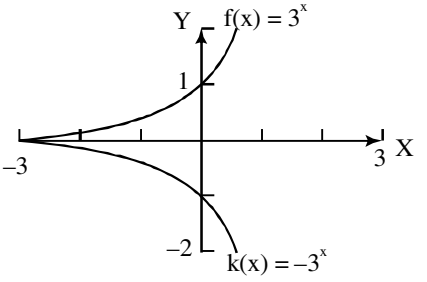
(iii) Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by reflecting the graph of f in the x-axis, as shown in Figure (c). The domain is $(-\infty, \infty)$ and the range is $(-\infty, 0)$.



(a)



(b)



(c)

www.pdfworld.net

www.pdfworld.net

Law of Indices

(i) $a^0 = 1, \quad (a \neq 0)$

(ii) $a^{-m} = \frac{1}{a^m}, \quad (a \neq 0)$

(iii) $a^{m+n} = a^m \cdot a^n$, where m and n are rational numbers

(iv) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$

(v) $(a^m)^n = a^{mn}$

(vi) $a^{p/q} = \sqrt[q]{a^p}$

Logarithm

The logarithm of a number b to a base a ($a > 0$ and $a \neq 1$) is the exponent indicating the power to which a must be raised to obtain b. Thus if $a^N = b$, N is called the logarithm of b to the base a.

Notation : $\log_a b$ (read : " The logarithm of the number b to the base a").

The equations $N = \log_a b$ and $a^N = b$ are equivalent.

For the logarithm of a number b to the base a = 10 we use the symbol log b to mean $\log_{10} b$

For instance, instead of $\log_{10} 27$ we write log 27.

By the definition of logarithm, $\log_2 16 = 4$, since $2^4 = 16$.

Similarly, $\log_3 27 = 3$, since $3^3 = 27$; $\log_2 \frac{1}{4} = -2$, since $2^{-2} = \frac{1}{4}$;

$\log_5 1 = 0$ since $5^0 = 1$; $\log 10 = 1$, $\log 1 = 0$, $\log \frac{1}{100} = -2$.

Note that such expressions as $\log_3(-27)$ or $\log_6 0$ have no sense, since the equations $3^x = -27$ and $6^x = 0$ have no roots.

In general, the expression $\log_a b$, where $a > 0$ and $a \neq 1$, has sense only for $b > 0$.

The base of the logarithm 'a' must not equal unity otherwise numbers not equal to unity will not have a logarithm and any number will be the logarithm of unity.

Properties of Logarithms

1. $\log_a 1 = 0$ because 0 is the power to which a must be raised to obtain 1.
2. $\log_a a = 1$ because 1 is the power to which a must be raised to obtain a.
3. $\log_a a^N = N$ because N is the power to which a must be raised to obtain a^N.
4. $a^{\log_a b} = b$ because $\log_a b$ is the power to which a must be raised to obtain b.

Logarithmic Function

A logarithmic function is a function of the form $y = \log_a x$ ($a > 0$, $a \neq 1$) with base a.

The logarithmic function can be considered as the inverse of an exponential function.

Let us express from the function $y = a^x$, the variable x in terms of y :

$x = \log_a y$

Passing over to the customary notation, we obtain

$y = \log_a x$

A logarithmic function is thus an inverse of the exponential function. Therefore, their graphs are symmetric about the straight line y = x. Knowing the graph of an exponential function, we obtain the graph of a logarithmic function.

Sketching the Graphs of Logarithmic Functions

On the same coordinate plane, let us sketch graphs of the following functions :

(i) $f(x) = 2^x$ (ii) $g(x) = \log_2 x$

(i) For $f(x) = 2^x$, let us construct a table of values.

x	−2	−1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

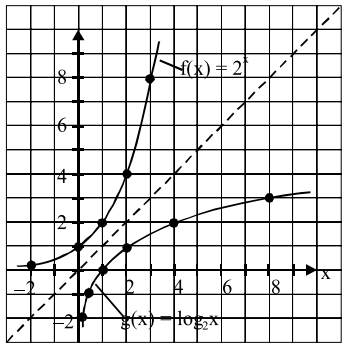
By plotting these points and connecting them with a smooth curve, we obtain the graph shown in Figure.

(ii) Because $g(x) = \log_2 x$ is the inverse of $f(x) = 2^x$, the graph of g is obtained by reflecting the graph of f in the line y = x, as shown in figure. This reflection

www.pdfworld.net

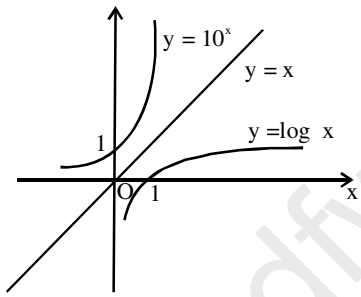
www.pdfworld.net

is formed by interchanging the x-and y-coordinates of the points in the above table of values.



Inverse Functions

Also, the graph of the function y = log x is symmetric to the graph of the function y = 10^x about the straight line y = x. The logarithmic function with base 10 is the common logarithmic function.

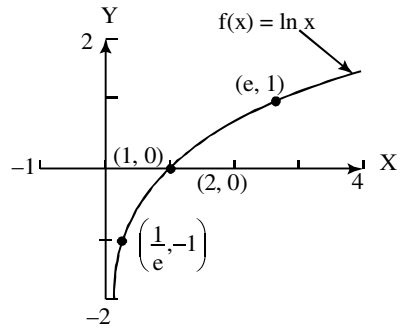


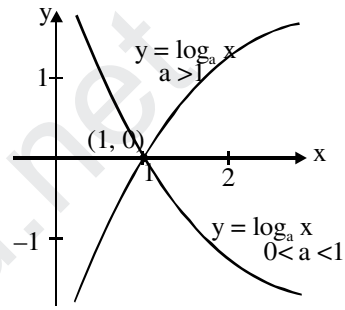
The properties of the logarithmic function y = logx can be obtained form its graph. They are listed below :

(1) The function y = log x is defined for all positive numbers (therefore, all negative numbers and zero are said to have no logarithms).
(2) The graph intersects the x–axis at the point (1,0)
(3) The function y = log x is an increasing function.
(4) The function y = log x attains all real values, that is, the range of values of the function is the set of all real numbers.

The Natural Logarithmic Function

The function defined by f(x) = $\log_e$ x = ln x, x > 0 is called the natural logarithmic function.





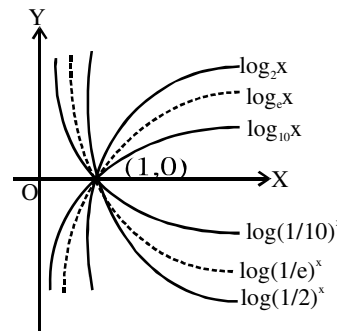
Graph of y = $\log_a$ x

Domain (0, ∞)
Range : (–∞, ∞)
Intercept : (1, 0)

Reflection of graph of y = a^x about the line y = x. Increasing if a > 1 and decreasing if 0 < a < 1.

At x = 0, $\log_a$ x is undefined and as x gets close to 0, the value of $\log_a$ x approaches infinity.

The figure shows the graphs of several logarithmic functions:



www.pdfworld.net

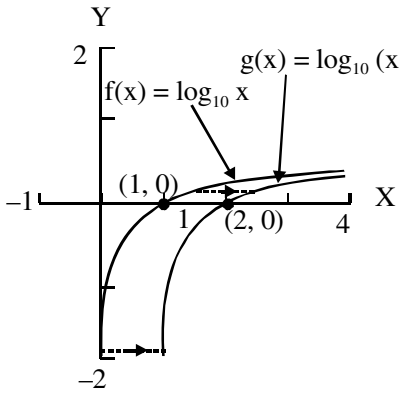
www.pdfworld.net

➤ **Example 4.** Compare the graphs of the following functions with the graph of $f(x) = \log_{10} x$.

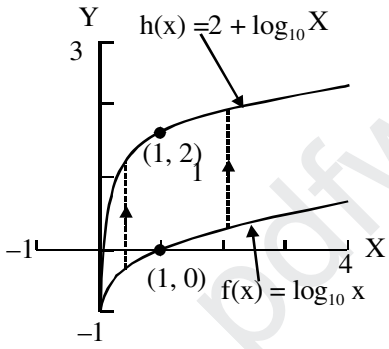
(i) $g(x) = \log_{10}(x - 1)$

(ii) $h(x) = 2 + \log_{10} x$

➤ ***Solution*** The graph of each of these functions is similar to the graph of $f(x) = \log_{10} x$, as shown in figure.



(a) Right shift of 1 unit



(b) Upward shift of 2 units

(i) Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the right.

(ii) Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units up.

1.21 LAWS OF LOGARITHM

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following laws hold good :

1. To find the logarithm of a product.

$$\log_a MN = \log_a M + \log_a N$$

Let MN be the product; let a be the base of the system, and suppose

$x = \log_a M$, $y = \log_a N$;

So that $a^x = M$, $a^y = N$

Thus the product $MN = a^x \times a^y = a^{x+y}$;

$\Rightarrow \log_a MN = x + y = \log_a M + \log_a N$

Similarly $\log_a MNP = \log_a M + \log_a N + \log_a P$;

and so on for any number of factors.

For example ,

$\log 42 = \log (2 \times 3 \times 7) = \log 2 + \log 3 + \log 7.$

2. To find the logarithm of a fraction.

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

Let $\frac{M}{N}$ be the fraction, and suppose

$x = \log_a M$, $y = \log_a N$;

So that $a^x = M$, $a^y = N$

Thus the fraction $\frac{M}{N} = \frac{a^x}{a^y}$

$= a^{x-y}$;

$\Rightarrow \log_a \frac{M}{N} = x - y = \log_a M - \log_a N$

For example : $\log \frac{30}{7} = \log 30 - \log 7$

$= \log (2 \times 3 \times 5) - \log 7 = \log 2 + \log 3 + \log 5 - \log 7$

3. To find the logarithm of a number raised to any power

$$\log_a (M^p) = p \log_a M$$

Let required log be $\log_a (M^p)$ and suppose

$x = \log_a M$, so that $a^x = M$;

then $M^p = (a^x)^p$

$= a^{px}$;

$\Rightarrow \log_a (M^p) = p \log_a M,$

Similarly, $\log_a \left(M^{\frac{1}{r}} \right) = \frac{1}{r} \log_a M.$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

For instance, the expression $\frac{1}{2}\log_{10} x - 3\log_{10}(x+1)$ can be written as the logarithm of a single quantity as :

$$\frac{1}{2}\log_{10} x - 3\log_{10}(x+1)$$

$$= \log_{10} x^{1/2} - \log_{10}(x+1)^3 = \log_{10}\frac{\sqrt{x}}{(x+1)^3}$$

➤ **Example 1.** Compute $\log_{ab}\frac{\sqrt[3]{a}}{\sqrt{b}}$ if $\log_{ab} a = 4$.

➤ ***Solution*** By the laws of logarithms we have

$$\log_{ab}\frac{\sqrt[3]{a}}{\sqrt{b}} = \frac{1}{3}\log_{ab} a - \frac{1}{2}\log_{ab} b = \frac{4}{3} - \frac{1}{2}\log_{ab} b$$

It remains to find the quantity $\log_{ab} b$. Since

$$1 = \log_{ab} ab = \log_{ab} a + \log_{ab} b = 4 + \log_{ab} b$$

it follows that $\log_{ab} b = -3$ and so

$$\log_{ab}\frac{\sqrt[3]{a}}{\sqrt{b}} = \frac{4}{3} - \frac{1}{2}\cdot(-3) = \frac{17}{6}$$

➤ **Example 2.** Prove that

$\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$, if $a^2 + b^2 = 7ab$, $a > 0$, $b > 0$

➤ ***Solution*** Let us transform the conditions by isolating a perfect square $a^2 + b^2 + 2ab = 9ab$,

i.e. $(a+b)^2 = 9ab$.

Taking logarithms of this equality to the base 10, we obtain

$$2\log(a+b) - 2\log 3 = \log a + \log b$$

$$\Rightarrow \quad \log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$$

➤ **Example 3.** If $\log 25 = a$ and $\log 225 = b$, then find the value of $\log\left(\frac{1}{9}\right)^2 + \log\left(\frac{1}{2250}\right)$ in terms of a and b (base of the log is 10).

➤ ***Solution*** $\log 25 = a$; $\log 225 = b$

$2\log 5 = a$; $\log(25\cdot 9) = b$ or $\log 25 + 2\log 3 = b$

$\Rightarrow$ $2\log 3 = b - a$

Now $\log\left(\frac{1}{9}\right)^2 + \log\left(\frac{1}{2250}\right)$

$= -2\log 9 - \log 2250$

$= -4\log 3 - [\log 225 + \log 10]$

$= -2(b-a) - [b+1]$

$= -2b + 2a - b - 1 = 2a - 3b - 1$.

4. To find the logarithm of a number with base raised to any power

$$\log_{a^q} M = \frac{1}{q}\log_a M$$

➤ **Example 4.** Suppose that *a* and *b* are positive real numbers such that

$$\log_{27} a + \log_9 b = \frac{7}{2} \text{ and } \log_{27} b + \log_9 a = \frac{2}{3}.$$

Find the value of the ab.

➤ ***Solution*** $\log_{27} a + \log_9 b = \frac{7}{2}$;

$$\log_{27} b + \log_9 a = \frac{2}{3}$$

$$\frac{1}{3}\log_3 a + \frac{1}{2}\log_3 b = \frac{7}{2}$$

$$\frac{1}{3}\log_3 b + \frac{1}{2}\log_3 a = \frac{2}{3}$$

adding the equation

$$\frac{1}{3}\log_3(ab) + \frac{1}{2}\log_3(ab) = \frac{7}{2} + \frac{2}{3} = \frac{25}{6}$$

$$\frac{5}{6}\log_3(ab) = \frac{25}{6} \quad \Rightarrow \quad \log_3(ab) = 5$$

$\Rightarrow$ $ab = 3^5 = 243$

5. To find the logarithm to a different base b.

(i) $\log_b M = \frac{\log_a M}{\log_a b}$

www.pdfworld.net

www.pdfworld.net

(ii) $\log_b a = \dfrac{1}{\log_a b}$

Let M be any number whose logarithm to base b is required.

Let $y = \log_b M$, so that $b^y = M$;

$\therefore \quad \log_a (b^y) = \log_a M$;

that is $\quad y \log_a b = \log_a M$;

$$y = \frac{1}{\log_a b} \times \log_a M,$$

or $$\log_b M = \frac{1}{\log_a b} \times \log_a M, \qquad \ldots. (1)$$

Now since M and b are given, $\log_a M$ and $\log_a b$ are known, and thus $\log_b M$ may be found.

In equation (1) put a for M; thus

$$\log_b a = \frac{1}{\log_a b} \times \log_a a = \frac{1}{\log_a b};$$

$$\log_b a \times \log_a b = 1$$

$\therefore$ This result may also be proved directly as follows ;

Let $x = \log_a b$, so that $a^x = b$;

then by taking logarithms to base b, we have

$x \log_b a = \log_b b = 1$;

$\log_a b \times \log_b a = 1$.

➤ **Example 5.** Compute the value of

$$\frac{1}{\log_2 36} + \frac{1}{\log_3 36}.$$

➤ ***Solution*** $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36} = \log_{36} 2 + \log_{36} 3$

$$= \log_{36} 6 = \frac{1}{2}.$$

➤ **Example 6.** Compute $\log_3 5.\log_{25} 27$.

➤ ***Solution*** $\log_3 5.\log_{25} 27$

$$= \log_3 5.\ \log_{5^2} 3^3 = \frac{3}{2}\log_3 5.\log_5 3 = 3/2$$

since $\log_3 5.\log_5 3 = 1$

➤ **Example 7.** Compute $\log_3 2.\log_4 3 \ldots\ldots \log_{10} 9.\ \log_{11} 10$

➤ ***Solution*** $\log_3 2 = \dfrac{\log_{11} 2}{\log_{11} 3}$; $\log_4 3 = \dfrac{\log_{11} 3}{\log_{11} 4}$; .....

$$\log_{10} 9 = \frac{\log_{11} 9}{\log_{11} 10}$$

Hence $\quad \log_3 2.\log_4 3 \ldots\ldots \log_{11} 10$

$$= \frac{\log_{11} 2}{\log_{11} 3}.\frac{\log_{11} 3}{\log_{11} 4} \ldots\ldots \frac{\log_{11} 9}{\log_{11} 10}.\log_{11} 10 = \log_{11} 2$$

➤ **Example 8.** Given that $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2}(8) = \dfrac{2}{s^3+1}$. Write $\log_2 \dfrac{a^2 b^5}{c^4}$ as a function of 's' (a, b, c > 0, $c \neq 1$).

➤ ***Solution*** Given

$\log_2 a = s$ ....(1)

$\log_2 b = 2s^2$ ....(2)

$$\log_8 c^2 = \frac{s^3+1}{2} \qquad \ldots.(3)$$

$$\Rightarrow \frac{2\log c}{3\log 2} = \frac{s^3+1}{2}$$

$\Rightarrow 4\log_2 c = 3(s^3+1)$ ....(4)

Now $2\log_2 a + 5\log_2 b - 4\log_2 c = 2s + 10s^2 - 3(s^3+1)$

6. Fundamental logarithmic identity

$$a^{\log_a x} = x$$

➤ **Example 9.** Compute $2^{\log_{2\sqrt{2}} 15}$

➤ ***Solution*** $\log_{2\sqrt{2}} 15 = \log_{2^{3/2}} 15 = \dfrac{2}{3}\log_2 15$

Applying the fundamental logarithmic identity, we get

$$2^{\log_{2\sqrt{2}} 15} = 2^{2/3 \log_2 15} = (2^{\log_2 15})^{2/3} = 15^{2/3} = \sqrt[3]{225}$$

➤ **Example 10.** Compute $\left(\sqrt[3]{9}\right)^{\frac{1}{5\log_5 3}}$

➤ ***Solution*** $\dfrac{1}{5\log_5 3} = \dfrac{1}{5}\log_3 5$

www.pdfworld.net

www.pdfworld.net

$$\left(\sqrt[3]{9}\right)^{\frac{1}{5\log_5 3}} = (9^{1/3})^{\frac{1}{5}\log_3 5} = (3^{2/3})^{\frac{1}{5}\log_3 5}$$
$$= 3^{\log_3 5^{2/15}} = 5^{2/15} = \sqrt[15]{25}$$

➤ **Example 11.** Calculate
$81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

➤ ***Solution*** We have $81^{1/\log_5 3} = 81^{\log_3 5} = (3^4)^{\log_3 5}$

$= (3^{\log_3 5})^4 = 5^4 = 625$ since $3^{\log_3 5} = 5$

Similarly $3^{4/\log_7 9} = 3^{4\log_9 7} = (3^2)^{2\log_9 7} = (9^{\log_9 7})^2 = 7^2 = 49$,

$$27^{\log_9 36} = 27^{\log_3 6} = 3^{3\log_3 6} = (3^{\log_3 6})^3 = 216.$$

Adding the resulting values together, we get the required number 890.

➤ **Example 12.** Compute $\sqrt{\left(\frac{1}{\sqrt{27}}\right)^{2-\frac{\log_5 13}{2\log_5 9}}}$

➤ ***Solution*** Using in succession the laws of logarithms and exponents we compute the radicand:

$$\left(\frac{1}{\sqrt{27}}\right)^{2-\frac{\log_5 13}{2\log_5 9}}$$

$$= \frac{1}{27}\cdot(\sqrt{27})^{\frac{1}{2}\log_9 13} = \frac{1}{27}\cdot(3^{\log_3 13})^{3/8} = 3^{-3}\cdot 13^{3/8}$$

whence it is clear that the given number is equal to $3^{-3/2}\cdot 13^{3/16}$.

7. $x^{\log_a b} = b^{\log_a x}$

➤ **Example 13.** Simplify the expression $a^{\frac{\log\log a}{\log a}}$ where all the logarithms are taken to the same base b.

➤ ***Solution*** $a^{\frac{\log_b \log_b a}{\log_b a}} = (a^{\log_a b})^{\log_b \log_b a}$

$= b^{\log_b \log_b a} = \log_b a$

CAUTION

There is no general property that can be used to write $\log_a (M \pm N)$.

Specifically, $\log_a(M+N)$ DOES NOT EQUAL $\log_a M + \log_a N$.

➤ **Example 14.** If $\log_{x-3}(2x-3)$ is a meaningful quantity then find the interval in which x must lie.

➤ ***Solution*** $x-3>0$, $x-3\neq 1$ and $2x-3>0$

$x>3$; $x\neq 4$ and $x>3/2$

Hence $x \in (3,4)\cup(4,\infty)$

➤ **Example 15.** For $x \geq 0$, what is the smallest possible value of the expression
$\log(x^3-4x^2+x+26) - \log(x+2)$?

➤ ***Solution*** $\log\frac{(x^3-4x^2+x+26)}{(x+2)}$

$$= \log\frac{(x^2-6x^2+13)(x+2)}{(x+2)}$$

$= \log(x^2-6x+13)$ $[\because\ x\neq -2]$

$= \log(x-3)^2+4$

$\therefore$ Minimum value is log 4 when $x=3$

➤ **Example 16.** Compute $\log_6 16$ if $\log_{12} 27 = a$

➤ ***Solution*** The chain of transformations

$$\log_a 16 = 4\log_6 2 = \frac{4}{\log_2 6} = \frac{4}{1+\log_2 3}$$

shows us that we have to know $\log_2 3$ in order to find $\log_6 16$.

We find it from the condition $\log_{12} 27 = a$:

$$a = \log_{12} 27 = 3\log_{12} 3 = \frac{3}{\log_2 12} = \frac{3}{1+2\log_3 2}$$

$$= \frac{3}{1+\frac{2}{\log_2 3}} = \frac{3\log_2 3}{2+\log_2 3}$$

which means that $\log_2 3 = \frac{2a}{3-a}$ (note that, obviously, $a\neq 3$).

We finally have $\log_6 16 = \frac{4(3-a)}{3+a}$.

www.pdfworld.net

www.pdfworld.net

➤ **Example 17.** If $\log_6 15 = \alpha$ and $\log_{12} 18 = \beta$, then compute the value of $\log_{25} 24$ in terms of α & β.

➤ ***Solution*** $\alpha = \dfrac{1+\log_3 5}{1+\log_3 2}$; $\beta = \dfrac{2+\log_3 2}{1+2\log_3 2}$

Let $\log_3 2 = x$ and $\log_3 5 = y$

$1 + y = \alpha(1 + x)$ ...(1)

$2 + x = \beta(2x + 1)$ ...(2)

From (2) $x = \dfrac{2-\beta}{2\beta-1}$ ...(3)

Putting this value of x in (1)

$$y = \frac{\alpha(1+\beta)-(2\beta-1)}{2\beta-1} \quad ...(4)$$

Now $\log_{25} 24 = \dfrac{3x+1}{2y}$.

Substitute the value of x and y to get

$$\log_{25} 24 = \frac{5-\beta}{2\alpha+2\alpha\beta-4\beta+2}$$

➤ **Example 18.** Simplify the expression

$$\frac{\log_a \sqrt{a^2-1}\,\log^2_{1/a}\sqrt{a^2-1}}{\log_{a^2}(a^2-1)\log_{\sqrt[3]{a}}\sqrt[6]{a^2-1}}$$

➤ ***Solution*** According to formulae, we have

$$(\log_{1/a}\sqrt{a^2-1})^2 = (-\log_a\sqrt{a^2-1})^2$$

$$= (\log_a\sqrt{a^2-1})^2 \quad ...(1)$$

$$\log_{\sqrt[3]{a}}\sqrt[6]{a^2-1} = \log_{(\sqrt[3]{a})^3}(\sqrt[6]{a^2-1})^3$$

$$= \log_a\sqrt{a^2-1} \quad ...(2)$$

$$\log_{a^2}(a^2-1) = \log_{(a^2)^{1/2}}(a^2-1)^{1/2}$$

$$\log_a\sqrt{a^2-1} \quad ...(3)$$

Substituting the right hand sides of expressions (1) to (3) into the initial fraction, we obtain

$$\frac{\log_a\sqrt{a^2-1}\,\log^2_a\sqrt{a^2-1}}{\log_a\sqrt{a^2-1}\,\log_a\sqrt{a^2-1}} = \log_a\sqrt{a^2-1}$$

➤ **Example 19.** If $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b}$.

Prove that $a^a b^b c^c = 1$

➤ ***Solution*** Let $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b} = k$

then $\log a = k(b-c)$, $\log b = k(c-a)$, $\log c = k(a-b)$

$\Rightarrow$ $a \log a + b \log b + c \log c = 0$

$\Rightarrow$ $\log(a^a \cdot b^b \cdot c^c) = 0$

$\Rightarrow$ $a^a \cdot b^b \cdot c^c = 1$

It should be noted that the above formulae also have drawbacks.

Their left and right members are meaningful for different restrictions on the values of the letters that enter into them.

For example, $\log_a MN$ has meaning when M and N are both positive or both negative. But $\log_a M + \log_a N$ has meaning for only positive M and N. For this reason, replacing $\log_a MN$ by $\log_a M + \log_a N$ when M and N are negative is meaningless. When solving equations this can lead to loss of solutions.

1*. $\log_a MN = \log_a|M| + \log_a|N| \quad (MN > 0)$

2*. $\log_a \dfrac{M}{N} = \log_a|M| - \log_a|N| \quad (MN > 0)$

3*. $\log_a N^{2k} = 2k\log_a|N| \quad (N \neq 0, k \text{ an integer})$

4*. $\log_{x^{2k}} N = \log_{|x|} N$

$(N > 0, k \neq 0$ an integer, $x \neq 0$, $|x| \neq 1)$

It should be noted that Formulae 1* and 2* also have drawbacks; their left and right members are meaningful for different restrictions on the values of the letters that enter into them. Namely the right side has meaning for arbitrary M and N different from zero, while the left hand side is only meaningful for M and N having the same sign, which means that they are subject to more stringent restrictions. For this reason, replacing $\log_a MN$ by $\log_a|M| + \log_a|N|$ when solving equations can lead to extraneous solutions but not to the loss of solutions, as can

www.pdfworld.net

www.pdfworld.net

happen when using formulae 1 – 4. Since acquiring extraneous solutions of an equation is preferable to losing solutions (superfluous solutions may be discarded by verification, but lost solutions cannot be found), one should use formulas 1* to 4* when manipulating literal expressions.

Here are some problems which illustrate the importance of utilizing these properties.

➤ **Example 20.** Simplify the expression

$\log_4 \frac{x^2}{4} - 2\log_4 4x^4$ and then compute its value for x = –2.

➤ ***Solution*** It is quite evident here that computations by formulae 1 and 3, that is

$$\log_4 \frac{x^2}{4} - 2\log_4 4x^4 = 2\log_4 x - \log_4 4$$

$$- 2\log_4 4 - 8\log_4 x = -3 - 6\log_4 x$$

are erroneous because the latter expression for x = – 2 is meaningless, whereas the original one is meaningful and is equal to –6.

This paradoxical result is due to the fact that formulae 1 and 3 are only applicable to positive values of the letters. Now if we use formulae 1* and 3* in which the values of the letters may be negative as well, we get

$$\log_4 \frac{x^2}{4} - 2\log_4 4x^4 = 2\log_4 |x| - 1 - 2 - 8\log_4 |x|$$

$$= -3 - 6\log_4 |x|$$

It is clear that for x = –2 this expression is equal to –6.

Common Logarithms

The integral part of a logarithm is called the characteristic and the decimal part is called the mantissa.

The characteristic of the logarithm of any number to the base 10 can be written down by inspection.

To determine the characteristic of the logarithm of any number greater than unity

Since $10^1 = 10$,

$10^2 = 100$,

$10^3 = 1000$,

.....

it follows that a number with two digits in its integral part lies between 10^1 and 10^2 ; a number with three digits in its integral part lies between 10^2 and 10^3 ; and so on. Hence a number with n digits in its integral part lies between 10^{n-1} and 10^n.

Let N be a number whose integral part contains n digits; then

$N = 10^{(n-1) + \text{a fraction}}$;

$\therefore$ $\log N = (n - 1) + \text{a fraction}$

Hence the characteristic is n – 1; that is, the characteristic of the logarithm of a number greater than unity is less by one than the number of digits in its integral part, and is positive.

To determine the characteristic of the logarithm of a decimal fraction.

Since $10^0 = 1$,

$10^{-1} = \frac{1}{10} = 0.1$,

$10^{-2} = \frac{1}{100} = 0.01$,

$10^{-3} = \frac{1}{1000} = 0.001$,

.....

It follows that a decimal with one zero immediately after the decimal point, such as 0.0324, being greater than 0.01 and less than 0.1, lies between 10^{-2} and 10^{-1}; a number with two zeros after the decimal point lies between 10^{-3} and 10^{-2}; and so on. Hence a decimal fraction with n zeros immediately after the decimal point lies between $10^{-(n+1)}$ and 10^{-n}.

Let D be a decimal beginning with n zeros ; then

$D = 10^{-(n+1) + \text{a fraction}}$;

$\log D = -(n + 1) + \text{a fraction}$;

Hence the characteristic is – (n + 1); that is, the characteristic of the logarithm of a decimal fraction is greater by unity than the number of zeros immediately after the decimal point, and is negative.

Let N be any number, then since multiplying or dividing by a power of 10 merely alters the position of the decimal point without changing the sequence of digits, it follows that $N \times 10^p$. and $N + 10^q$, where

www.pdfworld.net

www.pdfworld.net

p and q are any integers, are numbers whose significant digits are the same as those of N.

Now $\log(N \times 10^p) = \log N + p \log 10$

$= \log N + p$ ... (1)

Again $\log(N \div 10^q) = \log N - q \log 10$

$= \log N - p$ ... (2)

In (1) an integer is added to log N, and in (2) an integer is subtracted from log N; that is, the mantissa or decimal portion of the logarithm remains unaltered. Till now the mantissae have been supposed positive.

In the case of a negative logarithm the minus sign is written over the characteristic, and not before it, to indicate that the $\bar{4}.30103$, the logarithm of 0.0002, is equivalent to $-4 + 0.30103$, and must be distinguished from –4.30103, an expression in which both the integer and the decimal are negative. In working with negative logarithm it is necessary to make the mantissa positive. For instance, the logarithm –3.69897, in which the whole expression is negative, may be transformed by subtracting 1 from the characteristic and adding 1 to the mantissa. Thus

$$-3.69897 = -4 + (1 - .69897) = \bar{4}.30103$$

➤ **Example 21.** Let $x = (0.15)^{20}$. Find the characteristic and mantissa in the logarithm of x, to the base 10.

Assume $\log_{10}2 = 0.301$ and $\log_{10}3 = 0.477$.

➤ ***Solution.*** $\log x = \log(0.15)^{20} = 20 \log\left(\frac{15}{100}\right)$

$= 20[\log 15 - 2] = 20[\log 3 + \log 5 - 2]$

$= 20[\log 3 + 1 - \log 2 - 2]$

$= 20[-1 + \log 3 - \log 2]$

$= 20[-1 + 0.477 - 0.301]$

$= -20 \times 0.824 = -16.48 = \bar{17}.52$

Hence characteristic = – 17 and mantissa = 0.52

➤ **Example 22.** Find the number of positive integers which have the characteristic 3, when the base of the logarithm is 7.

➤ ***Solution*** $\log_7 N = x$ where $3 \le x < 4$

$\Rightarrow \quad 7^3 \le N < 7^4$

$\Rightarrow$ number of integers are 2058

➤ **Example 23.** How many digits are contained in the number 2^{75} ?

➤ ***Solution*** Computing $\log 2^{75}$, we have $\log 2^{75} = 75. \log 2 \approx 75. 0.3010 = 22.5750$. Consequently, the characteristic of this common logarithm is equal to 22. Therefore, $2^{75} = a. 10^{22}$, where $1 \le a < 10$, a is an integer, and, hence the number 2^{75} has 23 digits.

➤ **Example 24** Given log 3 = 0.4771, find

$$\log\left\{(2.7)^3 \times (0.81)^{\frac{4}{5}} \div (90)^{\frac{5}{4}}\right\}$$

➤ ***Solution*** The required value

$$= 3\log\frac{27}{10} + \frac{4}{5}\log\frac{81}{100} - \frac{5}{4}\log 90$$

$$= 3(\log 3^3 - 1) + \frac{4}{5}(\log 3^4 - 2) - \frac{5}{4}(\log 3^2 + 1)$$

$$= \left(9 + \frac{16}{5} - \frac{5}{2}\right)\log 3 - \left(3 + \frac{8}{5} + \frac{5}{4}\right)$$

$$= \frac{97}{10}\log 3 - 5\frac{17}{20} \quad = 4.62787 - 5.85 = -1.2213$$

We should notice that the logarithm of 5 and its powers can be obtained from log 2 as

$$\log 5 = \log\frac{10}{2} = \log 10 - \log 2 = 1 - \log 2$$

➤ **Example 25.** Find the number of digits in 875^{16}, given

log 2 = 0.3010, log 7 = 0.8451.

➤ ***Solution*** $\log(875^{16}) = 16(\log 7 + 3 \log 5)$

$= 16 (\log 7 + 3 - 3 \log 2)$

$= 16 \times 2.9421$

$= 47. 0736;$

Hence the number of digits is 48.

Practice Problems [P]

1. Draw the graphs of $y = e^x + 1$ and $y = e^x - 2$
2. Find all real values of x for which the expression

$$\sqrt{\log_{\frac{1}{2}}\frac{x}{x^2 - 1}}$$ is a real number.

www.pdfworld.net

www.pdfworld.net

3. Compute $\sqrt[3]{5^{\frac{1}{\log_7 5}} + \frac{1}{\sqrt{-\log_{10} 0.1}}}$

4. Compute $2^{\log_3 5} - 5^{\log_3 2} + \left(\frac{1}{49}\right)^{1+\log_7 2} + 5^{-\log_{1/5} 7}$

5. Prove that $\log \frac{75}{16} - 2\log \frac{5}{9} + \log \frac{32}{243} = \log 2$

6. Simplify

(i) $e^{\ln \ln 3}$

(ii) $\ln ab - \ln |b|$

(iii) $\log_3 5 \cdot \log_4 9 \cdot \log_5 2$

(iv) $\sqrt{\log_{0.5}^2 4}$

7. Find the natural number whose square is the expression $(0.05)^{\log_{\sqrt{20}}(0.\overline{1})}$.

8. Simplify

(i) $\frac{\log_5 250}{\log_{50} 5} - \frac{\log_5 10}{\log_{1250} 5}$

(ii) $\log_3 2 \cdot \log_8 7 \cdot \log_4 3 \cdot \log_7 6 \cdot \log_5 4$

(iii) $\log_3 \log_2 \log_{\sqrt{3}} 81$

(iv) Simplify $x^{\ln y - \ln z} \cdot y^{\ln z - \ln x} \cdot z^{\ln x - \ln y}$

9. Show that if $x > 0, y > 0$, then it follows from the equation $x^2 + 4y^2 = 12xy$ that

$\log(x + 2y) - 2\log 2 = \frac{1}{2}(\log x + \log y)$

10. Prove that $\log_{a+b} m + \log_{a-b} m = 2 \log_{a+b} m \log_{a-b} m$ if it is known that $m^2 = a^2 - b^2$.

11. Prove the identity $\log_{a/b} x = \frac{\log_a x \log_b x}{\log_b x - \log_a x}$

12. If x is equal to (1998) !, then find the sum of the series $\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \ldots\ldots + \frac{1}{\log_{1998} x}$.

13. If $\log_{18} 36 = a$ & $\log_{24} 72 = b$, then find the value of $4(a+b) - 5ab$.

14. Given, $\log_a x = \alpha$; $\log_b x = \beta$; $\log_c x = \gamma$ & $\log_d x = \delta$ $(x \neq 1)$, then find the value of $\log_{abcd} x$

15. Simplify

$b.a^{\frac{2}{\log_b a} + 1} - 2a^{\log_a b + 1} . b^{\log_b a + 1} + a.b^{\frac{2}{\log_a b} + 1}$.

16. Simplify

(i) $\frac{81^{1/\log_5 9} + 3^{3/\log_{\sqrt{6}} 3}}{409} \left((\sqrt{7})^{2/\log_{25} 7} - 125^{\log_{25} 6}\right)$

(ii) $\left(2^{\log_{\sqrt[4]{2}} a} - 3^{\log_{27}(a^2+1)^3} - 2a\right) / \left(7^{4\log_{49} a} - a - 1\right)$

(iii) $\frac{\log_a b + \log_a (b^{\frac{1}{2}\log_b a^2})}{\log_a b - \log_{ab} b} \cdot \frac{\log_{ab} b \log_a b}{b^{2\log_b \log_a b} - 1}$

(iv) $5^{\log_{1/5}\left(\frac{1}{2}\right)} + \log_{\sqrt{2}} \frac{4}{\sqrt{7}+\sqrt{3}} + \log_{1/2} \frac{1}{10+2\sqrt{21}}$

17. (i) Find the characteristics of the logarithms (base 10) of 23.8, 0.035

(ii) How many digits are there in the integral part of the numbers whose logarithms are 4.3010, 1.4771.

18. Given $\log 194.8445 = 2.2896883$, find the eleventh root of $(39.2)^2$.

19. Show that $\left(\frac{21}{20}\right)^{100}$ is greater than 100.

20. If $\log_{10} 2 = 0.301$ & $\log_{10} 3 = 0.477$, then find the number of digits in the number $N = 3^{12} \times 2^8$.

21. (i) Find the seventh root of 0.00324, having given that $\log 4409 = 3.6443$.

(ii) How many zeros are there between the decimal point and the first significant digit in $\left(\frac{1}{2}\right)^{1000}$.

www.pdfworld.net

www.pdfworld.net

1.22 EXPONENTIAL EQUATION

If we have an equation of the form $\mathbf{a^x = b}$ $(a > 0)$, then

(i) $x \in \phi$ if $b \le 0$
(ii) $x = \log_a b$ if $b > 0$ $a \ne 1$
(iii) $x \in \phi$ if $a = 1$, $b \ne 1$
(iv) $x \in R$ if $a = 1$, $b = 1$
(Since $1^x = 1 \Rightarrow 1 = 1$, $x \in R$)

Consider equations of the form $a^{f(x)} = a^{g(x)}$, where $a > 0$ and $a \ne 1$ and equations which can be reduced to them. The solution of such equations is based on the following result :

If $a > 0$ and $a \ne 1$, then the equation $a^{f(x)} = a^{g(x)}$ is equivalent to the equation $f(x) = g(x)$.

CAUTION A common mistake made by students is the incorrect use of the following statement: " if two exponential expressions are equal and if their bases are equal (different from 0 and ±1), then their exponents are equal as well." What is usually forgotten is the phrase "different from 0 and ±1". The result is a loss of roots, namely those for which the base is equal to 0 or ±1.

If we have an equation of the form $(a(x))^{f(x)} = (a(x))^{g(x)}$ called exponential-power equation, then

(i) $a(x) > 0$ and $a(x) \ne 1$, $f(x) = g(x)$, or
(ii) $a(x) = 1$, or
(iii) $a(x) = -1$, $(-1)^{f(x)} = (-1)^{g(x)}$, or
(iv) $a(x) = 0$, $f(x) > 0$, $g(x) > 0$.

Note that no value of x should lead to 0^0 form.

➤ **Example 1.** Solve the equation $2^{x^2 - 2x} = 2^{3x-6}$.

➤ ***Solution*** The given equation is equivalent to the equation $x^2 - 2x = 3x - 6$,
and therefore the roots of the last equation $x_1 = 2$ and $x_2 = 3$ are also roots of the original equation.

➤ **Example 2.** Solve $(18)^{8-4x} = (54\sqrt{2})^{3x-2}$.

➤ ***Solution*** Here $(3^2 . 2)^{8-4x} = (3^3 . 2\sqrt{2})^{3x-2}$

or $\{(3\sqrt{2})^2\}^{8-4x} = \{(3\sqrt{2})^3\}^{3x-2}$

or $(3\sqrt{2})^{16-8x} = (3\sqrt{2})^{9x-6}$

or $16 - 8x = 9x - 6$

$\therefore \quad x = \frac{22}{17}$

➤ **Example 3.** Solve the equation $\frac{0.2^{x-0.5}}{\sqrt{5}} = 5 \times 0.04^{x-1}$.

➤ ***Solution*** We reduce all the powers to the same base $\frac{1}{5}$.

$$\left(\frac{1}{5}\right)^{x-0.5} \times \left(\frac{1}{5}\right)^{0.5} = \left(\frac{1}{5}\right)^{-1} \times \left[\left(\frac{1}{5}\right)^2\right]^{x-1}$$

Further, we have : $\left(\frac{1}{5}\right)^x = \left(\frac{1}{5}\right)^{2x-3}$.

The last equation is equivalent to the equation $x = 2x - 3$, from where we find : $x = 3$.
Thus, $x = 3$ is the only root of the given equation.

➤ **Example 4.** Solve the equation
$6^{2x+4} = (3^{3x}) . (2^{x+8})$.

➤ ***Solution*** We rewrite the given equation as
$(3^{2x+4})(2^{2x+4}) = 3^{3x}(2^{x+8})$.

$$\Rightarrow \quad \frac{3^{2x+4}}{3^{3x}} = \frac{2^{x+8}}{2^{2x+4}} \qquad \Rightarrow 3^{4-x} = 2^{4-x}.$$

We reduce this equation to the form $\left(\frac{2}{3}\right)^{4-x} = 1$

and get $4 - x = 0 \Rightarrow x = 4$

➤ **Example 5.** Solve the equation $4^x + 2^{x+1} - 24 = 0$.

➤ ***Solution*** Let us apply the method of introducing new variables.
Since $4^x = (2^2)^x = (2^x)^2$ and $2^{x+1} = 2 \times 2^x$, the equation can be rewritten in the following way :

$(2^x)^2 + 2 \times 2^x - 24 = 0$.

Setting $u = 2^x$, we get the quadratic equation

$u^2 + 2u - 24 = 0$, whose roots are :

$u_1 = 4$ and $u_2 = -6$

$\Rightarrow \quad 2^x = 4$; $2^x = -6$.

From the first equation we get : $x = 2$. The second equation has no solutions since $2^x > 0$ for any values of x. Thus, the root of the equation is $x = 2$.

www.pdfworld.net

www.pdfworld.net

➤ **Example 6.** Solve the equation
$2^x + (0.5)^{2x-3} - 6(0.5)^x = 1.$

➤ ***Solution*** Since $(0.5)^{2x-3} = 2^{3-2x} = \frac{8}{2^{2x}}$

and $6(0.5)^x = \frac{6}{2^x}$,

we have $2^x + \frac{8}{2^{2x}} - \frac{6}{2^x} - 1 = 0$

Setting $u = 2^x$, we get : $u + \frac{8}{u^2} - \frac{6}{u} - 1 = 0$

$\Rightarrow \quad u^3 - u^2 - 6u + 8 = 0 \Rightarrow (u-2)(u^2 + u - 4) = 0.$
The last equation has three roots :

$$u_1 = 2, u_2 = \frac{-1+\sqrt{17}}{2}, u_3 = \frac{-1-\sqrt{17}}{2}$$

Now, $2^x = 2$; $2^x = \frac{\sqrt{17}-1}{2}$; $2^x = \frac{-1-\sqrt{17}}{2}$.

From the first equation we find : $x_1 = 1$, from the second:

$$-x_2 = \log_2 \frac{\sqrt{17}-1}{2}.$$

The third equation has no solution since $\frac{-1-\sqrt{17}}{2} < 0$ and $2^x > 0$ for $x \in R$.

Hence, the original equation has the following roots:

$$x_1 = 1 \text{ and } x_2 = \log_2 \frac{\sqrt{17}-1}{2}.$$

➤ **Example 7.** Solve the equation

$$4^{\sqrt{x^2-2}+x} - 5.2^{x-1+\sqrt{x^2-2}} = 6$$

➤ ***Solution*** Assuming $2^{\sqrt{x^2-2}+x} = y$,
we get a quadratic equation

$$y^2 - \frac{5}{2}y - 6 = 0$$

whose roots are $y_1 = 4$ and $y_2 = -3/2$.

$\Rightarrow \quad 2^{x+\sqrt{x^2-2}} = 4, \quad 2^{x+\sqrt{x^2-2}} = -3/2.$

The second equation has no solutions since $2^{x+\sqrt{x^2-2}} > 0$ for all values of x.

From the first equation we get $x + \sqrt{x^2-2} = 2$

Separating out the radical and squaring both sides of the equation, we obtain

$x^2 - 2 = 4 - 4x + x^2$

Collecting terms, we get the only root x = 3/2. Verification shows that this root satisfies the initial equation.

➤ **Example 8.** Solve the equation
$6 \times 3^{2x} - 13 \times 6^x + 6 \times 2^{2x} = 0.$

➤ ***Solution*** Since $6^x = 3^x \times 2^x$,
we have $6 \times 3^{2x} - 13 \times 3^x \times 2^x + 6 \times 2^{2x} = 0.$
Setting $u = 3^x$, $v = 2^x$, we get the equation

$6u^2 - 13uv + 6v^2 = 0,$

which is a homogeneous equation of the second degree in two variables u and v.

Since $v = 2^x$ does not vanish for any values of x, dividing both sides of the equation by v^2,

we get $6\left(\frac{u}{v}\right)^2 - 13\frac{u}{v} + 6 = 0.$

Setting $z = \frac{u}{v}$, we get $6z^2 - 13z + 6 = 0$

$\Rightarrow z_1 = \frac{2}{3}, z_2 = \frac{2}{3}.$

Taking into account that $z = \frac{u}{v} = \left(\frac{3}{2}\right)^x$, we have :

$$\left(\frac{3}{2}\right)^x = \frac{3}{2}; \quad \left(\frac{3}{2}\right)^x = \frac{2}{3}, \Rightarrow x_1 = 1, x_2 = -1.$$

Hence, the equation has two roots : $x_1 = x_2 = -1$.

➤ **Example 9.** Solve the equation $3^x \cdot 8^{\frac{x}{x+2}} = 6$

➤ ***Solution*** Some students solve this equation like this:

$$3^x \cdot 2^{\frac{3x}{x+2}} = 3^1 \cdot 2^1$$

www.pdfworld.net

www.pdfworld.net

CAUTION

They equate the exponents of the respective bases as:

$$x = 1, \frac{3x}{x+2} = 1$$

The "answer" is x = 1.

But this solution is incorrect in the sense that only one root of the equation is found and nothing has been said about any other roots. Actually, if the exponents on the appropriate bases are equal, then the products of these powers are equal, however the converse is simply incorrect.

For instance, the equation $3^1 \cdot 2^1 = 3^2 \cdot 2^{\log_2(2/3)}$ is valid, but $1 \neq 2$ and $1 \neq \log^2(2/3)$. Therefore, the foregoing reasoning may lead to a loss of roots, and this is exactly what occurred in the equation at hand. Taking logarithms of both members of the original equation to the base 3, we get

$$x + \frac{3x}{x+2}\log_3 2 = \log_3 6$$

or $x^2 + x(3\log_3 2 + 2 - \log_3 6) - 2\log_3 6 = 0$

We now have to solve this quadratic equation. This can be done using a familiar formula, but we have already seen, by trial and error, that $x_1 = 1$ is a root of the original equation and, consequently, satisfies the quadratic equation. Hence by product of roots the second root of the quadratic equation is $x_2 = -2\log_3 6$ and so the original equation has two roots:

$x_1 = 1$, $x_2 = -2\log_3 6$

Thus, it is useful to be able to guess a root, but never consider the guessing as the whole solution.

➤ **Example 10.** Solve the equation $1 + 3^{x/2} = 2^x$

➤ ***Solution*** We have $1 + 3^{x/2} = 2^x$

Dividing both sides by 2^x, then we obtain

$$\Rightarrow \quad \frac{1}{2^x} + \frac{3^{x/2}}{2^x} = 1$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^x + \left(\frac{3}{4}\right)^{x/2} = 1$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^x + \left(\frac{\sqrt{3}}{2}\right)^x = 1$$

$$\Rightarrow \quad (\cos \pi/3)^x + (\sin \pi/3)^x = 1$$

which is possible only when x = 2.

Hence x = 2 is only solution of the equation.

➤ **Example 11.** Solve the equation $7^{6-x} = x + 2$

➤ ***Solution*** By trial we find the root x = 5.

The equation has no other solutions since the function $f(x) = 7^{6-x}$ decreases and $g(x) = x + 2$ increases and consequently, the graphs of these functions cannot intersect more than once.

➤ **Example 12.** Solve the equation $x^{\sqrt{x}} = \sqrt{x^x}$

➤ ***Solution*** This equation may be rewritten as

$$x^{\sqrt{x}} = x^{x/2}$$

We see that the bases are equal.

So as not to lose any roots, let us see whether the base can be 0 or 1.

Since the expression 0^0 is meaningless, the number 0 is not an element in the domain set and therefore x = 0 is not a root of the equation. However, x = 1 is a root.

Now let us seek roots that are different from 0 and 1.

Equating the exponents, we obtain $\sqrt{x} = x/2$, whereby we find the second root of the equation, x = 4.

CAUTION

One sometimes hears the erroneous assertion : " If an exponential expression is 1, then the exponent is equal to zero." This is only true if the base is different from 1, but if the base is 1, then for any exponent the power will be 1.

➤ **Example 13.** Solve the equation

$$(x^2 + x - 57)^{3x^2+3} = (x^2 + x - 57)^{10x}. \qquad ...(1)$$

➤ ***Solution*** When solving the given exponential-power equation, we have to consider four cases :

Case (i) $x^2 + x - 57 = 1$, that is, $x^2 + x - 58 = 0$.

In this case equation (1) takes the form

$$1^{3x^2+3} = 1^{10x}, \text{ i.e. } 1 = 1.$$

Hence, the roots of the equation $x^2 + x - 58 = 0$ are also roots of equation (1).

From the equation $x^2 + x - 58 = 0$

we find $x_{1,2} = \frac{-1 \pm \sqrt{233}}{2}$.

www.pdfworld.net

www.pdfworld.net

Case (ii) $x^2 + x - 57 = -1$, that is, $x^2 + x - 56 = 0$. In this case equation (1) takes the form

$$(-1)^{3x^2+3} = (-1)^{10x}. \quad ...(2)$$

Equation (2) can be satisfied only by those values of x for which $3x^2 + 3$ and $10x$ are integers (since the negative number (–1) can be raised only to an integer power of equal parity (both even or both odd).

From the equation $x^2 + x - 56 = 0$ we find : $x_1 = -8$, $x^2 = 7$. The value $x_1 = -8$ does not satisfy equation (2), while the value $x_2 = 7$ satisfies this equation. Hence, x = 7 is a root of equation (1).

Case (iii) $x^2 + x - 57 = 0$. In this case equation (1) takes the form $0^{3x^2+3} = 0^{10x}$ ...(3)

Equation (3) can be satisfied only by those values of x for which $3x^2 + 3 > 0$ (this is true for all x's) and $10x > 0$; in this case equation (3) takes the form 0 = 0 (let us recall that the expression 0^r has sense only for $r > 0$).

From the equation $x^2 + x - 57 = 0$

we find $x_{1.2} = \dfrac{-1 \pm \sqrt{229}}{2}$

The value $x_1 = \dfrac{-1 - \sqrt{229}}{2}$ does not satisfy the condition $10x > 0$, while $x_2 = \dfrac{-1 - \sqrt{229}}{2}$ does.

Hence, $x = \dfrac{-1 + \sqrt{229}}{2}$ is a root of equation (1).

Case (iv) If $x^2 + x - 57 > 0$ and $x^2 + x - 57 \neq 1$, then from equation (1) we conclude that

$3x^2 + 3 = 10x$, we find : $x_1 = 3$, $x_2 = \dfrac{1}{3}$. Both of these values must be checked by substituting into equation (1). For x = 3 we get : $(-45)^{30} = (-45)^{30}$ which is a true equality.

Summing up, we conclude that equation (1) has five roots:

$$x_{1.2} = \frac{-1 \pm \sqrt{233}}{2}, \; x_3 = 7, \; x_4 = \frac{-1 \pm \sqrt{229}}{2},$$

$$x_5 = 3, \; x_6 = \frac{1}{3}.$$

Practice Problems [Q]

Solve the following equations :

1. $2^{2x-3} = 4^{x^2-3x-1}$
2. $0.6^x \cdot \left(\dfrac{25}{9}\right)^{x^2-12} = \left(\dfrac{27}{125}\right)^3$
3. $3\sqrt[x]{81} - 10\sqrt[x]{9} + 3 = 0$
4. $7^{3x} + 9.5^{2x} = 5^{2x} + 9.7^{3x}$
5. $4^{x+1.5} + 9^x = 6^{x+1}$
6. $2^{2x^2} + 2^{x^2+2x+2} = 2^{5+4x}$
7. $4^{\sqrt{3x^2-2x+1}} + 2 = 9.2^{\sqrt{3x^2-2x}}$
8. $\left(\sqrt{5+\sqrt{24}}\right)^x + \left(\sqrt{5-\sqrt{24}}\right)^x = 10$
9. $\sqrt{2^x \sqrt[3]{4^x \times 0.125^{\frac{1}{x}}}} = 4\sqrt[3]{2}$
10. $9^{x^2-1} - 36 \times 3^{x^2-3} + 3 = 0$
11. $3^{3x+1} - 4 \times 27^{x-1} + 9^{1.5x-1} - 80 = 0$
12. $(2+\sqrt{3})^x + (2-\sqrt{3})^x = 4$
13. $(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \dfrac{4}{2-\sqrt{3}}$
14. $5^{x-2} \times 2^{\frac{3x}{x+1}} = 4$
15. $(x^2 - x - 1)^{x^2-1} = 1$
16. $(x-2)^{x^2-x} = (x-2)^{12}$
17. $4^x + (x-1)\,2^x = 6 - 2x$

1.23 EXPONENTIAL INEQUALITIES

The elementary exponential inequalities are inequalities of the form

$$a^x > b, \; a^x < b, \quad (1)$$

where a and b are certain numbers ($a > 0$, $a \neq 1$).

www.pdfworld.net

www.pdfworld.net

Depending on the values of the parameters a and b, the set of solutions of the inequality $a^x > b$ can be in the following forms :

1. $x \in (\log_a b, \infty)$ for $a > 1, b > 0$;
2. $x \in (-\infty, \log_a b)$ for $0 < a < 1, b > 0$;
3. $x \in R$ for $a > 0, b < 0$;

Depending on the values of a and b, the set of solutions of the inequality $a^x < b$ can be in the following forms :

1. $x \in (-\infty, \log_a b)$ for $a > 1, b > 0$;
2. $x \in (\log_a b, \infty)$ for for $0 < a < 1, b > 0$;
3. $x \in \phi$ for $a > 0, b < 0$ (i.e. the inequality has no solutions)

➤ **Example 1.** Solve the inequality $\sqrt[3]{2^{\frac{3x-1}{x-1}}} < 8^{\frac{x-3}{3x-7}}$

➤ ***Solution*** We transform the inequality

$$\sqrt[3]{2^{\frac{3x-1}{x-1}}} < 8^{\frac{x-3}{3x-7}} \qquad ...(1)$$

to $2^{\frac{3x-1}{3(x-1)}} < 2^{\frac{3(x-3)}{3x-7}}$

This is equivalent to the inequality

$$\frac{3x-1}{3(x-1)} < \frac{3(x-3)}{3x-7} \qquad ...(2)$$

$$\Rightarrow \quad \frac{3x-1}{3x-3} - \frac{3x-9}{3x-7} < 0$$

$$\Rightarrow \quad \frac{12x-20}{(3x-3)(3x-7)} < 0$$

$$\Rightarrow \quad \frac{x-\frac{5}{3}}{(x-1)\left(x-\frac{7}{3}\right)} < 0$$

Solving the last inequality by the method of intervals, we get $(-\infty, 1) \cup \left(\frac{5}{3}, \frac{7}{5}\right)$ which is the solution of inequality (1)

➤ **Example 2.** Solve the inequality

$$2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}.$$

➤ ***Solution*** We get $2^{x+2}(1-2-2^2) > 5^{x+2}(5^{-1}-1)$,

$$2^{x+2}(-5) > 5^{x+2}\left(-\frac{4}{5}\right),$$

$$\frac{2^{x+2}}{5^{x+2}} < \frac{4}{25} \text{ or } \left(\frac{2}{5}\right)^{x+2} < \left(\frac{2}{5}\right)^2.$$

The last inequality is equivalent to the inequality $x+2>2$, whence we find $(0, \infty)$ which is the solution.

➤ **Example 3.** Solve the inequality

$$\left(\frac{1}{2}\right)^{(x^6-2x^3+1)^{1/2}} < \left(\frac{1}{2}\right)^{1-x}$$

➤ ***Solution*** If we disregard the exponents, we can say that this is an elementary exponential inequality with base less than unity : $(1/2)^a < (1/2)^b$. Solving it, we find that the original inequality is equivalent to the inequality $(x^6 - 2x^3 + 1)^{1/2} > 1 - x$.

Since $(x^6 - 2x + 1)^{1/2} = \sqrt{(x^3-1)^2} = |x^3 - 1|$, it follows that we have yet to solve the inequality

$|x^3 - 1| > 1 - x$

Since the left member here is nonnegative, it is automatically satisfied for $1 - x < 0$, that is, when $x > 1$. We now consider $x \le 1$. In this case, $x^3 \le 1$, and so $|x^3 - 1| = 1 - x^3$ and we have the inequality $1 - x^3 > 1 - x$ or $x(x-1)(x+1) < 0$.

Solving this inequality by the method of intervals, we find that it is true for $x < -1$ and for x located in the domain $x \le 1$ under consideration and so are solutions of the original inequality.

Thus, the original inequality is valid for $x < -1, 0 < x < 1, x > 1$.

➤ **Example 4.** Solve the inequation

$$4^x \le 3.2^{x+\sqrt{x}} + 4^{1+\sqrt{x}}$$

➤ ***Solution*** The inequation is defined if $x \ge 0$

writing the inequation as $2^{2x} \le 3.2^{x+\sqrt{x}} + 4.2^{2\sqrt{x}}$

and dividing by 2^{2x} we get $1 \le 3.2^{\sqrt{x}-x} + 4.2^{2(\sqrt{x}-x)}$

On putting $t = 2^{\sqrt{x}-x}$ we get $4t^2 + 3t - 1 \ge 0$

www.pdfworld.net

www.pdfworld.net

$\Rightarrow$ $(4t - 1)(t + 1) \geq 0$

$\Rightarrow$ $t \geq \frac{1}{4}$ $(\because\ t + 1 > 0)$

$\Rightarrow$ $2^{\sqrt{2}-2} \geq 2^{-2}$

$\Rightarrow$ $\sqrt{x} - x \geq -2$ $(\because\ 2^{\alpha}$ increases as α increases)

On putting $\sqrt{x} = u$, we get $u^2 - u - 2 \leq 0$

$\Rightarrow$ $(u - 2)(u + 1) \leq 0$

$\Rightarrow$ $u \leq 2$ $(\because u = \sqrt{x} \geq 0)$

$\Rightarrow$ $\sqrt{x} \leq 2 \Rightarrow x \leq 4$

$\Rightarrow$ Solution set is $[0, 4]$.

Practice Problems [R]

Solve the following inequations :

1. $0.1^{4x^2-2x-2} \leq 0.1^{2x-3}$
2. $x^2 . 5^x - 5^{2+x} < 0.$
3. $\left(\frac{1}{3}\right)^{\frac{|x|+1}{x-2}} > 9$
4. $4^x + 2^{x+1} - 6 \leq 0$
5. $\left(\frac{1}{3}\right)^{\sqrt{x+4}} > \left(\frac{1}{3}\right)^{\sqrt{x^2++3x+4}}$
6. $\frac{2^{1-x} - 2^x + 1}{2^x - 1} \leq 0$
7. $9^{\sqrt{x^2-3}} + 3 < 3^{\sqrt{x^2-3}-1} . 28$
8. $\sqrt{9^x + 3^x - 2} \geq 9 - 3^x$
9. $3^{\sqrt{x}} > 2^a$
10. $2^x + 2^{|x|} \geq 2\sqrt{2}$
11. $|x|^{x^2-x-2} < 1$

1.24 LOGARITHMIC EQUATION

1. Consider logarithmic equations of the form $\log_a f(x) = \log_a g(x)$, where $a > 0$ and $a \neq 1$. ...(1)

The equation $\log_a f(x) = \log_a g(x)$ is equivalent to the system :

$$\begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0 \end{cases} \quad ...(2)$$

Note that for solving equation (1) we have not necessarily to solve system (2).

We may proceed in a different way, namely, to solve the equation $f(x) = g(x)$ and from the found solutions to choose those which satisfy the system of inequalities

$\begin{cases} f(x) > 0 \\ g(x) > 0, \end{cases}$ that is, those which belong to the domain of equation (1).

2. Now, consider equations of the form $\log_{a(x)} f(x) = \log_{a(x)} g(x)$.

It is equivalent to the system :

$$\begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0 \\ a(x) > 0 \\ a(x) \neq 1 \end{cases}$$

In other words, the roots of equation are represented by those and only those roots of the equation $f(x) = g(x)$ which simultaneously satisfy the conditions :

$f(x) > 0$, $g(x) > 0$, $a(x) > 0$, $a(x) \neq 1$

(these conditions specify the domain of equation).

If the given equation includes logarithms to different bases, it is first necessary to reduce all the logarithms to the same base.

3. When solving logarithmic equations, we use the properties of logarithms. Consider, for instance, the equation

$\log_a f(x) + \log_a g(x) = \log_a h(x)$. ...(1)

It is transformed to :

$\log_a (f(x)\, g(x)) = \log_a h(x)$. ...(2)

but equations (1) and (2) may be non-equivalent.

Indeed, the domain of the expression $\log_a f(x) + \log_a g(x)$

www.pdfworld.net

www.pdfworld.net

is given by the system of inequalities $\begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases}$, whereas the domain of the expression $\log_a f(x)\ g(x))$ is specified by the inequality $f(x)\ g(x) > 0$ which is, in turn, equivalent to the collection of systems of inequalities :

$$\begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases}; \qquad \begin{cases} f(x) < 0 \\ g(x) < 0 \end{cases}$$

Thus, when passing from equation (1) to equation (2), we can encounter an extension of the domain of equation (1) (at the expense of the solutions of the last system of inequalities), and, hence, extraneous roots may appear. Therefore, on solving equation (2), we have to choose those of its roots which belong to the domain of the original equation (1), that is, which satisfy the system of inequalities $\begin{cases} f(x) > 0 \\ g(x) > 0 \\ h(x) > 0 \end{cases}$. This check is an essential part of the solution of a logarithmic equation.

It is clear that the check may also be realized by a direct substitution of the found solutions into the original equation.

Now let us summarize the above equations and some more forms.

	Original form	***Equivalent form***	***Conditions to be applied***
1.	$\log_a f(x) = b$	$f(x) = a^b$	$a > 0, \neq 1$
2.	$\log_{g(x)} f(x) = b$	$f(x) = g(x)^b$	$g(x) > 0, \neq 1$
3.	$\log_{f_1(x)} \{\log_{f_2(x)} f(x)\} = 0$	$f_1(x) = f_2(x)$	$f_1(x) > 0, f_1(x) \neq 1\ f_2(x) > 0, f_2(x) \neq 1$
4.	$\log_a f_1(x) = \log_a f_2(x)$ $(a > 0, \neq 1)$	$f_1(x) = f_2(x)$	$f_1(x) > 0$ or $f_2(x) > 0$ (whichever is easier to solve)
5.	$\log_{f_1(x)} A = \log_{f_2(x)} A$ $(A > 0)$	$f_1(x) = f_2(x)$	$f_1(x) > 0 \neq 1$ or $f_2(x) > 0, \neq 1$ (whichever is easier to solve)
6.	$\log_{f(x)} g_1(x) = \log_{f(x)} g_2(x)$	$g_1(x) = g_2(x)$	(i) $g_1(x) > 0$ (ii) $f(x) > 0, \neq 1$ or (i) $g_2(x) > 0$ (ii) $f(x) > 0, \neq 1$ (whichever system of inequalities is easier to solve)
7.	$\log_{g_1(x)} f(x) = \log_{g_2(x)} f(x)$	$g_1(x) = g_2(x)$	(i) $g_1(x) > 0, \neq 1$ (ii) $f(x) > 0$ or (i) $g_2(x) > 0, \neq 1$ (ii) $f(x) > 0$ (whichever system of inequalities is easier to solve)

pdfworld.net

www.pdfworld.net

www.pdfworld.net

8.	$2n \log_a f_1(x) = \log_a f_2(x)$ $(a > 0, \neq 1, n \in N)$	$f_1^{2n}(x) = f_2(x)$	$f(x) > 0$
9.	$(2n+1) \log_a f_1(x) = \log_a f_2(x)$ $(a > 0, \neq 1, n \in N)$	$f_1^{2n+1}(x) = f_2(x)$	$f(x) > 0$
10.	$\log_a f_1(x) + \log_a f_2(x) = \log_a f_3(x)$	$f_1(x).f_2(x) = f_3(x)$	$f_1(x) > 0$, $f_2(x) > 0$

➤ **Example 1.** Solve the equation
$\log_3 (x^2 - 3x - 5) = \log_3(7 - 2x)$.

➤ ***Solution*** The equation is equivalent to the following mixed system :

$$\begin{cases} x^2 - 3x - 5 = 7 - 2x \\ x^2 - 3x - 5 > 0 \\ 7 - 2x > 0 \end{cases}$$

Solving the equation of this system, we get : $x_1 = 4, x_2 = -3$.
Of these two values only $x = -3$ satisfies both inequalities of system
(that is, the value $x = 4$ does not belong to the domain of equation).
Therefore, $x = -3$ is the solution.

➤ **Example 2.** Solve the equation
$\log(x + 4) + \log (2x + 3) = \log(1 - 2x)$.

➤ ***Solution*** We transform the equation to the form
$\log ((x + 4) (2x + 3)) = \log (1 - 2x)$,
and further $(x + 4) (2x + 3) = 1 - 2x$
$\Rightarrow \quad x_1 = -1, x_2 = -5.5$
The domain is given by the system of inequalities :

$$\begin{cases} x + 4 > 0 \\ 2x + 3 > 0 \\ 1 - 2x > 0 \end{cases}$$

Substituting the found roots of the equation into the system , we make sure that $x_1 = -1$ satisfies this system, while $x_2 = -5.5$ does not. Thus, $x = -1$ is the only root of the equation.

➤ **Example 3.** Solve the equation

$$\log_5 (5^{\frac{1}{x}} + 125) = \log_5 6 + 1 + \frac{1}{2x}.$$

➤ ***Solution*** Let us first consider the given equation as a logarithmic one. Since $1 + \frac{1}{2x} = \log_5 5^{1+\frac{1}{2x}}$, we write equation in the form :

$$\log_5 (5^{\frac{1}{x}} + 125) = \log_5 6 + \log_5 5^{1+\frac{1}{2x}}.$$

Further, we have :

$$\log_5 (5^{\frac{1}{x}} + 125) = \log_5 (6 \times 5 \times 5^{\frac{1}{2x}})$$

$$\Rightarrow \quad 5^{\frac{1}{x}} + 125 = 30 \times 5^{\frac{1}{2x}}.$$

We have obtained an exponential equation which can be solved by introducing a new variable. Setting $u = 5^{\frac{1}{2x}}$, we get the equation $u^2 - 30u + 125 = 0$, whose roots are : $u_1 = 5, u_2 = 25$.

$$\Rightarrow \quad 5^{\frac{1}{2x}} = 5; \quad 5^{\frac{1}{2x}} = 25 \cdot$$

From the first equation we get : $\frac{1}{2x} = 1 \Rightarrow x_1 = \frac{1}{2}$.

From the second equation we get:

$$\frac{1}{2x} = 2 \Rightarrow x_2 = \frac{1}{4}.$$

Thus, the equation has two roots : $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{4}$.

➤ **Example 4.** Solve the equation

$$\log \sqrt{x - 1} + \frac{1}{2} \log(2x + 15) = 1$$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** The set of permissible values of the unknown x can be found as a solution of the system $x-1>0, 2x+15>0$ and is an interval $(1, \infty)$. Using the properties of logarithms, we reduce the given logarithmic equation to the form.

$$\log(\sqrt{(x-1)}\ .\ \sqrt{2x+15}\,)=1$$

$$\Rightarrow \quad \sqrt{(x-1)(2x+15)}=10$$

whose solutions are $x_1 = 5$, $x_2 = -23/2$. The set of permissible values of x of the initial equation includes only the root $x_1 = 5$ which is indeed a solution of the initial equation.

➤ **Example 5.** Solve the equation

$$2 - x + 3\log_5 2 = \log_5(3^x - 5^{2-x}),$$

➤ ***Solution*** Let us transfer the logarithm appearing on the left hand side to the right hand side and making use of the properties of logarithms,

$$2 - x = \log_5\left(\frac{3^x - 5^{2-x}}{8}\right)$$

$$\Rightarrow \quad \left(\frac{3^x - 5^{2-x}}{8}\right) = 5^{2-x}$$

which can be written in the form

$$3^x = 9.5^{2-x} \text{ or } 3^{x-2} = 5^{2-x} \text{ or } 15^{x-2} = 1$$

$\Rightarrow$ $x - 2 = 0$ whose solution is $x = 2$.

The set of permissible values of x of the given equation can be found as the solution of the inequality

$$3^x - 5^{2-x} > 0$$

For $x = 2$ the given equation holds true and consequently, $x = 2$ is a solution of the initial logarithmic equation.

➤ **Example 6.** Suppose that x and y are positive numbers for which $\log_9 x = \log_{12} y = \log_{16}(x + y)$. If the value of $\frac{y}{x} = 2\cos\theta$, where $\theta \in (0, \pi/2)$ find θ.

➤ ***Solution*** Given $\log_9 x = \log_{12} y = \log_{16}(x + y) = k$ (say)

$$x = 9^k\ ;\ y = 12^k\ ;\ x + y = 16^k$$

we have to find $\frac{y}{x} = \left(\frac{4}{3}\right)^k$

now $9^k + 12^k = 16^k$

divide by 9^k, $\quad 1 + \left(\frac{4}{3}\right)^k = \left(\frac{16}{9}\right)^k = \left(\frac{4}{3}\right)^{2k}$;

put $\left(\frac{4}{3}\right)^k = z = \frac{y}{x} > 0$

$$\therefore \quad z^2 - z - 1 = 0 \qquad \Rightarrow \qquad z = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad z = \frac{\sqrt{5}+1}{2} \text{ (as } \frac{1-\sqrt{5}}{2} \text{ is rejected);}$$

$$\therefore \quad \left(\frac{4}{3}\right)^k = \frac{y}{x} = \frac{\sqrt{5}+1}{2}\ ;$$

$$\therefore \quad \frac{y}{x} = \left(\frac{\sqrt{5}+1}{2}\right)\cdot 2 = 2\cos\frac{\pi}{5}.$$

Hence $\quad \theta = \pi/5$

➤ **Example 7.** Solve the equation

$$x^{0.5\log_{\sqrt{x}}(x^2-x)} = 3^{\log_9 4}.$$

➤ ***Solution*** $x^{\frac{1}{2}\log_{(l/2)\log x}(x^2-x)} = 3^{\log_3 2}$

$$\Rightarrow \quad x^{\log_x(x^2-x)} = 2 \qquad \Rightarrow \qquad x^2 - x = 2$$
$$\Rightarrow \quad x^2 - x - 2 = 0 \qquad \Rightarrow \qquad (x-2)(x+1) = 0$$
$$\Rightarrow \quad x = 2 \text{ or } \quad x = -1 \text{ (rejected)}$$
$$\therefore \quad x = 2$$

➤ **Example 8.** Solve the equation

$$x + \log_{10}(1 + 2^x) = x\log_{10}5 + \log_{10}6$$

➤ ***Solution*** Transposing $x\log_{10}5$ to the left side of the equation and utilizing the laws of logarithms, we get

$$x + \log_{10}(1+2^x) - x\log_{10}5$$
$$= x\log_{10}10 - x\log_{10}5 + \log_{10}(1 + 2^x)$$
$$= \log_{10}2x(1 + 2^x)$$

The equation can thus be rewritten as

www.pdfworld.net

www.pdfworld.net

$\log_{10} 2x(1+2x) = \log_{10} 6$,

$\Rightarrow \quad (2^x)^2 + 2^x - 6 = 0$

Denoting $z = 2^x$, we arrive at the quadratic equation

$z^2 + z - 6 = 0$

which has the roots $z_1 = -3$, $z_2 = 2$.

Since the equation $2^x = -3$ is impossible (because 2^x is positive for all values of x), it remains to solve the equation $2^x = 2$. It has the root $x = 1$, which is the only root of the equation.

➤ **Example 9.** Solve the equation

$$\log_x (3x^{\log_5 x} + 4) = 2 \log_5 x.$$

➤ ***Solution*** Using the definition of logarithm, we transform the equation to: $x^{2\log_5 x} = 3x^{\log_5 x} + 4$

Setting $u = x^{\log_5 x} > 0$, we get the equation

$u^2 - 3u - 4 = 0$, whose roots are : $u_1 = -1$, $u_2 = 4$.

Now, the problem is reduced to solving the following collection of equations:

$$x^{\log_5 x} = -1; \; x^{\log_5 x} = 4.$$

Since $x^{\log_5 x} > 0$, and $-1 < 0$, the first equation of this collection has no solution. Taking the logarithms to the base 5 of both sides of the second equation, we get :

$$\log_5^2 x = \log_5 4, \text{ i.e. } \log_5 x = \pm\sqrt{\log_5 4},$$

whence we find : $x_{1,2} = 5^{\pm\sqrt{\log_5 4}}$

➤ **Example 10.** Find $x \in R$ satisfying the equation

$$|x-1|^{\log_3 x^2 - 2\log_x 9} = (x-1)^7$$

➤ ***Solution*** If $x > 1$, $(x-1)^{\log_3 x^2 - 2\log_x 9} = (x-1)^7$

and $x - 1 \neq 1$, i.e., $x \neq 2$

we have $\log_3 x^2 - 2\log_x 9 = 7$

or $2\log_3 x - 4\log_x 3 = 7$

or $2\log_3 x - 4.\dfrac{1}{\log_3 x} = 7$

or $(2\log_3 x)^2 - 7(\log_3 x) - 4 = 0$

or $(2\log_3 x + 1)(\log_3 x - 4) = 0$

or $\log_3 x = \dfrac{-1}{2}, 4$

$\Rightarrow \quad x = 3^{-1/2}, \; 3^4 = \dfrac{1}{\sqrt{3}}, 81$

But $\dfrac{1}{\sqrt{3}} > 1$; so $x = 81$

If $x = 2$, the equation is satisfied.

If $0 < x < 1$, $(1-x)^{\log_3 x^2 - 2\log_x 9} = -(1-x)^7$.

Here LHS is positive but the RHS is negative. So they cannot be equal. So, there is no solution in the interval (0, 1).

Clearly, $x \neq 1$ for log to be defined.

Hence, the solutions are $x = 81, 2$

➤ **Example 11.** Solve the equation

$$\log_{10}\sqrt{1+x} + 3\log_{10}\sqrt{1-x} = \log_{10}\sqrt{1-x^2} + 2$$

➤ ***Solution*** $\log_{10}\sqrt{1+x} + 3\log_{10}\sqrt{1-x}$

$= \log_{10}\sqrt{1-x} + \log_{10}\sqrt{1+x} + 2$

It is easy to see that this manipulation does not change the domain of the variable.

Eliminating $\log_{10}\sqrt{1+x}$ from both sides, we obtain the equation $2\log_{10}\sqrt{1-x} = 2$

whose domain consists of the numbers $x < 1$, which, as is evident, is greater than that of the original equation. We should thus expect extraneous roots. Solving the last equation we get the root $x = -99$, which does not lie in the domain of the original equation and therefore is not its root. Thus, the given equation does not have any roots.

One of the sources of mistakes is the clearing of fractions. But this causes an extension of the domain of the variable: those values of x are included for which the denominator is equal to 0.

Mistakes that occur in solving equations in which the left member is a fraction and the right member is zero are due to this disregard for the domain of the

www.pdfworld.net

www.pdfworld.net

variable. Frequently, the student simply discards the denominator in such cases and equates the numerator to zero. For a correct solution, one should equate the numerator to zero, find the roots of the resulting equation and discard those for which the denominator vanishes.

➤ **Example 12.** Solve the equation

$$\frac{1}{\log_6(3+x)}+\frac{2\log_{0.25}(4-x)}{\log_2(3+x)}=1$$

➤ ***Solution*** Taking all logarithms to the base 2 and manipulating, we get the equivalent equation

$$\frac{\log_2 6-\log_2(4-x)}{\log_2(3+x)}=1 \qquad (1)$$

$$\Rightarrow \quad \log_2 6-\log_2(4-x)=\log_2(3+x)$$

$$\Rightarrow \quad \frac{6}{4-x}=3+x \qquad (2)$$

This last equation is reduced to a quadratic equation and its roots are found to be : $x_1 = 3$, $x_2 = -2$

During the solution process, extraneous roots could have appeared only due to an extension of the domain of the variable because of clearing of fractions in equations (1) and (2). It is therefore sufficient to test the resulting roots for membership in the domain of the variable of the original equation. We thus find that x_2 does not lie in the domain, but x_1 does and hence, it is a root of the original equation.

➤ **Example 13.** Solve the equation

$$\log_{0.5x}x^2-14\log_{16x}x^3+40\log_{4x}\sqrt{x}=0$$

➤ ***Solution*** Taking advantage of the change of base rule and taking x as the new logarithmic base, we have

$$\frac{\log_x x^2}{\log_x 0.5x}-\frac{14\log_x x^3}{\log_x 16x}+\frac{40\log_x\sqrt{x}}{\log_x 4x}=0$$

CAUTION But it is quite evident that the new equation is devoid of meaning for x = 1, whereas the original equation is not only meaningful for x = 1 but has unity as its root. This is precisely where most students lose the root x = 1.

We must therefore reason as follows; we want to pass to the base x; to do this we must be sure that $x > 0$ and $x \neq 1$. Since all the x of our domain are positive, the first condition $x > 0$ is satisfied; on the other hand, unity lies in the domain and substitution shows that x = 1 is a root. thus, one root of the original equation has been found x = 1. Now let us seek roots that differ from unity. Then we can pass to the base x without losing roots.

Using the properties of logarithms and denoting $\log_x 2$ by y we have

$$\frac{2}{1-y}-\frac{42}{1+4y}+\frac{20}{1+2y}=0$$

This equation is reduced to the quadratic $2y^2 + 3y - 2 = 0$, whose roots are $y_1 = 1/2$, $y_2 = -2$. Then we get $\log_x 2 = 1/2 \Rightarrow x = 4$ and $\log_x 2 = -2 \Rightarrow x = 1/\sqrt{2}$.

Both of these values, 4 and $1/\sqrt{2}$, are roots of the original equation.

Hence, the original equation has three roots.

CAUTION A common and very grave mistake that results in a loss of roots is the cancelling of a common factor from both sides of an equation. It is clear that in the process, roots may be lost which make the common factor vanish.

In such cases it is best to transpose all terms to the left side, take out the common factor and consider two cases : (1) the common factor is equal to zero; (2) the common factor is not equal to zero; then of course the expression in the bracket is zero. It is also possible to consider, first the case when the common factor is equal to zero and, then cancel the common factor.

➤ **Example 14.** Solve the equation

$$\log_{3x+4}(4x^2+4x+1)+\log_{2x+1}(6x^2+11x+4)=4$$

➤ ***Solution*** $\log_{3x+4}(2x+1)+\log_{2x+1}(6x^2+11x+4)$

$$=2\log_{3x+4}(2x+1)+\log_{2x+1}(3x+4)(2x+1)$$

$$=2\log_{3x+4}(2x+1)+1+\log_{2x+1}(3x+4)$$

www.pdfworld.net

www.pdfworld.net

and hence putting $t = \log_{3x+4}(2x+1)$,

the given equation can be written as $2t + \frac{1}{t} = 3$.

Thus $t = \log_{3x+4}(2x+1) = 1$ or $\frac{1}{2}$.

Clearly, $\log_{3x+4}(2x+1) = 1$ has no solution

and hence $\log_{3x+4}(2x+1) = \frac{1}{2}$ which yields

$$(2x+1)^2 = 3x+4 \Rightarrow x = -1 \text{ or } x = \frac{3}{4}$$

obviously, $x = -1$ is inadmissible. Thus $x = \frac{3}{4}$ is the only solution.

Practice Problems [S]

Solve the following equations :

1. $\log_5[2 + \log_3(3+x)] = 0$

2. $\log(5-x) - \frac{1}{3}\log(35 - x^3) = 0$

3. $\log_3(3^x - 8) = 2 - x$

4. $\log_{\sqrt{5}}(4^x - 6) - \log_{\sqrt{5}}(2^x - 2) = 2$

5. $\log(3x^2 + 12x + 19) - \log(3x+4) = 1$

6. $\log_{x+1}(x^2 + x - 6)^2 = 4$

7. $2\log_2\log_2 x + \log_{1/2}\log_2(2\sqrt{2}\,x) = 1$

8. $\frac{1}{10}\log\sqrt[3]{x^2 - 4x + 4} - \frac{1}{2}\log x - \log\frac{1}{\sqrt{x}} = 0$

9. $|1 - \log_{1/5} x| + 2 = |3 - \log_{1/5} x|$

10. $x^{\log x + 5} = 10^{15 + 3\log x}$

11. $\frac{x^{\frac{1}{2}\log_2 x}}{4} = 2^{\frac{\log_2^2 x}{4}}$

12. $9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$

13. $4^{\log_9 x} - 6.2^{\log_9 x} + 2^{\log_3 27} = 0$

14. $\log_{10}^2 x + \log_{10} x^2 = \log_{10}^2 2 - 1$

15. $5^{\log_{10} x} = 50 - x^{\log_{10} 5}$

16. $|x-1|^{(\ell n|x|)^2 - \ell n x^2} = |x-1|^3$.

17. $\log_{x^3+2x^2-3x+5}(x^3 + 3x^2 + 2x - 1)$

$= \log_{2x} x + \log_{2x} 2.$

18. $\log_{(3-4x^2)}(9 - 16x^4) = 2 + \frac{1}{\log_2(3-4x^2)}$

19. $20\log_{4x}\sqrt{x} + 7\log_{16x} x^3 - 3\log_{x/2} x^2 = 0$

20. $(x+1)\log_3^2 x + 4x\log_3 x - 16 = 0$

1.25 LOGARITHMIC INEQUALITIES

Consider the following elementary logarithmic inequalities :

(i) If $a > 1$, then the inequalities $0 < x_1 < x_2$ and $\log_a x_1 < \log_a x_2$ are equivalent

(ii) If $0 < a < 1$, then the inequalities $0 < x_1 < x_2$ and $\log_a x_1 > \log_a x_2$ are equivalent.

(iii) If $a > 1$, then the inequalities $\log_a x < k$ and $0 < x < a^k$

(iv) If $a > 1$, then the inequalities $\log_a x > k$ and $x > a^k$ are equivalent.

(v) If $0 < a < 1$, then the inequalities $\log_a x < k$ and $x > a^k$ are equivalent.

(vi) If $0 < a < 1$, then the inequalities $\log_a x > k$ and $0 < x < a^k$ are equivalent.

An inequality of the form

$\log_a f(x) > b$

is equivalent to the following systems of inequalities :

www.pdfworld.net

www.pdfworld.net

(i) $f(x) > 0, f(x) > a^b$ for $a > 1$

(ii) $f(x) > 0, f(x) < a^b$ for $0 < a < 1$,

and an inequality of the form

$\log_a f(x) < b$

is equivalent to the following systems of inequalities:

(i) $f(x) > 0, f(x) < a^b$ for $a > 1$;

(ii) $f(x) > 0, f(x) > a^b$ for $0 < a < 1$.

Now consider some more forms of logarithmic inequations given below:

	Original Form	*Equivalent Form*	
1.	(i) $\log_a f(x) \geq 0$	$f(x) \geq 1$ when $a > 1$	$0 < f(x) \leq 1$ when $0 < a < 1$
	(ii) $\log_a f(x) \leq 0$	$0 < f(x) \leq 1$ when $a > 1$	$f(x) \geq 1$ when $0 < a < 1$
	(iii) $\log_a f(x) \geq b$	$f(x) \geq a^b$ when $a > 1$	$0 < f(x) \leq 1$ when $0 < a < 1$
	(iv) $\log_a f(x) \leq b$	$0 < f(x) \leq a^b$ when $a > 1$	$f(x) \geq a^b$ when $0 < a < 1$
2.	(i) $\log_{g(x)} f(x) \geq 0$	$f(x) \geq 1$, $g(x) > 1$	$0 < f(x) \leq 1$, $0 < g(x) < 1$
	(ii) $\log_{g(x)} f(x) \leq 0$	$f(x) \geq 1$, $0 < g(x) < 1$	$0 < f(x) \leq 1$, $g(x) > 1$
	(iii) $\log_{g(x)} f(x) > b$	$f(x) > 0$, $g(x) > 1$, $f(x) > [g(x)]^b$	$f(x) > 0$, $0 < g(x) < 1$, $f(x) < [g(x)]^b$.
3.	(i) $\log_{g(x)} f_1(x) \geq \log_{g(x)} f_2(x)$	$f_1(x) \geq f_2(x)$, $f_1(x) > 0$, $g(x) > 1$	$f_1(x) \leq f_2(x)$, $f_1(x) > 0$, $0 < g(x) < 1$
	(ii) $\log_{g(x)} f_1(x) \leq \log_{g(x)} f_2(x)$	$f_1(x) \leq f_2(x)$, $f_1(x) > 0$, $g(x) > 1$	$f_1(x) \geq f_2(x)$, $f_2(x) > 0$, $0 < g(x) < 1$
4.	$\log_a f_1(x) \geq \log_a f_2(x)$	$f_1(x) \geq f_2(x)$, $f_2(x) > 0$, $a > 0$	$f_1(x) \leq f_2(x)$, $f_1(x) < 0$, $0 < a < 1$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

➤ **Example 1.** Prove that $\log_{\log_3 2}\frac{1}{2} > 0$

➤ ***Solution*** Since $0 < \log_3 2 < 1$ and $1/2 < 1$, it follows that the inequality is valid.

➤ **Example 2.** What can be said about the number x if it is known that for ever real $a \neq 0$, $\log_x(a^2+1) < 0$?

➤ ***Solution*** For every $a \neq 0$ the number $1 + a^2 > 1$. Since the logarithm of a number greater than unity is negative only to a base less than unity, it follows that $0 < x < 1$.

➤ **Example 3.** Solve the inequality

$$\log_2 \frac{4}{x+3} > \log_2 (2-x).$$

➤ ***Solution*** The given inequality is equivalent to the following system of inequalities :

$$\begin{cases} \dfrac{4}{x+3} > 0 \\ 2-x > 0 \\ \dfrac{4}{x+3} > 2-x, \end{cases}$$

whence we get : $\begin{cases} -3 < x < 2 \\ \dfrac{(x+2)(x-1)}{x+3} > 0, \end{cases}$

and further we have the solution of the given inequality :$(-3, -2) \cup (1, 2)$

➤ **Example 4.** Find all x such that $\log_{1/2}x > \log_{1/3}x$

➤ ***Solution*** $\log_{1/3}x = \dfrac{\log_{1/2}x}{\log_{1/2}\frac{1}{3}} = \log_{1/3}\frac{1}{2}.\log_{1/2}x$

and so our inequality can be rewritten as

$$\log_{1/2}x.(1 - \log_{1/3}\frac{1}{2}) > 0$$

Since $1 - \log_{1/3}\frac{1}{2} > 0$, from the latter inequality we obtain $\log_{1/2}x > 0 \Rightarrow \ 0 < x < 1$.

➤ **Example 5.** Solve the logarithmic inequality

$$\log_4^2 x - \log_2 x - 15 > 0.$$

➤ ***Solution*** The set of permissible values of the unknown is $x > 0$. Reducing the logarithms to the same base 2 we get $\frac{1}{4}\log_2^2 x - \log_2 x - 15 > 0$

We solve this inequality as a quadratic inequality with respect to the new unknown $y = \log_2 x$. As a result we find that logarithmic inequality is equivalent to

$$\log_2 x > 10, \log_2 x < -6$$

The union of their sets of solutions yields the set of solutions of inequality : $x \in (0, 2^{-6}) \cup (2^{10}, \infty)$

➤ **Example 6.** Solve the inequality

$$\log_2^2 (x-1)^2 - \log_{0.5}(x-1) > 5.$$

➤ ***Solution*** Since $\log_2 (x-1)^2 = 2\log_2 |x-1|$

and $\log_{0.5}(x-1) = \dfrac{\log_2(x-1)}{\log_2 0.5} = -\log_2 (x-1)$,

the inequality can be rewritten as :

$$4\log_2^2 |x-1| + \log_2(x-1) > 5. \qquad ...(1)$$

Let us set $y = \log_2 (x-1)$.

Since $x - 1 > 0$ and hence $|x-1| = x-1$, inequality (1) takes the form : $4y^2 + y - 5 > 0$,

whence we find: $y < -\frac{5}{4}$; $y > 1$.

Now, the problem is reduced to solving the collection of logarithmic inequalities :

$$\log_2 (x-1) < -\frac{5}{4}; \quad \log_2(x-1) > 1$$

or $\log_2(x-1) < \log_2 2^{-5/4}$; $\log_2 (x-1) > \log_2 2$. ...(2)

From the first inequality of collection (2) we get: $0 < x - 1 < 2^{-5/4}$, and, consequently, $x - 1 < 2$, that is, $x > 3$.

Thus, $\left(1, 1 + \dfrac{1}{2\sqrt[4]{2}}\right) \cup (3, \infty)$ is the solution of the inequality.

www.pdfworld.net

www.pdfworld.net

➤ **Example 7.** Solve the inequality,

$$2\log_{1/2}(x-1) \le \frac{1}{3} - \frac{1}{\log_{x^2-x} 8}.$$

➤ ***Solution*** Converting on base 2,

$\log_2 \dfrac{2(x-1)^6}{x(x-1)} \ge 0$ note that $x > 1$;

$$\frac{2(x-1)^5}{x} \ge 1$$

put $x - 1 = y \Rightarrow y > 0$

$$\frac{2y^5}{y+1} - 1 \ge 0 \quad \Rightarrow \quad \frac{2y^5 - y - 1}{y+1} \ge 0$$

$$\Rightarrow \quad \frac{2y^5 - 2y + y - 1}{y+1} \ge 0$$

$$\Rightarrow \quad \frac{2y(y^4-1) + y - 1}{y+1} \ge 0$$

$$\Rightarrow \quad \frac{(y-1)[2y(y+1)(y^2+1)+1]}{y+1} \ge 0$$

$$\Rightarrow \quad \frac{y-1}{y+1} \ge 0$$

$$\Rightarrow \quad y \ge 0$$

$$\Rightarrow \quad x \ge 2$$

➤ **Example 8.** Solve the inequality

$$\log_{\frac{1}{2}} x + \log_3 x > 1$$

➤ ***Solution*** Using the logarithmic identity

$\log_b a = \dfrac{\log_c a}{\log_c b}$ we transform the left member of the given inequality in the following way :

$$\log_{\frac{1}{2}} x + \log_3 x = \frac{\log_3 x}{\log_3 \frac{1}{2}} + \log_3 x$$

$$= \log_3 x\left(\log_{\frac{1}{2}} 3 + 1\right) = \log_3 x \cdot \log_{\frac{1}{2}} \frac{3}{2}$$

$$= \frac{\log_3 x}{\log_{\frac{3}{2}} \frac{1}{2}} = -\frac{\log_3 x}{\log_{\frac{3}{2}} 2}$$

Then the given inequality takes the form

$$-\frac{\log_3 x}{\log_{\frac{3}{2}} 2} > 1$$

We have $2 > 1$ and $\dfrac{3}{2} > 1$ and hence, $\log_{\frac{3}{2}} 2 > 0$.

Consequently, the above inequality is equivalent to

$$\log_3 x < -\log_{\frac{3}{2}} 2$$

Hence, noting that $x > 0$, we finally have $0 < x < 3^{-\log_{\frac{3}{2}} 2}$.

➤ **Example 9.** Solve the inequality

$$\frac{1}{\log_2 x} - \frac{1}{\log_2 x - 1} < 1$$

➤ ***Solution*** Reducing the fractions on the left hand side to a common denominator, we find

$$\frac{-1}{\log_2 x(\log_2 x - 1)} < 1$$

$$\Rightarrow \quad \frac{1 + \log_2 x(\log_2 x - 1)}{\log_2 x(\log_2 x - 1)} > 0$$

The numerator of the last expression is positive since we have

$$1 + \log_2^2 x - \log_2 x = \left(\log_2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$\Rightarrow \quad \log_2 x(\log_2 x - 1) > 0,$

which is fulfilled for $x > 2$ and $0 < x < 1$.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

➤ **Example 10.** Solve the inequality
$\log_{0.2}(x^3+8) - 0.5\log_{0.2}(x^2+4x+4) \le \log_{0.2}(x+58)$.

➤ ***Solution*** The inequality is equivalent to the following system of inequalities :

$$\begin{cases} x^3+8>0 \\ x^2+4x+4>0 \\ x+58>0 \\ \log_{0.2}(x^3+8)-0.5\log_{0.2}(x+2)^2 \le \log_{0.2}(x+58) \end{cases}$$

Further, we have:

$$\begin{cases} x>-2 \\ x\ne -2 \\ x>-58 \\ \log_{0.2}(x^3+8)-\log_{0.2}\sqrt{(x+2)^2} \le \log_{0.2}(x+58) \end{cases}$$

whence

$$\begin{cases} x>-2 \\ \log_{0.2}\dfrac{(x+2)(x^2-2x+4)}{|x+2|} \le \log_{0.2}(x+58) \end{cases}$$

Since $x > -2$, we have : $|x + 2| = x + 2$, and

$$\begin{cases} x>-2 \\ \log_{0.2}(x^2-2x+4) \le \log_{0.2}(x+58). \end{cases}$$

Finally, we get the system :

$$\begin{cases} x>-2 \\ x^2-2x+4 \ge +58, \end{cases}$$

and further $\begin{cases} x>-2 \\ x^2-3x-54, \ge 0, \end{cases}$

whence we obtain : $[9, \infty)$ which is the solution of inequality.

➤ **Example 11.** Solve the inequality $\log_{2x+3} x^2 < 1$

➤ ***Solution*** $\log_{2x+3} x^2 < 1$

$x^2 > 0 \Rightarrow x \in (-\infty, 0) \cup (0, \infty)$ ...(1)

$2x < 3 \ne 1 \Rightarrow x \ne -1$ ...(2)

Case A : If $0 < 2x + 3 < 1$

i.e. $-\dfrac{3}{2} < x < -1$ ...(3)

then $\log_{2x+3} x^2 < 1 \Rightarrow x^2 > 2x + 3$

or $x^2 = 2x - 3 > 0$ i.e. $(x - 3)(x + 1) > 0$

$\Rightarrow x \in (-\infty, -1) \cup (3, \infty)$ ...(4)

from (1), (2), (3) and (4)

$$x \in \left(-\frac{3}{2}, -1\right)$$

Case B : If $2x + 3 > 1$

$\Rightarrow x > -1$ ...(5)

$\log_{2x+3} x^2 < 1 \Rightarrow x^2 < 2x + 3$

or $x^2 - 2x - 3 < 0$ or $(x - 3)(x + 1) < 0$

$\Rightarrow x \in (-1, 3)$ ...(6)

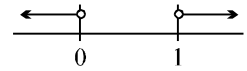
from (1), (2), (5) and (6)

$x \in (-1, 0) \cup (0, 3)$

Hence $x \in \left(-\dfrac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$

➤ **Example 12.** Find the solution set of inequality, $\log_{x+3}(x^2 - x) < 1$.

➤ ***Solution*** $x(x - 1) > 0$



Case A: $x + 3 > 1$

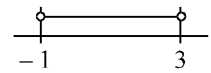
$\therefore x > -2$

$\therefore x \in (-2, 0) \cup (1, \infty)$

$x^2 - x < x + 3$

$x^2 - 2x - 3 < 0$

$(x - 3)(x + 1) < 0$



$\therefore x \in (-1, 0) \cup (1, 3)$

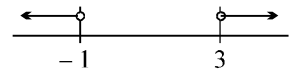
Case B: $0 < x + 3 < 1$

$-3 < x < -2$

$x^2 - x > x + 3$

$x^2 - 2x - 3 > 0$

$(x - 3)(x + 1) > 0$



www.pdfworld.net

www.pdfworld.net

$\therefore \quad x \in (-3, -2)$

$\therefore \quad x \in (-3, -2) \cup (-1, 0) \cup (1, 3)$

➤ **Example 13.** Solve the inequation

$\log_{(x-3)} (2(x^2 - 10x + 24)) \geq \log_{(x-3)} (x^2 - 9)$

➤ ***Solution*** The inequation is equivalent to the collection of systems

$$\begin{cases} 2(x^2 - 10x + 24) \geq x^2 - 9, \\ x^2 - 9 > 0, \\ x - 3 > 1, \end{cases}$$

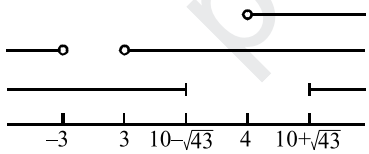
$$\begin{cases} 2(x^2 - 10x + 24) \leq x^2 - 9, \\ 2(x^2 - 10x + 24) > 0, \\ 0 < x - 3 < 1, \end{cases}$$

Solving the first system, we have

$$\begin{cases} x^2 - 20x + 57 \geq 0, \\ (x + 3)(x - 3) > 0, \\ x > 4, \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \in (-\infty, 10 - \sqrt{43}] \cup [10 + \sqrt{43}, \infty] \\ x \in (-\infty, -3) \cup (3, \infty) \\ x \in (4, \infty) \end{cases}$$

Therefore the system has solution



$x \geq 10 + \sqrt{43}$

i.e. $x \in [10 + \sqrt{43}, \infty)$

Solving the second system, we have

$$\begin{cases} x^2 - 20x + 57 \leq 0, \\ (x - 6)(x - 4) > 0, \\ 3 < x < 4, \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x \in [10 - \sqrt{43}, 10 + \sqrt{43}] \\ x \in (-\infty, 4) \cup (6, \infty) \\ x \in (3, 4) \end{cases}$$

Therefore the system has solution

$10 - \sqrt{43} \leq x < 4$

i.e. $x \in [10 - \sqrt{43}, 4)$

Combining the both systems, then solution of the original inequation is

$x \in [10 - \sqrt{43}, 4) \cup [10 + \sqrt{43}, \infty)$

➤ **Example 14.** Solve the inequation

$$\log_{\left(\frac{x^2 - 12x + 30}{10}\right)} \left(\log_2 \frac{2x}{5}\right) > 0$$

➤ ***Solution*** This inequation is equivalent to the collection of two systems

$$\begin{cases} \dfrac{x^2 - 12x + 30}{10} > 1, \\ \log_2\left(\dfrac{2x}{5}\right) > 1, \end{cases} ; \quad \begin{cases} 0 < \dfrac{x^2 - 12x + 30}{10} < 1 \\ 0 < \log_2\left(\dfrac{2x}{5}\right) < 1 \end{cases}$$

Solving the first system, we have

$$\Rightarrow \quad \begin{cases} x^2 - 12x + 20 > 0 \\ \dfrac{2x}{5} > 2 \end{cases} \Rightarrow \begin{cases} x < 2, \quad x > 10 \\ x > 5 \end{cases}$$

$\Rightarrow \quad x > 10$

Solving the second system, we have

$$\Rightarrow \quad \begin{cases} 0 < x^2 - 12x + 30 < 10 \\ 1 < \dfrac{2x}{5} < 2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x^2 - 12x + 30 > 0 \text{ and } x^2 - 12x + 20 < 0 \\ 5/2 < x < 5 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x < 6 - \sqrt{6} \text{ and } x > 6 + \sqrt{6} \ \& 2 < x < 10 \\ 0 < x < 5 \end{cases}$$

$\Rightarrow \quad 2 < x < 6 - \sqrt{6}$.

www.pdfworld.net

www.pdfworld.net

Combining both systems, the solution of the original inequation is

$$x \in (2,\ 6 - \sqrt{6}\ ) \cup (10,\ \infty).$$

➤ **Example 15.** Solve the inequality

$$(x^2 + x + 1)^x < 1$$

➤ ***Solution*** Since both sides of the original inequality are positive for all x,
we take logs to the base 10 to get the equivalent inequality $x.\log_{10}(x^2 + x + 1) < 0$.
This inequality holds true in two cases :
when x satisfies the system of inequalities.

$x > 0$

$\log_{10}(x^2 + x + 1) < 0$

and when x satisfies the system of inequalities

$x < 0$

$\log_{10}(x^2 + x + 1) > 0$

Let us solve the first system of inequalities. We find that it is equivalent to the system

$x > 0$

$x^2 + x + 1 < 1$

Since the solution of the second inequality of the system is $-1 < x < 0$ and the solution of the first is $x > 0$, this system is inconsistent, which means that in this case the original inequality does not have a solution.

The second system is equivalent to

$x < 0$

$x^2 + x + 1 > 1$

The solution set of this system consists of all $x < -1$, whereby the solution of the original inequality is the set of all values of $x < -1$

Alternative

Since the properties of a power depend on whether the base is greater or less than unity, it is natural to consider two cases.

(a) Suppose that $x^2 + x + 1 < 1$, then $x > 0$.
or $-1 < x < 0$, and $x > 1$.
Hence, these values of x cannot be solutions of the inequality.

(b) Suppose that $x^2 + x + 1 > 1$., then $x < 0$
or $x > 0$, $x < -1$, and $x < 0$.
Thus, the solution set of the original inequality consists of all values of $x < -1$.

➤ **Example 16.** Solve the inequality

$$\log_{x^2}\left(\frac{4x-5}{|x-2|}\right) \geq \frac{1}{2}$$

➤ ***Solution*** The domain of x here is defined from the conditions : $(4x - 5) / |x - 2| > 0$, $x^2 > 0$, $x^2 \neq 1$
$\Rightarrow$ $x > 5/4$ and $x \neq 2$.
But for all these values of x we have $x^2 > 1$ and so our inequality,by the property of logarithms to a base exceeding unity, is equivalent to $\dfrac{4x-5}{|x-2|} \leq x$

Since $x \neq 2$, the expression $|x - 2|$ is positive and therefore $4x - 5 \geq x\,|x - 2|$
We now consider two cases.

(a) Let $x > 2$. Then our inequality will be rewritten as $4x - 5 \geq x^2 - 2x$
or $x^2 - 6x + 5 \leq 0 \Rightarrow 1 \leq x \leq 5$, and the solution set is $2 < x \leq 5$.

(b) Now let $5/4 \leq x < 2$. Then our inequality takes the form $4x - 5 \geq -x^2 + 2x$ or $x^2 + 2x - 5 \geq 0$.

$\Rightarrow x \geq \sqrt{6} - 1$ and $x \leq -\sqrt{6} - 1$, and the solution set is $\sqrt{6} - 1 \leq x < 2$.

Combining both cases, we find the solution of the original inequality as $\sqrt{6} - 1 \leq x < 2$ and $2 < x \leq 5$.

Practice Problems [T]

Solve the following inequations:

1. $\log_{0.5}(x^2 - 5x + 6) > -1$.

2. $\log_8(x^2 - 4x + 3) \leq 1$.

3. $\log_{1/4}\left(\dfrac{35 - x^2}{x}\right) \geq -\dfrac{1}{2}$.

4. $\dfrac{\log^2 x - 3\log x + 3}{\log x - 1} < 1$.

www.pdfworld.net

www.pdfworld.net

5. $\dfrac{x-1}{\log_3(9-3^x)-3} \le 1.$

6. $\dfrac{\log_2(x+1)}{x-1} > 0.$

7. $\log_{0.5}\left(\log_8 \dfrac{x^2-2x}{x-3}\right) < 0.$

8. $\dfrac{(x-0.5)(3-x)}{\log_2|x-1|} > 0.$

9. $\log_x(x^3-x^2-2x) < 3.$

10. $\log_{(x-3)}(2(x^2-10x+24)) \ge \log_{(x-3)}(x^2-9)$

11. $\sqrt{\log^2_{1/2} x + 4\log_2\sqrt{x}} < \sqrt{2}\ (4-\log_{16}x^4).$

12. $\log_a(x-1)+\log_a x > 2.$

13. $\log_x 2x \le \sqrt{\log_x(2x^3)}$

14. $\log_x 2 \cdot \log_{2x} 2 \cdot \log_2 4x > 1.$

15. $\log_3 \dfrac{|x^2-4x|+3}{x^2+|x-5|} \ge 0.$

16. $\dfrac{1}{\log_3(x+1)} < \dfrac{1}{2\log_9\sqrt{x^2+6x+9}}$

17. $\log_a(1-8a^{-x}) \ge 2(1-x)$

18. $\log_{\frac{2}{3}|x-2|} 2^{1-x^2} \ge 0$

19. $\dfrac{x^2-4}{\log_{1/2}(x^2-1)} < 0$

20. $\log_{x^2} \dfrac{4x-5}{|x-2|} \ge \dfrac{1}{2}$

1.26 GREATEST INTEGER FUNCTION

Greatest Integer Function

The function $y = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer just less than or equal to x. It is also known as integer floor function.

Thus $[3.245] = 3$, $[0.75] = 0$, $[3] = 3$

$[-5.86] = -6$, $[-0.57] = -1$, $[\pi] = 3$,

$[e] = 2$, $[-\pi] = -4$

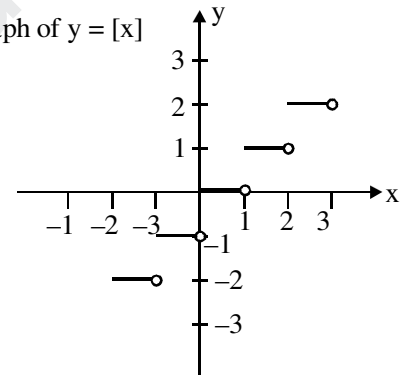
In general, if n is an integer and x is any real number between n and $(n+1)$

i.e. $n \le x < n+1$

then $[x] = n$

$$y = [x] = \begin{cases} -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \end{cases}$$ and so on.

graph of $y = [x]$



Note

(i) $x - [x]$ is the fractional part of x and is denoted by $\{x\}$

(ii) $-[-x]$ is the least integer greater than or equal to x.

(iii) $[x+0.5]$ is the nearest integer to x. If two integers are equally near to x, $[x+0.5]$ denotes the larger of the two.

Properties of Greatest Integer Function

Let x and y be real numbers. Then we have

(i) $[x] \le x < [x]+1$

(ii) $x-1 < [x] \le x, \qquad 0 \le x-[x] < 1$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

(iii) $[x + m] = [x] + m$ if m is an integer.

(iv) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(v) $[-x] = -[x], \; x \in I$

$= -[x] - 1, \; x \notin I$

In another form,

$$[x] + [-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$$

(vi) $[x] \ge n \Rightarrow x \ge n, \; n \in I$

(vii) $[x] > n \Rightarrow x \ge n + 1, \; n \in I$

(viii) $[x] \le n \Rightarrow x < n + 1, \; n \in I$

(ix) $[x] < n \Rightarrow x < n, \; n \in I$

(x) $\left[\frac{[x]}{m}\right] = \left[\frac{x}{m}\right]$ if m is a positive integer

(xi) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + ..$

$... + \left[x + \frac{n-1}{n}\right] = [nx]$

(xii) If n and a are positive integers, [n/a] is the number of integers among 1,2....., n that are divisible by a.

(xiii) **de Polignac's Formula :** To find the highest power of a prime p which is contained in n!.

Let p denote a prime. Let E be the largest exponent of p such that p^E divides n!.

Then $E = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right]$

Out of the first n integers, as many are divisible by p as the number of times p is contained in n, that is, p, 2p, 3p,..... out of these, some may contain the factor p again, namely, p^2, $2p^2$, $3p^2$.... The number of such integers is the number of times p^2 is contained in n. Some of these may contain p a third time. The number of such integers is the number of times p^3 is contained in n. We proceed in this manner examining higher and higher powers of p, stopping when $p^r > n$.

The highest power required

$= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ...$ where [ ] is the greatest integer function.

➤ **Example 1.** Prove the following properties of the greatest integer function:

(i) $[x + y] \ge [x] + [y]$

(ii) $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$ where n is a natural number

(iii) $\left[x + \frac{1}{2}\right] = [2x] - [x]$

➤ ***Solution***

(i) Let x be an arbitrary real number. Then we can write $x = [x] + \alpha$ where α is a nonnegative number less than 1. Now let us represent y in the form $y = [y] + \beta (0 \le \beta < 1)$.

Then $x + y = [x] + [y] + \alpha + \beta$.

Since $\alpha + \beta \ge 0$ the last equality shows that [x] + [y] is an integer not exceeding x + y. Further, since (x + y) is the greatest of the integers not exceeding x + y we have $[x + y] \ge [x] + [y]$.

(ii) Let us represent x in the form $x = [x] + \alpha$ where $0 \le \alpha < 1$. The division of the integer [x] by n results in a quotient q and a remainder r, that is $[x] = qn + r \; (0 \le r \le n - 1)$. Thus we have

$\frac{[x]}{n} = q + \frac{r}{n}$, $\left[\frac{[x]}{n}\right] = q$ and

$x = qn + r + \alpha = qn + r_1$ where $r_1 = r + \alpha < n$. Hence $x/n = q + r_1/n \; (0 \le r_1/n < 1)$ and $[x/n] = q = [[x]/n]$, which is what we intended to prove.

Alternative

Let us consider all whole numbers which do not exceed x and are divisible by n. The number of these whole numbers is obviously equal to [x/n]. Let us also consider all whole numbers which do not exceed [x] and are divisible by n. Their number is equal to [[x]/n]. Now ,since these groups of whole numbers coincide, the numbers in these groups are equal.

www.pdfworld.net

www.pdfworld.net

Hence $[[x]/n] = [x/n]$.

(iii) If $x - [x] < 1/2$ then $[x + 1/2] = [x]$,

and we have $[2x] = 2[x]$ and

$[2x] - [x] = 2[x] - [x] = [x] = [x + 1/2]$.

If $x - [x] \geq 1/2$ then $[x + 1/2] = [x] + 1$, $[2x] = 2[x] + 1$,

and we again have $[2x] - [x]$

$= 2[x] + 1 - [x] = [x] + 1 = [x + 1/2]$

➤ **Example 2.** Let $x = \log_4 9 + \log_9 28$

show that $[x] = 3$, where $[x]$ denotes the greatest integer less than or equal to x.

➤ ***Solution*** Now $x = \log_4 9 + \log_9 28$

$$= \log_2 3 + \frac{\log_3 28}{\log_3 9} > \log_2 3 + \frac{\log_3 27}{2} = \log_2 3 + \frac{3}{2}$$

$$> \log_2 2\sqrt{2} + \frac{3}{2} = \frac{3}{2} + \frac{3}{2} = 3$$

Hence $x > 3$ ....(1)

again $x = \log_2 3 + \log_9 28 < \log_2 4 + \log_9 81$

$= 2 + 2 = 4$

hence $x < 4$ ...(2)

from (1) and (2), $3 < x < 4 \Rightarrow [x] = 3$

➤ **Example 3.** If $[x]$ denotes the integral part of x for real x, then find the value of

$$\left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{200}\right] + \left[\frac{1}{4} + \frac{1}{100}\right] + \left[\frac{1}{4} + \frac{3}{200}\right] + \ldots\ldots + \left[\frac{1}{4} + \frac{199}{200}\right]$$

➤ ***Solution*** The given expression can be written as

$$0 + \ldots + 0 + \left[\frac{1}{4} + \frac{150}{200}\right] + \left[\frac{1}{4} + \frac{151}{200}\right] + \ldots\ldots + \left[\frac{1}{4} + \frac{199}{200}\right]$$

$= 1 \times 50 = 50$

➤ **Example 4.** Find the highest power of 5 that divides 518!

➤ ***Solution*** The highest power of 5 less than 518 is $125 = 5^3$. Therefore the highest power of 5 in 518! is

$$\left[\frac{518}{5}\right] + \left[\frac{518}{25}\right] + \left[\frac{518}{125}\right] = 103 + 20 + 4 = 127.$$

➤ **Example 5.** How many zeros are there at the end of 60! ?

➤ ***Solution*** Let $[x]$ be the greatest integer function. With this notation, the higher power of 2 in 60!

$$= \left[\frac{60}{2}\right] + \left[\frac{60}{2^2}\right] + \left[\frac{60}{2^3}\right] + \left[\frac{60}{2^4}\right] + \left[\frac{60}{2^5}\right],$$

$= 30 + 15 + 7 + 3 + 1,$

Also, the highest power of 5 in 60!

$$= \left[\frac{60}{5}\right] + \left[\frac{60}{5^2}\right],$$

$= 12 + 2 = 14$

The highest power of $10 (= 2 \times 5)$ in $60! = \min.(56, 14) = 14$

➤ **Example 6.** For how many positive integral values of n does n! end with precisely 25 zeros?

➤ ***Solution*** The number of final zeros would equal the highest power of 10 that divides n!, which should equal the highest power of 5 that divides n!.

Then $\left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \left[\frac{n}{125}\right] + \ldots. = 25.$

It's easy to see that n = 105 in the smallest satisfactory value of n. The next four values of n will also work. The answer is 5.

➤ **Example 7.** Prove that for $n = 1, 2, 3, \ldots,$

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \left[\frac{n+8}{16}\right] + \ldots. = n.$$

➤ ***Solution*** Note that $[x] = \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right]$. Hence,

$$n = \left[\frac{n}{2}\right] + \left[\frac{n+1}{2}\right]; \quad \left[\frac{n}{2}\right] = \left[\frac{n}{4}\right] + \left[\frac{n+2}{4}\right];$$

$$\left[\frac{n}{4}\right] = \left[\frac{n}{8}\right] + \left[\frac{n+4}{8}\right]; \ldots\ldots\ldots$$

Adding these equations we get the required result.

www.pdfworld.net

www.pdfworld.net

➤ **Example 8.** Let $[\sqrt{n^2+1}] = [\sqrt{n^2+\lambda}]$ where $n, \lambda \in N$. Show that λ can have $2n$ different values.

➤ ***Solution*** We have n^2+1
$= (n+1)^2 - 2n < (n+1)^2 \ \forall \ n \in N$

i.e. $\sqrt{n^2+1} < n+1.$

Thus, we have $n < \sqrt{n^2+1} < n+1$

i.e. $[\sqrt{n^2+1}] = n.$

Now, the given equation reduces to $[\sqrt{n^2+\lambda}] = n$

i.e. $n < \sqrt{n^2+\lambda} < (n+1)$

i.e. $n^2 < n^2+\lambda < (n+1)^2$

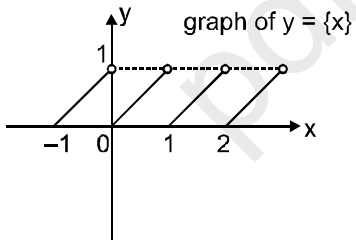
i.e. $0 < \lambda < 2n+1.$

Hence, λ can have $2n$ different values.

Fractional Part Function

The function $y = \{x\}$ is called the fractional part function where $\{x\} = x - [x]$.

Thus $\{3.245\} = 3.245 - [3.245]$
$= 3.245 - 3 = 0.245,$
$\{-5.86\} = -5.86 - [-5.86] = -5.86 + 6 = 0.14.$
$\{3\} = 0$



Properties of Fractional Part Function

Let x a be real number. Then we have

(i) $0 \le \{x\} < 1$

(ii) $\{x+m\} = \{x\}$ if m is an integer.

(iii) $\{-x\} = 1 - \{x\}, x \notin I$
$= 0, \ x \in I$

➤ **Example 9.** If $\{x\}$ and $[x]$ represent fractional and integral part of x, then find the value of

$$[x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000}.$$

➤ ***Solution*** $[x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000}$ we know that

$\{x+r\} = \{x\}$ as $r \in$ Integer

$$= [x] + \sum_{r=1}^{2000} \frac{\{x\}}{2000}$$

$$= [x] + \left[\frac{\{x\}}{2000} + \frac{\{x\}}{2000} + \ldots\ldots + \text{upto 2000 times}\right]$$

$$= [x] + \frac{2000\{x\}}{2000} = [x] + \{x\} = x$$

Thus, $[x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000} = x.$

Least Integer Function

(x) or $\lceil x \rceil$ denotes the least integer greater than or equal to x,
i.e., $(x) \le [x] + 1$. It is known as integer ceiling function.

Thus $(3.55) = 4, (0.62) = 1, \in$

$(-0.265) = 0, (-6.28) = -6$

Properties of Least Integer Function

(i) $(x) = [x]$, if $x \in I$ and $(x) = [x] + 1$, if $x \notin I$, in fact $(x) = -[-x]$

(ii) $(x + n) = n + (x), n \in I$

(iii) $(-x) = -(x), x \in I$
$(-x) = -(x) + 1, x \notin I$

➤ **Example 10.** Solve the equation $(x)^2 = [x]^2 + 2x$ where $[x]$ and (x) are the integer just less than or equal to x and just greater than or equal to x repectively.

➤ ***Solution*** **Case A :** If $x \in I$
then $(x) = [x]$
$\therefore \quad (x)^2 = [x]^2 + 2x$
$\Rightarrow \quad [x]^2 = [x]^2 + 2x$
$\therefore \quad x = 0$

www.pdfworld.net

www.pdfworld.net

Case B : If $x \notin I$

then $(x) = [x] + 1$

$\therefore \quad (x)^2 = [x]^2 + 2x$

$\Rightarrow \quad [x]^2 + 1 + 2[x] = [x]^2 + 2x$

$\Rightarrow \quad x = [x] + \frac{1}{2} = n + \frac{1}{2}, \; n \in I$

Hence the solution of the original equation is

$$x = 0, \; n + \frac{1}{2}, \; n \in I$$

Practice Problems [U]

1. Find the value of $\left[\frac{3}{4}\right] + \left[\frac{3}{4}+\frac{1}{100}\right] + \left[\frac{3}{4}+\frac{2}{100}\right] + \ldots\ldots\ldots \left[\frac{3}{4}+\frac{99}{100}\right]$

2. Prove that $f(x) = [x - [x] + 1]$ is a constant function.

3. If $y = 2[x] + 3$ and $y = 3[x - 2] + 5$ then find the value of $[x + y]$.

4. How many zeros are there at the end of 400!

5. Prove that there exists no value of $n \in N$ such that $[e] + [e^2] + \ldots\ldots [e^n] = 36$.

6. If x be a real number for which
(x) = the integer just greater than or equal to x,
$[x]$ = the integer just less than or equal to x
then solve $(x)^2 = [x]^2 + 2x$

7. Solve $[x] + |x - 2| \leq 0$ and $-1 \leq x \leq 3$ where $[x]$ is the greatest integer function.

8. Find the number of different values which $\left[\frac{2x^2}{x^2+1}\right]$ can assume.

9. Prove that the equation $\{x\} + x^2 = 0$ has only one solution.

Target Problems for JEE Advanced

➤ **Problem 1.** Prove that every natural number $n \geq 12$ is a sum of two composite numbers.

➤ ***Solution*** If n is odd, write $n = (n - 9) + 9$
since n is odd, therefore $n - 9$ is even. Also $n \geq 12 \Rightarrow n - 9 \geq 3$, so that $n - 9$ is an even number greater than 2, and consequently $n - 9$ is a composite number. Also 9 is a composite number. Thus n is the sum of two composite numbers.
If n is even, write $n = (n - 4) + 4$

Since $n \geq 12$, $n - 4$ and 4 are both composite even numbers. Thus n is the sum of two composite even numbers.

➤ **Problem 2.** Prove that the number $b(2b + 1)(7b + 1)$ can be divided by three for any integer b.

➤ ***Solution*** We divide the set of integers into three classes, i.e. (a) 3q, (b) 3q + 1, (c) 3q + 2, where q is an integer.

Assume that b is a number belonging to class (a). Then $b(2b + 1) \times (7b + 1) = 3q(6q + 1)(21q + 1)$, whence we see that this number is divisible by 3 for any integer q.
Assume that b is a number belonging to class (b). Then $b(2b + 1) \times (7b + 1) = (3q + 1).3.(2q + 1)(21q + 8)$. We see that this number is divisible by 3 for any integer q.

Assume that b is a number belonging to class (c). Then $b(2b + 1) \times (7b + 1) = (3q + 2)(6q + 5).3.(7q + 5)$. We see that this number is divisible by 3 for any integer q.

➤ **Problem 3.** If a and b are two integers such that 7 divides $a^2 + b^2$, prove that 49 divides $a^2 + b^2$.

➤ ***Solution*** When an integer is divided by 7, it leaves as remainder one of the possible remainders 0, 1, 2, 4, (being the residues when $0, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2$ are divided by 7). Therefore when $a^2 + b^2$ (a, b being both integers) is divided by 7, the possible remainders are $i + j$ (mod. 7) where $i, j = 0, 1, 2, 4$. Since the sum of two of the numbers 0, 1, 2, 4, is a multiple of 7 iff each of them is zero, it follows that $a^2 + b^2$ is a multiple of 7 iff a^2 and b^2 are both multiples

www.pdfworld.net

www.pdfworld.net

of 7. But a^2 and b^2 are multiples of 7 iff a and b are multiples of 7 (because 7 is a prime), i.e., iff a^2 and b^2 are both multiples of 49, and this implies that $a^2 + b^2$ is a multiple of 49.

➤ **Problem 4.** Show that $15x^2 - 7y^2 = 9$ has no integral solutions.

➤ ***Solution*** Since the R.H.S. is odd, x and y must be opposite parity (i.e. one even and the other odd). As 3|15 and 3|9, 3 must divide $7y^2$.

$\therefore \quad y = 3y_1$

Substituting and simplifying we, get $5x^2 - 21y_1^2 = 3$.

Again, $3 \mid 5x^2$, therefore, $x = 3x_1$, leading to the new equation $15x_1^2 - 7y_1^2 = 1$.

$\therefore \quad 15x_1^2 = 7y_1^2 + 1.$

The last digit of a perfect square y_1^2 ends in 0, 1, 4, 9, 6, 5 and, hence, the last digit of $7y_1^2 + 1$ ends in 1, 8, 9, 4, 3, 6.

But $15x_1^2$ either ends in 0 or 5.

Therefore $15x_1^2 = 7y_1^2 + 1$ has no solution in integers.

Hence, the given equation has no integral solution.

➤ **Problem 5.** Find all the integral solutions of $y^2 = 1 + x + x^2$.

➤ ***Solution*** If $x > 0$, then

$x^2 < x^2 + 1 + x < x^2 + 2x + 1 = (x + 1)^2$.

So $x^2 + x + 1$ lies between the two consecutive square integers and hence, can not be a square.

If $x = 0$, $y^2 = 1 + 0 + 0 = 1$ is a square number. Thus, the solutions in this case are is (0, 1), (0, –1).

Again if $x < -1$, then $x^2 > x^2 + x + 1 > x^2 + 2x + 1$, and hence, there exist no solution.

For $x = -1$, we have $y^2 = 1 - 1 + (-1)^2 = 1$

$\therefore \quad y = \pm 1$

Thus, the only integral solutions are

(0, 1) (0, –1), (–1, +1), (–1, –1).

➤ **Problem 6.** Prove that for $n > 1$, $n! < \left(\frac{n+1}{2}\right)^n$

(n! = 1.2.3.....n)

➤ ***Solution*** For n = 2 we obtain the true inequality $2 < 9/4$.

Suppose that $k! < \left(\frac{k+1}{2}\right)^k$.

Then by the induction hypothesis,

$$(k+1)! = k!\,(k+1) < \left(\frac{k+1}{2}\right)^k (k+1).$$

If we now prove that

$$\left(\frac{k+1}{2}\right)^k (k+1) < \left(\frac{k+2}{2}\right)^{k+1} \qquad ...(1)$$

the theorem is proved because then

$$(k+1)! < \left(\frac{k+1}{2}\right)^k (k+1) < \left(\frac{k+2}{2}\right)^{k+1}$$

that is our inequality holds true for n = k + 1.

Inequality (1) can clearly be rewritten as

$$2 < \left(1 + \frac{1}{k+1}\right)^{k+1}$$

but the binomial theorem yields

$$\left(1 + \frac{1}{k+1}\right)^{k+1} = 1 + (k+1)\frac{1}{k+1} + ... > 2$$

so that inequality (1) holds and the original inequality is thus proved.

➤ **Problem 7.** Prove the inequality

$$\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{3n+1} > 1.$$

➤ ***Solution*** For n =1, the inequality is true since

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1.$$

Suppose that the inequality is true for n = k, i.e.

$$\frac{1}{k+1} + \frac{1}{k+2} + ... + \frac{1}{3k+1} > 1$$

Let us add the sum of the three fractions

www.pdfworld.net

www.pdfworld.net

$$\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}$$

to both sides of the inequality and transfer the first member $\frac{1}{k+1}$, from the left hand side to the right hand side. Then we obtain

$$\frac{1}{k+2}+.....+\frac{1}{3k+4}$$

$$>1+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}-\frac{1}{k+1}.$$

The right hand side of the inequality exceeds unity since

$$1+\frac{1}{3k+2}+\frac{1}{3k+3}-\frac{1}{3k+4}+\frac{1}{k+1}$$

$$=1+\frac{2}{(3k+2)(3k+3)(3k+4)}>1.$$

Consequently, the left hand side exceeds unity i.e.

$$\frac{1}{(k+1)+1}+.....+\frac{1}{3(k+1)+1}>1.$$

The last inequality can be obtained from the original inequality for n = k + 1.

Thus, having assumed that the inequality is true for n = k, we have proved its truth for n = k + 1; we have proved the inequality by induction.

➤ **Problem 8.** Show by using induction that for all $n \in N$, $(\sqrt{2}-1)^n=\sqrt{m}-\sqrt{m-1}$, where m is a positive integer.

➤ ***Solution*** First of all we prove this for all odd positive integers n.

Let $P(n):(\sqrt{2}-1)^n=\sqrt{m}-\sqrt{m-1}$, where $m \in N$

When n = 1, L.H.S. $=\sqrt{2}-1=\sqrt{2}-\sqrt{1}$,

which is of the form $\sqrt{m}-\sqrt{m-1}$

Hence P(1) is true ...(A)

Let P(2k – 1) be true $\Rightarrow (\sqrt{2}-1)^{2k-1}$

$=\sqrt{m}-\sqrt{m-1}$, $m \in N$ ...(1)

But on expansion $(\sqrt{2}-1)^{2k-1}$ reduces to the form $a\sqrt{2}-b$, where a, b ∈ N.

Hence from (1), $a\sqrt{2}-b=\sqrt{m}-\sqrt{m-1}$

$\Rightarrow \quad \sqrt{2a^2}-\sqrt{b^2}=\sqrt{m}-\sqrt{m-1}$

$\therefore \quad 2a^2 = m$ and $b^2 = m - 1$ ...(2)

$\therefore$ Now $(\sqrt{2}-1)^{2k+1}=(\sqrt{2}-1)^{2k-1}(\sqrt{2}-1)^2$

$=(\sqrt{2}a-b)(3-2\sqrt{2})$

$=(3a+2b)\sqrt{2}-(4a+3b)$

$=\sqrt{2(3a+2b)^2}-\sqrt{(4a+3b)^2}$ ...(3)

Let $x = 2(3a + 3b)^2$ and $y = (4a + 3b)^2$

Clearly $x, y \in N$

Also $x - y = 2(9a^2 + 4b^2 + 12ab) - 16a^2 - 9b^2 - 24ab$

$= 2a^2 - b^2$

$= 1 \quad \therefore y = x - 1$

$\therefore$ from (3), $(\sqrt{2}-1)^{2k+1}=\sqrt{x}-\sqrt{x-1}$, where $x \in N$

Hence P(2k + 1) is true whenever P(2k – 1) is true ...(B)

From (A) and (B) it follows that P(n) is true for all odd natural numbers n ...(X)

Now we prove the result for all even natural number n :

Let $P(n):(\sqrt{2}-1)^n=\sqrt{m}-\sqrt{m-1}$

When n = 2, L.H.S. $=(\sqrt{2}-1)^2=3-2\sqrt{2}$

$=\sqrt{9}-\sqrt{8}$

Hence P(2) is true. ...(C)

Let P(2k) be true $\Rightarrow (\sqrt{2}-1)^{2k}=\sqrt{m}-\sqrt{m-1}$, $m \in N$ ...(4)

But on expansion $(\sqrt{2}-1)^{2k}=c-\sqrt{2}d$, where $c, d \in N = \sqrt{c^2}-\sqrt{2d^2}$

From (4), $\sqrt{c^2}-\sqrt{2d^2}=\sqrt{m}-\sqrt{m-1}$

$\therefore \quad c^2 = m$ and $2d^2 = m - 1$

$\Rightarrow \quad c^2 - 2d^2 = 1$ ...(5)

www.pdfworld.net

www.pdfworld.net

Now $(\sqrt{2}-1)^{2k+2} = (\sqrt{2}-1)^{2k}(\sqrt{2}-1)^2$

$= (c - \sqrt{2}\,d)(3 - 2\sqrt{2})$

$= 3c + 4d - (2c + 3d)\sqrt{2}$

$= \sqrt{(3c+4d)^2} - \sqrt{2(2c+3d)^2}$ ...(6)

Let $u = (3c + 4d)^2$, $v = 2(2c + 3d)^2$

then $u - v = 9c^2 + 16d^2 + 24cd - 8c^2 - 18d^2 - 24cd$

$= c^2 - 2d^2 = 1$ [from (5)]

$\therefore \quad v = u - 1$

Thus from (6), $(\sqrt{2}-1)^{2k+2} = \sqrt{u} - \sqrt{u-1}$, where $u \in N$

Hence P(2k + 2) is true whenever P(2k) is true ...(D)

From (C) and (D) it follows that

P(n) is true for all even natural numbers n ...(Y)

From (X) and (Y) it follows that P(n) is true for all natural numbers n.

➤ **Problem 9.** Prove that the square of any prime number $p \geq 5$, when divided by 12, gives 1 as remainder.

➤ ***Solution*** When divided by 6, a natural number can give as a remainder only the numbers 0, 1, 2, 3, 4 and 5. Therefore, any natural number has one of the following forms :

6k, 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5.

It is obvious that the numbers 6k, 6k + 2, 6k + 3, 6k + 3, and 6k + 4 are composite.

Therefore, the prime number $p \geq 5$ has the form 6k + 1 or 6k + 5. If p = 6k + 1, then

$p^2 = (6k + 1)^2 = 36k^2 + 12k + 1$. If p = 6k + 5, then

$p^2 = (6k + 5)^2 = 36k^2 + 60k + 25$

$= 12(3k^2 + 5k + 2) + 1.$

Thus, in both cases, when dividing p^2 by 12, the remainder is equal to 1.

➤ **Problem 10.** Prove that for every positive integer n, $1^n + 8^n - 3^n - 6^n$ is divisible by 10.

➤ ***Solution*** Since 10 is the product of two primes 2 and 5, it will suffice to show that the given expression is divisible both by 2 and 5. To do so, we shall use the simple fact that if a and b be any positive integers, then $a^n - b^n$ is always divisible by a –b.

Writing $A \equiv 1^n + 8^n - 3^n - 6^n$, $= (8^n - 3^n) - (6^n - 1^n)$, we find that $8^n - 3^n$ and $6^n - 1^n$ are both divisible by 5,

and consequently A is divisible by 5 (= 8 – 3 = 6 – 1).

Again, writing $A = (8^n - 6^n) - (3^n - 1^n)$, we find that A is divisible by 2(= 8 – 6 = 3 – 1). Hence A is divisible by 10.

➤ **Problem 11.** Solve the inequality

$$\frac{7x-11}{a+3} > (1 + 3a)\frac{x}{4}.$$

➤ ***Solution*** $\dfrac{7x-11}{a+3} > (1 + 3a)\dfrac{x}{4}$ ...(1)

Setting a + 3 = 0, we find : a = –3 . Hence, we have to consider the following cases :

(i) a < –3 ; (ii) a = –3 ; (iii) a > –3.

(i) Consider the case a < –3. In this case a + 3 < 0, and inequality (1) is equivalent to the inequality $4(7x - 11) < (a + 3) \times (1 + 3a)x$, that is, to the inequality : $(3a^2 + 10a - 25)\,x > -44$. ...(2)

Setting $3a^2 + 10a - 25 = 0$, we find $a = \dfrac{5}{3}$; a = –5

Thus, the solution of inequality (2) has to be considered in the following cases;

$$\begin{cases} a < -5; \\ a < -3 \end{cases} \quad \begin{cases} a > \frac{5}{3}; \\ a < -3 \end{cases} \quad \begin{cases} a = -5; \\ a < -3 \end{cases} \quad \begin{cases} a = \frac{5}{3}; \\ a < -3 \end{cases} \quad \begin{cases} -5 < a < \frac{5}{3} \\ a < -3 \end{cases}$$

that is, in the cases : a < –5; a = –5; –5 < a < –3.

In the first case $3a^2 + 10a - 25 > 0$, and from inequality (2) we find : $x > -\dfrac{44}{3a^2 + 10a - 25}$

In the second case inequality (2) takes the form:

0 . x > – 44 which is true for any x.

Finally, if –5 < a < –3, then $3a^2 + 10a - 25 < 0$, and from inequality (2)

we find that $x < -\dfrac{44}{3a^2 + 10a - 25}$

(ii) Consider the case a = –3. In this case inequality (1) has no solution.

www.pdfworld.net

www.pdfworld.net

(iii) Consider the case $a > -3$. In this case $a + 3 > 0$, and inequality (1) is equivalent to the inequality

$$4(7x - 11) > (a + 3)(1 + 3a)x$$

$$\Rightarrow \quad (3a^2 + 10a - 25)x < -44 \qquad ...(3)$$

The same as for inequality (2), $3a^2 + 10a - 25 = 0$ gives $a = \frac{5}{3}$ and -5. Since we now consider the case $a > -3$, we have to take into account only one of the indicated two values of $a = \frac{5}{3}$. Thus, when solving inequality (3), we must consider the following cases:

$a > \frac{5}{3}$; $a = \frac{5}{3}$; $-3 < a < \frac{5}{3}$.

In the first case we find : $x < -\frac{44}{3a^2 + 10a - 25}$, in the second, inequality (3) has no solution, and in the third we get : $x > -\frac{44}{3a^2 + 10a - 25}$.

Finally (1) if $a = -3; a = \frac{5}{3}$, then the inequality has no solution ;

(2) if $a < -5; -3 < a < \frac{5}{3}$, then $x > -\frac{44}{3a^2 + 10a - 25}$;

(3) if $-5 < a < -3; a > \frac{5}{3}$, then $x < -\frac{44}{3a^2 + 10a - 25}$;

(4) if $a = -5$, then $-\infty < x < \infty$.

➤ **Problem 12.** Solve the linear inequality

$$x - 2\frac{a-1}{a} \le \frac{2}{3a}(x + 1)$$

➤ ***Solution*** It should be noted at once that for $a = 0$ the inequality has no solution since both sides of the inequality lose sense.

We transform the inequality as follows :

$$\left(1 - \frac{2}{3a}\right)x \le \frac{2}{3a} + 2 - \frac{2}{a},$$

$$x\left(1 - \frac{2}{3a}\right) \le 2\left(1 - \frac{2}{3a}\right)$$

If $1 - \frac{2}{3a} > 0$, then $x \le 2$.

Solving the inequality $1 - \frac{2}{3a} > 0$, we get $a < 0$ or $a > 2/3$.

Thus, if $a < 0$ or $a > 2/3$, then $x \le 2$.

If $1 - \frac{2}{3a} < 0$, i.e. $0 < a < 2/3$, then $x \le 2$.

Finally, if $a = 2/3$, then x is any number.

Hence if $a < 0$, then $x \in (-\infty, 2]$

if $a = 0$, then there are no solutions,

if $0 < a < 2/3$, then $x \in (-\infty, \infty)$

If $a > 2/3$, then $x \in (-\infty, 2]$

➤ **Problem 13.** Solve the equation

$$\sqrt{x^2 + x - 5} + \sqrt{x^2 + 8x - 4} = 5$$

➤ ***Solution*** Transforming the equation to the form

$$\sqrt{x^2 + x - 5} = 5 - \sqrt{x^2 + 8x - 4} \qquad ...(1)$$

and squaring both sides of the obtained equation, we get

$$x^2 + x - 5 = 25 - 10\sqrt{x^2 + 8x - 4} + x^2 + 8x - 4$$

We then single out the radical and collect like terms :

$$10\sqrt{x^2 + 8x - 4} = 7x + 26, \qquad ...(2)$$

squaring both sides of equation (2), we get :

$$100(x^2 + 8x - 4) = (7x + 26)^2$$

$$\Rightarrow \quad 51x^2 + 436x - 1076 = 0$$

$$\Rightarrow \quad x_1 = 2, x_2 = -\frac{538}{51}$$

The first of the found roots is readily checked by substitution into the original equation. Such a check shows that $x_1 = 2$ is a root of equation (1). The attempt to check the second root using the same method leads to difficult computations. However, it is possible to proceed in a different way. Let us find out whether $x_2 = -\frac{538}{51}$ is a solution of equation (2). Note that for this value the left hand side of equation (2) is positive, as opposed to the right hand side. Hence $x_2 = -\frac{538}{51}$ is not a root of equation (2). But equation (2) is a

www.pdfworld.net

www.pdfworld.net

consequence of equation (1), and hence x_2 is not a root of equation (1). Thus, the only root of the equation is x = 2.

➤ **Problem 14.** Solve the equation

$$\sqrt{2x^2+3x+5}+\sqrt{2x^2-3x+5}=3x \quad ..(1)$$

➤ ***Solution*** We multiply both sides of the given equation by the function

$$\varphi(x) = \sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5},$$

conjugate to $\sqrt{2x^2+3x+5}+\sqrt{2x^2-3x+5}$.

Since $(\sqrt{2x^2+3x+5}+\sqrt{2x^2-3x+5})$

$(\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5})$

$= (2x^2+3x+5)-(2x^2-3x+5) = 6x$, equation (1) takes the form :

$$6x = 3x\ (\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}-2)$$

$$= 0. \quad ...(2)$$

As is easily seen, $x_1 = 0$ is one of the roots of equation (2). It remains to solve the equation

$$\sqrt{2x^2+3x+5}-\sqrt{2x^2-3x+5}=2. \quad ...(3)$$

Adding together equations (1) and (3), we get the consequence :

$$2\sqrt{2x^2+3x+5}=3x+2. \quad ...(4)$$

Solving equation (4) by squaring we get :

$8x^2 + 12x + 20 = 9x^2 + 12x + 4$,

and further : $x^2 = 16$, i.e. $x_2 = 4, x_3 = -4$.

Substituting the found values $x_1 = 0$, $x_2 = 4$, $x_3 = -4$ into equation (1), we see that it is satisfied only by the value $x_2 = 4$. Thus, x = 4 is the only root of equation (1).

➤ **Problem 15.** Prove that the equation

$$(\sqrt{x+1}+1)(\sqrt{x+16}+4)$$

$$-(\sqrt{x+4}+2)(\sqrt{x+9}+3)=0$$

has no solution.

➤ ***Solution*** Observe that x = 0 is not a solution.

$$(\sqrt{x+1}+1)(\sqrt{x+16}+4)$$

$$=(\sqrt{x+4}+2)(\sqrt{x+9}+3)$$

$$\Rightarrow \quad \frac{x^2}{(\sqrt{x+1}-1)(\sqrt{x+16}-4)}$$

$$= \frac{x^2}{(\sqrt{x+4}-2)(\sqrt{x+9}-3)} \quad ....(1)$$

$$\Rightarrow \quad (\sqrt{x+1}-1)\ (\sqrt{x+16}-4)$$

$$= (\sqrt{x+4}-2)\ (\sqrt{x+9}-3) \quad ....(2)$$

since $x \neq 0$. From (1) and (2) we obtain

$$\Rightarrow \quad \sqrt{x+1}\ \sqrt{x+16} = \sqrt{x+4}\ \sqrt{x+9}+2$$

$$\Rightarrow \quad (x+1)(x+16)$$

$$= (x+4)(x+9)+4\sqrt{x+4}\ \sqrt{x+9}+4$$

$$\Rightarrow \quad x-6 = \sqrt{x+4}\ \sqrt{x+9}$$

$$\Rightarrow \quad x^2-12x+36 = x^2+13x+36 \Rightarrow x = 0$$

Thus the given equation has no solution.

➤ **Problem 16.** Solve the equation

$$|x+1|-|x|+3|x-1|-2|x-2| = x+2$$

➤ ***Solution*** (i) Let $x < -1$. Then $|x+1| = -(x+1)$, $|x| = -x$, $|x-1| = -(x-1)$ and $|x-2| = -(x-2)$ we have

$$-x-1+x-3(x-1)+2(x-2) = x+2.$$

We obtain the root x = –2 from this interval.

(ii) Let $-1 \le x < 0$. Then $|x+1| = x+1$, $|x| = -x$, $|x-1| = -(x-1)$ and $|x-2| = -(x-2)$ we have

$x+1+x-3(x-1)+2(x-2) = x+2$, which does not hold true.

(iii) Let $0 \le x < 1$. Then $|x+1| = x+1$, $|x| = x$, $|x-1| = x-1$ and $|x-2| = x-2$ we have

$x+1-x-3(x-1)+2(x-2) = x+2$

$x = -1$

$x = -1$ lies outside the interval $0 \le x < 1$.

www.pdfworld.net

www.pdfworld.net

(iv) Let $1 \le x < 2$. then $x + 1 > 0$, $x < 0$, $x - 1 \ge 0$, and $x - 2 < 0$. which implies

$$|x + 1| = x + 1,\ |x| = x,\ |x - 1| = x - 1,\ |x - 2| = -(x - 2)$$

We obtain, for this case, the equation

$$x + 1 - x + 3(x-1) + 2(x-2) = x + 2$$

This yields $4x = 8$, or $x = 2$. This value does not lies in the interval considered. Consequently, there is no root found between 1 and 2 for the given equation.

(v) Let $x \ge 2$. Then $x + 1 > 0$, $x - 1 > 0$, $x - 2 \ge 0$; hence $|x + 1| = x + 1, |x| = x, |x - 1| = x - 1, |x - 2| = x - 2$, and we have the equation

$$x + 1 - x + 3(x - 1) - 2(x - 2) = x + 2$$ which is an identity.

Accordingly, all real numbers $x \ge 2$ are roots of the given equation.

Therefore the equation is satisfied by $x \in \{-2\} \cup [2, \infty)$

➤ **Problem 17.** Solve $y - |x^2 - 2x| + \frac{1}{2} > 0$, $y + |x - 1| < 2$

➤ ***Solution*** From second equation of the system,

$$|x - 1| < 2 - y$$

$\Rightarrow \quad 2 - y > 0 \Rightarrow y < 2$

From first equation of the system,

$$|x^2 - 2x| < y + \frac{1}{2}$$

$$\Rightarrow \quad y + \frac{1}{2} > 0 \quad \Rightarrow y > -\frac{1}{2}$$

On combining we get $-\frac{1}{2} < y < 2$. Since y is an integer $y = 0, 1$.

Case-I :

If $y = 0$ the second inequation gives $|x - 1| < 2$

$\Rightarrow \quad -2 < x - 1 < 2 \quad \Rightarrow \quad -1 < x < 3$

$\Rightarrow \quad x = 0, 1, 2$ (x is an integer)

Out of these three values only $x = 0$, $x = 2$ satisfy the first inequation of the system.

This one set of solutions is $x_1 = 0$, $y_1 = 0$; $x_2 = 2$, $y_2 = 0$

Case-II :

If $y = 1$ the second inequation gives $|x - 1| < 1$

$\Rightarrow \quad -1 < x - 1 < 1 \quad \Rightarrow \quad 0 < x < 2$

$\Rightarrow \quad x = 1$

We note that for $x = 1$ the first equation is satisfied

$$\left(1 - \left|1^2 - 2 \times 1\right| + \frac{1}{2} > 0\right)$$

Thus another solution of the system is $x_3 = 1$, $y_3 = 1$.

➤ **Problem 18.** Prove that from the equalities

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$$

follows $x^y \cdot y^x = z^y \cdot y^z = x^z \cdot z^x$

➤ ***Solution*** Put

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z} = \frac{1}{t}$$

$\Rightarrow \quad \log x = tx\,(y + z - x),\ \log y = ty\,(z + x - y),$

$$\log z = tz\,(x + y - z)$$

$\Rightarrow \quad y \log x + x \log y = 2txyz,\ y \log z + z \log y$

$$= 2txyz,\ z \log x + x \log z = 2txyz.$$

$\Rightarrow \quad y \log x + x \log y = y \log z + z \log y$

$= z \log x + x \log z,\ \log x^y y^x = \log z^y y^z = \log x^z z^x$

Finally $\quad x^y y^x = z^y y^z = x^z z^x$

➤ **Problem 19.** Given that $y = 10^{\frac{1}{1-\log x}}$, $z = 10^{\frac{1}{1-\log y}}$ (logarithms are taken to the base 10), prove that $x = 10^{\frac{1}{1-\log z}}$.

➤ ***Solution*** From the given relations it follows that

$$y^{1-\log x} = 10,\ z^{1-\log y} = 10$$

Taking logarithms of these equalities to the base 10, we get

$$(1 - \log x) \log y = 1,\ (1 - \log y) \log z = 1.$$

$$\Rightarrow \quad \log x = 1 - \frac{1}{\log y} = 1 - \frac{1}{1 - \frac{1}{\log z}} = \frac{1}{1 - \log z}$$

$$\Rightarrow \quad x = 10^{\frac{1}{1-\log z}}$$

➤ **Problem 20.** Find all the solutions of the equation

$$x^2 2^{x+1} + 2^{|x-3|+2} = x^2 2^{|x-3|+4} + 2^{x-1}$$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** We consider two cases.

(a) Let $x \geq 3$,

Here we have the equation

$$x^2 2^{x+1} + 2^{x-1} = x^2 2^{x+1} + 2^{x-1}$$

which is evidently satisfied for every x, and so in this case the solutions of the given equation will be all values of $x \geq 3$

(b) Let $x < 3$.

Then the equation takes the form

$$x^2 2^{x+1} + 2^{5-x} = x^2 2^{7-x} + 2^{x-1}$$

$$\Rightarrow \quad 2^{x-1}(4x^2 - 1) = 2^{5-x}(4x^2 - 1)$$

It was precisely at this point where many students are taken in by the exponential ("main") expressions and disregard the "insignificant" power expressions and simply cancel them obtaining the equation $2^{x-1} = 2^{5-x}$. They then obtain the root $x = 3$. We notice that it does not satisfy the condition $x < 3$.

It is clear that before cancelling out $4x^2 - 1$ the students should consider the case of $4x^2 - 1 = 0$. Then they would get the roots $x_{1,2} = \pm 1/2$ which also satisfy the condition $x < 3$.

Thus, the solutions of the given equation are : any $x \geq 3$, $x_1 = 1/2$, $x_2 = -1/2$.

➤ **Problem 21.** Solve $(\sqrt{3}+1)^{2x} + (\sqrt{3}-1)^{2x} = 2^{3x}$

➤ ***Solution***

Here $(\sqrt{3}+1)^{2x} + (\sqrt{3}-1)^{2x} = (2^3)^x = (2\sqrt{2})^{2x}$ ... (1)

$$\Rightarrow \quad \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^{2x} + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^{2x} = 1$$

Let $\frac{\sqrt{3}+1}{2\sqrt{2}} = \cos\theta$, $\frac{\sqrt{3}-1}{2\sqrt{2}} = \sin\theta$

$\therefore \cos^{2x}\theta + \sin^{2x}\theta = 1 \quad \Rightarrow x = 1$

for $\cos^n\theta + \sin^n\theta = 1$ holds only when $n = 2$.

Hence $x = 1$

➤ **Problem 22.** Find the value(s) of x satisfying the equation ,

$$\log_4\left(6 + \sqrt{x} - \left|\sqrt{x} - 2\right|\right) = \frac{1}{2} + \log_2\left|\sqrt{x} - \left|\sqrt{x} - 2\right|\right|$$

➤ ***Solution*** Put $\sqrt{x} - \left|\sqrt{x} - 2\right| = y$

$$\Rightarrow \quad \log_4(6 + y) = \frac{1}{2} + \log_4 |y|^2$$

$$\Rightarrow \quad \log_4\left(\frac{6+y}{y^2}\right) = \frac{1}{2} \quad \Rightarrow \quad \frac{6+y}{y^2} = 2$$

$$\Rightarrow \quad y = 2 \quad \text{or} \quad y = -\frac{3}{2}$$

Let $\sqrt{x} - \left|\sqrt{x} - 2\right| = 2$

$$\sqrt{x} - 2 = \left|\sqrt{x} - 2\right|$$

If $\sqrt{x} \geq 2$ i.e. $x \geq 4$ then $\sqrt{x} - 2 = \sqrt{x} - 2$ which is an identity

$\Rightarrow x \geq 4$ is a solution .

If $\sqrt{x} < 2$ i.e. $0 \leq x < y \Rightarrow$ no solution;

similarly $\sqrt{x} - \left|\sqrt{x} - 2\right| = 2$ gives $x = 1/16$

Hence the solution is $[4, \infty) \cup \left\{\frac{1}{16}\right\}$

➤ **Problem 23.** Solve for x: $\log^2(4-x) + \log(4-x) \cdot \log\left(x + \frac{1}{2}\right) - 2\log^2\left(x + \frac{1}{2}\right) = 0.$

➤ ***Solution*** $\log^2(4-x) + \log(4-x) \cdot \log\left(x + \frac{1}{2}\right) - 2\log^2\left(x + \frac{1}{2}\right) = 0$

let $\log(4-x) = A$ & $\log\left(x + \frac{1}{2}\right) = B$

$$A^2 + AB - 2B^2 = 0$$
$$A^2 + 2AB - AB - 2B^2 = 0$$
$$A(A + 2B) - B(A + 2B) = 0$$
$$A = B \quad \text{or} \quad A = -2B$$

$$\therefore \quad \log(4-x) = \log\left(x + \frac{1}{2}\right)$$

$$\therefore \quad 4 - x = x + \frac{1}{2} \Rightarrow 2x = \frac{7}{2} \Rightarrow x = \frac{7}{4}$$

www.pdfworld.net

www.pdfworld.net

$$\log(4-x) = -2\log\left(x+\frac{1}{2}\right)$$

$$4-x = \frac{1}{(x+(1/2))^2}$$

$$\Rightarrow \quad (4-x)(x^2+\frac{1}{4}+x) = 1$$

$$4x^2 - x^3 + 1 - \frac{x}{4} + 4x - x^2 - 1 = 0$$

$$x^3 - 3x^2 - \frac{15x}{4} = 0$$

$$x(4x^2 - 12x - 15) = 0$$

$x = 0$ or $4x^2 - 12x - 15 = 0$

$$x = \frac{12 \pm \sqrt{144+240}}{8} = \frac{12 \pm \sqrt{384}}{8}$$

$$= \frac{12 \pm 4\sqrt{24}}{8} = \frac{3 \pm \sqrt{24}}{2}$$

Rejecting $x = \frac{3-\sqrt{24}}{2}$ we have $x = \frac{3+\sqrt{24}}{2}$

$$\therefore \quad x = \left\{0, \frac{7}{4}, \frac{3+\sqrt{24}}{2}\right\}$$

➤ **Problem 24.** Find all real numbers x and y satisfying

$\log_3 x + \log_2 y = 2$; $3^x - 2^y = 23$.

➤ ***Solution*** A solution by trial is $x = 3$, $y = 2$ since

$\log_3 3 + \log_2 2 = 2$ and $3^3 - 2^2 = 23$

If $x < 3$, then $\log_3^x < 1$ and since $\log_3 x + \log_2 y = 2$, $\log_2 y > 1$ and $y > 2$ and hence

$3^x < 3^3 = 27$ and $2^y > 2^2 = 4$

$\Rightarrow \quad 3^x - 2^y < 27 - 4 = 23$

So x cannot be less than 3.

If $x > 3$, then $\log_3 x > 1$ and $\log_2 y < 1$ and so $y < 2$

$3^x < 3^3 = 27$, and $2^y < 2^2 = 4$.

So $3^x - 2^y > 27 - 4 = 23$

So x cannot be greater than 3. Thus $x \nless 3$ and $x \ngtr 3$

$\Rightarrow \quad x = 3 \qquad \Rightarrow \qquad y = 2$

So, the only solution for the given equation is $x = 3$ and $y = 2$.

➤ **Problem 25.** Find the value of x, y, z satisfying the equations

$\log_2 x + \log_4 y + \log_4 z = 2$

$\log_9 x + \log_3 y + \log_9 z = 2$

and $\log_{16} x + \log_{16} y + \log_4 z = 2$.

➤ ***Solution*** from (1) $\log_2(x^2yz) = 4$

$\Rightarrow \quad x^2yz = 2^4$ ....(1)

Similarly $y^2zx = 3^4$ ....(2)

$z^2xy = 4^4$ ....(3)

$(1) \times (2) \times (3)$

$\Rightarrow \quad x^4y^4z^4 = (2 \cdot 3 \cdot 4)^4$

$\Rightarrow \quad xyz = 24$

from (1) $\Rightarrow \quad x = \frac{16}{24} = \frac{2}{3}$

from (2) $\Rightarrow \quad y = \frac{81}{24} = \frac{27}{8}$

from (3) $\Rightarrow \quad z = \frac{256}{24} = \frac{32}{3}$

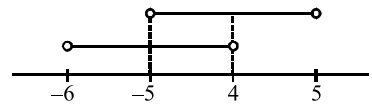
➤ **Problem 26.** Solve the inequation

$$\log_{\left(\frac{25-x^2}{16}\right)}\left(\frac{24-2x-x^2}{14}\right) > 1$$

➤ ***Solution*** The given inequation is valid for

$$\frac{2x-2x-x^2}{14} > 0 \text{ and } \frac{25-x^2}{16} > 0$$

$\therefore \quad x^2 + 2x - 24 < 0$ and $x^2 - 25 < 0$



$\Rightarrow \quad (x+6)(x-4) < 0$ and $-5 < x < 5$

$\Rightarrow \quad -6 < x < 4$ and $-5 < x < 5$

Combining both inequality $-5 < x < 4$ ...(1)

www.pdfworld.net

www.pdfworld.net

Now consider the following cases:

Case A : If $0 < \frac{25 - x^2}{16} < 1 \Rightarrow 9 < x^2 < 25$

$\therefore$ $x \in (-5, -3) \cup (3, 5)$ ...(2)

$\therefore$ From the given inequation

$$\frac{24 - 2x - x^2}{14} < \frac{25 - x^2}{16}$$

$\Rightarrow$ $x^2 + 16x - 17 > 0$

$\therefore$ $x \in (-\infty, -17) \cup (1, \infty)$ ...(3)

Combining (2) and (3), we get

$x \in (3, 5)$ but $-5 < x < 4$ from (1)

$x \in (3, 4)$ ...(4)

Case B : If $\frac{25 - x^2}{16} < 1 \Rightarrow x \in (-3, 3)$ ...(5)

$\therefore$ The given inequation convert in the form

$$\frac{24 - 2x - x^2}{14} > \frac{25 - x^2}{16}$$

$\Rightarrow$ $x^2 + 16x - 17 < 0$

$\therefore$ $x \in (-17, 1)$ ...(6)

combining (5) and (6), we get

$x \in (-3, 1)$ but $-5 < x < 4$ from (1)

$x \in (-3, 1)$

Now combining (4) and (7), the inequation has the final solution : $x \in (-3, 1) \cup (3, 4)$

➤ **Problem 27.** Find all values of the parameter a for which the following inequality is valid for all $x \in R$

$1 + \log_5 (x^2 + 1) \geq \log_5 (ax^2 + 4x + a)$.

➤ ***Solution*** Here $1 \geq \log_5 \frac{ax^2 + 4x + a}{x^2 + 1}$

$\therefore$ we have $5 \geq \frac{ax^2 + 4x + a}{x^2 + 1}$ ...(1)

Also, for logarithms to be defined,

$x^2 + 1 > 0$ and $ax^2 + 4x + a > 0$

$x^2 + 1 > 0$ is true for all $x \in R$.

Now, $ax^2 + 4x + a > 0$ ...(2)

$\therefore$ the values of the parameter a should be such that both (1) and (2) hold for all $x \in R$

Now (2) holds for all $x \in R$ if

$D = 4^2 - 4.\, a\, .\, a < 0$ and coefficient of $x^2 = a > 0$,

i.e., $16 - 4a^2 < 0$, $a > 0$

$\Rightarrow$ $a^2 > 4$, $a > 0$ $\Rightarrow$ $a > 2$

Now, from (1), $5(x^2 + 1) \geq ax^2 + 4x + a$ $\{\because x^2 + 1 > 0\}$

or $(5 - a)x^2 - 4x + 5 - a \leq 0$

This holds for all $x \in R$ if

$D = (-4)^2 - 4(5 - a)(5 - a) \leq 0$

and the coefficient of $x^2 = 5 - a > 0$,

i.e., $16 - 4(5 - a)^2 \leq 0$, $5 - a > 0$

$\Rightarrow$ $(5 - a)^2 \geq 4$, $a < 5$

$\Rightarrow$ $\{5 - a \leq -2$ or $5 - a \geq 2\}$, $a < 5$

$\Rightarrow$ $\{a \geq 7$ or $a \leq 3\}$, $a < 5$

$\Rightarrow$ $a \leq 3$

Hence, the required values of a satisfy $a > 2$ and $a \leq 3$.

$\therefore$ the solution set = (2,3]

➤ **Problem 28.** Solve the equation $x^3 - [x] = 3$, where [x] denotes the integral part of the number x.

➤ ***Solution*** $\because$ $x = [x] + f$, $0 \leq f < 1$

And given equation is $x^3 - [x] = 3$

$\Rightarrow$ $x^3 - (x - f) = 3$

$\Rightarrow$ $x^3 - x = 3 - f$

$\Rightarrow$ $2 < x^3 - x \leq 3$

further for $x \geq 2$, we have

$x^3 - x = x(x^2 - 1) \geq 2\ (4 - 1) = 6 > 3$

for $x < -1$ we have $x^3 - x = x(x^2 - 1) < 0 < 2$;

for $x = -1$, we have $x^3 - x = 0 < 2$;

for $-1 < x \leq 0$ we have $x^3 - x \leq -x < 1$ and

for $0 < x \leq 1$ we have $x^3 - x < x < x^3 \leq 1$

Therefore x must be such that $1 < x < 2$

$\therefore$ $[x] = 1$

Now the original equation can be written as

$x^3 - 1 = 3$

$\Rightarrow$ $x^3 = 4$

$\Rightarrow$ $\therefore$ $x = (4)^{1/3}$ is the solution of the given equation.

Things to Remember

1. Let a, b and m be integers, $m > 0$. Then we say that **a is congruent to b modulo m** if $(a - b)$ is divisible by m. We denote this by $a \equiv b$ (mod. m).
2. If $a + b\sqrt{p} + c\sqrt{q} = 0$ where a, b, c are rationals and $\sqrt{p}$, $\sqrt{q}$ are unlike surds, then $a = 0, b = 0, c = 0$.

www.pdfworld.net

www.pdfworld.net

3. In operations involving square roots of negative numbers, the first thing to do is to express $\sqrt{-b}$ as $i\sqrt{b}$.

4. To prove the truth of the proposition P(n) for all natural n, we must verify the truth of the statement P(1), and then, assuming that the sentence P(k) is true, try to prove the truth of the statement P(k + 1).

5. (i) If a : b = c : d, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$
(Componendo and dividendo)

(ii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....$, then each

$$\frac{a+c+e+.....}{b+d+f+.....} = \frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$$

(iii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....$, then each

$$= \frac{xa+yc+ze+.....}{xb+yb+zf+.....}$$

(iv) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....$, then each

$$= \left(\frac{xa^n+ye^n+ze^n}{xb^n+yd^n+zf^n}\right)^{1/n}$$

6. Remainder theorem : The remainder of the division of the polynomial P(x) by the binomial $(x - \alpha)$ is equal to the value of the polynomial P(x) for $x = \alpha$, i.e. $r = P(\alpha)$
Factor theorem : The polynomial P(x) is exactly divisible by the binomial $(x - \alpha)$ if and only if the value of the polynomial is zero for $x = \alpha$,, i.e. $P(\alpha) = 0$

7. Cross multiplication rule: The solution of the equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ is

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}.$$

8. Quadratic Inequalities

a	D	Inequality	Solution of inequality
$a>0$	$D>0$	$ax^2+bx+c<0$	$(-\infty, \frac{-b-\sqrt{D}}{2a}) \cup (\frac{-b+\sqrt{D}}{2a}, \infty)$
$a>0$	$D>0$	$ax^2+bx+c<0$	$(\frac{-b-\sqrt{D}}{2a}, \frac{-b+\sqrt{D}}{2a})$
$a>0$	$D=0$	$ax^2+bx+c>0$	$(-\infty, \frac{-b}{2a}) \cup (\frac{-b}{2a}, \infty)$
$a>0$	$D=0$	$ax^2+bx+c<0$	no solution
$a>0$	$D<0$	$ax^2+bx+c>0$	$(-\infty, \infty)$
$a>0$	$D<0$	$ax^2+bx+c<0$	no solution
$a<0$	$D>0$	$ax^2+bx+c>0$	$(\frac{-b+\sqrt{D}}{2a}, \frac{-b-\sqrt{D}}{2a})$

9. Irrational Inequalities

(a) An inequation of the form $\sqrt[2n]{f(x)} < g(x)$, $n \in N$ is equivalent to the system

$$\begin{cases} f(x) \geq 0 \\ g(x) > 0 \\ f(x) < g^{2n}(x) \end{cases},$$

(b) An inequation of the form $\sqrt[2n]{f(x)} > g(x)$, $n \in N$ is equivalent to the collection of two systems of inequations :

$$\begin{cases} g(x) \geq 0 \\ f(x) > g^{2n}(x) \end{cases} \text{ and } \begin{cases} g(x) < 0 \\ f(x) \geq 0 \end{cases}$$

www.pdfworld.net

www.pdfworld.net

10. Inequations containing absolute values

(a) The inequation of the form $f(|x|) < g(x)$ is equivalent to the collection of systems

$$\begin{cases} f(x) < g(x), \text{ if } x \geq 0 \\ f(-x) < g(x), \text{ if } x < 0 \end{cases}$$

(b) The inequation of the form $|f(x)| < g(x)$

is equivalent to $\begin{cases} -g(x) < f(x) < g(x) \\ g(x) > 0 \end{cases}$

In particular $|f(x)| < a$
has no solution for $a \leq 0$ and for $a > 0$
it is equivalent to $-a < f(x) < a$

(c) The inequation of the form $|f(x)| > g(x)$ is equivalent to the

$$\begin{cases} f(x) > g(x), \text{ if } g(x) < 0 \\ -f(x) > g(x), \text{ if } g(x) < 0 \end{cases}$$

$$\Rightarrow \begin{cases} f(x) < -g(x) \text{ or } f(x) > g(x), & \text{if } g(x) > 0 \\ \text{always true}, & \text{if } g(x) < 0 \end{cases}$$

In particular $|f(x)| > a$ has no solution for $a \leq 0$ and for $a > 0$ is equivalent to $f(x) < -a$, $f(x) > a$

(d) The inequation of the form $|f(x)| \geq |g(x)|$ is equivalent to the system $f^2(x) \geq g^2(x)$

11. Properties of Modulus

(a) $|a + b| \leq |a| + |b|$
(b) $|a + b| = |a| + |b|$ only if $ab \geq 0$
(c) $|a - b| \geq ||a| - |b||$
(d) $|a - b| = |a| - |b|$ only if $ab \geq 0$
(e) $|a + b + c| \leq |a| + |b| + |c|$
(f) $|a + b + c| = |a| + |b| + |c|$ only if $ab \geq 0$, $bc \geq 0$ and $ca \geq 0$
(g) $|a \pm b \pm c| \leq |a| + |b| + |c|$

12. Transformations of graph

(a) The graph of function $y = -f(x)$ is obtained from the graph of the function $y = f(x)$ by reflection of the graph of $y = f(x)$ about the x–axis.

(b) The graph of function $y = f(-x)$ is obtained from the graph of the function $y = f(x)$ by reflection of the graph of $y = f(x)$ about the y–axis.

(c) The graph of function $y = |f(x)|$ is obtained from the graph of the function $y = f(x)$ as follows : the portion of the graph of the function $y = f(x)$ lying above the x–axis remains unchanged and its portion located below the x–axis, is reflected above the x-axis.

(d) The graph of function $y = f(|x|)$ is obtained from the graph of the function $y = f(x)$ as follows :
For $x \geq 0$, the graph of the function is retained, and this retained portion of the graph is reflected symmetrically about the y–axis, thus determining the graph of the function for $x \leq 0$.

(e) The graph of function $|y| = f(x)$ is obtained from the graph of the function $y = f(x)$ as follows :
First delete its portion located below the x–axis. The portion of the graph of the function $y = f(x)$ lying above the x–axis remains unchanged. Then take the image of the positive part of $f(x)$ about the x–axis considering it as a plane mirror.

(f) The graph of function $|y| = |f(x)|$ is obtained from the graph of the function $y = f(x)$ as follows :
First draw the graph of $y = f(x)$. Then take the image of the graph of $f(x)$ about the x–axis considering it as a plane mirror.The union of the graphs is the required graph.

13. (a) $\max(f(x), g(x)) = \dfrac{f(x)+g(x)}{2} + \left|\dfrac{f(x)-g(x)}{2}\right|$

(b) $\min(f(x), g(x)) = \dfrac{f(x)+g(x)}{2} - \left|\dfrac{f(x)-g(x)}{2}\right|$

14. Laws of logarithm

(i) $\log_a MN = \log_a M + \log_a N$

(ii) $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

(iii) $\log_a (M^p) = p \log_a M$

(iv) $\log_{a^q} M = \dfrac{1}{q} \log_a M$

(v) $\log_b M = \dfrac{\log_a M}{\log_a b}$

www.pdfworld.net

www.pdfworld.net

(vi) $\log_b a = \dfrac{1}{\log_a b}$

(vii) $a^{\log_a x} = x$

(viii) $x^{\log_a b} = b^{\log_a x}$

15. Exponential Inequalities

Depending on the values of the parameters a and b, the set of solutions of the inequality $a^x > b$ can be in the following forms :

(i) $x \in (\log_a b, \infty)$ for $a > 1, b > 0$;

(ii) $x \in (-\infty, \log_a b)$ for $0 < a < 1, b > 0$;

(iii) $x \in R$ for $a > 0, b < 0$;

Depending on the values of a and b, the set of solutions of the inequality $a^x < b$ can be in the following forms :

(i) $x \in (-\infty, \log_a b)$ for $a > 1, b > 0$;

(ii) $x \in (\log_a b, \infty)$ for $0 < a < 1, b > 0$;

(iii) $x \in \phi$ for $a > 0, b < 0$ (i.e. the inequality has no solutions)

16. Logarithmic Inequalities

An inequality of the form $\log_a f(x) > b$ is equivalent to the following systems of inequalities :

(i) $f(x) > 0, f(x) > a^b$ for $a > 1$

(ii) $f(x) > 0. f(x) < a^b$ for $a < 1$,

and an inequality of the form $\log_a f(x) < b$ is equivalent to the following systems of inequalities :

(i) $f(x) > 0, f(x) < a^b$ for $a > 1$;

(ii) $f(x) > 0, f(x) > a^b$ for $0 < a < 1$.

17. Properties of Greatest Integer Function

(i) $[x] \le x < [x] + 1$

(ii) $x - 1 < [x] \le x, \quad 0 \le x - [x] < 1$

(iii) $[x + m] = [x] + m$ if m is an integer.

(iv) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(v) $[-x] = -[x], x \in I$
$= -[x] - 1, x \notin I$

(vi) $[x] \ge n \Rightarrow x \ge n, n \in I$

(vii) $[x] > n \Rightarrow x \ge n + 1, n \in I$

(viii) $[x] \le n \Rightarrow x < n + 1, n \in I$

(ix) $[x] < n \Rightarrow x < n, n \in I$

(x) $\left[\dfrac{[x]}{m}\right] = \left[\dfrac{x}{m}\right]$ if m is a positive integer

(xi) $[x] + \left[x + \dfrac{1}{n}\right] + \left[x + \dfrac{2}{n}\right] + \ldots$
$\ldots + \left[x + \dfrac{n-1}{n}\right] = [nx]$

(xii) If n and a are positive integers, [n/a] is the number of integers among 1,2....., n that are divisible by a.

(xiii) **de Polignac's Formula :** Let E be the largest exponent of p such that p^E divides n!.

Then $E = \sum_{i=1}^{\infty} \left[\dfrac{n}{p^i}\right]$

18. Properties of Fractional Part Function

(i) $0 \le \{x\} < 1$

(ii) $\{x + m\} = \{x\}$ if m is an integer.

(iii) $\{-x\} = 1 - \{x\}, x \notin I$
$= 0, x \in I$

19. Properties of Least Integer Function :

(i) $(x) = [x]$, if $x \in I$ and $(x) = [x] + 1$, if $x \notin I$, in fact $(x) = -[-x]$

(ii) $(x + n) = n + (x), n \in I$

(iii) $(-x) = -(x), x \in I$
$(-x) = -(x) + 1, x \notin I$

[Objective Exercises]

Single Correct Answer Type for JEE Main

1. STATEMENT - 1 : If $\dfrac{1}{|a|} > \dfrac{1}{b}$, then $|a| < b$, where a, b are non-zero real numbers.

STATEMENT - 2 : If $\dfrac{1}{a} > \dfrac{1}{|b|}$, then $a < |b|$, where a & b are non-zero real numbers.

(A) Statement-1 and Statement-2 both are correct

(B) Statement-1 is correct and Statement-2 is incorrect

(C) Statement-1 is incorrect and Statement-2 is correct

(D) Statement-1 and Statement-2 both are incorrect

www.pdfworld.net

www.pdfworld.net

2. Which one of the following is true?
(A) Some integers are not rational numbers.
(B) Irrational numbers cannot be located on a real number line because in decimal form they neither terminate nor repeat.
(C) The graph of the union of two intervals can never be the same as the graph of the intersection of the same two intervals.
(D) If $x < -1$, then $\frac{|x+1|}{x+1}$ simplifies to -1.

3. The number of pairs of natural numbers, the difference of whose square is 45, is
(A) 3 (B) 4 (C) 5 (D) 7

4. If the number 497A is divisible by 3 and the number 562B is divisible by 4 the number of possible ordered pairs (A, B) is (where A and B are digits)
(A) 6 (B) 9 (C) 2 (D) 4

5. If $4x^4 - (a-1)x^3 + ax^2 - 6x + 1$ is divisible by $(2x-1)$, then 'a' is equal to
(A) 13 (B) –13 (C) 11 (D)–11

6. The values of p so that the equation $3x - p + 1 = 2x - 1$ has positive solution are
(A) $p \geq \frac{2}{3}$ (B) $p < -2$
(C) $p > 2$ (D) $p > 1$

7. If $x = 3 - \sqrt{8}$, then $x^3 + \frac{1}{x^3}$ is equal to
(A) 6 (B) 198
(C) $6\sqrt{2}$ (D) 102

8. Which one of the following is true ?
(A)The solution set to $(2x-3)(x+2) < 0$ is $(-\infty, -2) \cup (1.5, \infty)$
(B) Every quadratic inequality contains atleast one real number in its solution set.
(C) The solution set to $\frac{x-3}{(x+4)^2} > 0$ is $(3, \infty)$
(D) None of these statements is true.

9. If a is negative and is not equal to –2, then the equation $x^2 + a|x| + 1 = 0$
(A) cannot have any real roots
(B) must have either exactly four real roots or no real roots
(C) must have exactly two real roots
(D) must have either exactly two real roots or no real roots

10. The inequality $\frac{|x|^2 - |x| - 2}{2|x| - |x|^2 - 2} > 0$ holds if and only if
(A) $-1 < x < -\frac{2}{3}$ or $\frac{2}{3} < x < 1$
(B) $-1 < x < 1$
(C) $\frac{2}{3} < x < 1$
(D) $x > 1$ or $x < -1$ or $-\frac{2}{3} < x < \frac{2}{3}$

11. Number of real solution(s) of the equation $|x-3|^{3x^2 - 10x + 3} = 1$ is
(A) exactly four (B) exactly three
(C) exactly two (D) exactly one

12. The number of integral solutions (x, y) of the equation $|2x - y| + |x - y| = 1$, is
(A) 2 (B) 4
(C) 8 (D) None

13. The solution set of $x - \sqrt{1 - |x|} < 0$, is
(A) $\left[-1, \frac{-1+\sqrt{5}}{2}\right)$ (B) $[-1, 1]$
(C) $\left[-1, \frac{-1+\sqrt{5}}{2}\right]$ (D) $\left(-1, \frac{-1+\sqrt{5}}{2}\right)$

14. $7\log_{10}\frac{16}{15} + 5\log_{10}\frac{25}{24} + 3\log_{10}\frac{81}{80}$ equals
(A) $\log_{10} 2$ (B) $\log_{10} 3$
(C) $\log_{10} 5$ (D) zero

15. If $a = \log_{24} 12$, $b = \log_{36} 24$, $c = \log_{48} 36$ then $abc + 1$ equals
(A) 2ac (B) 2bc
(C) 2ab (D) None

www.pdfworld.net

www.pdfworld.net

16. Let p be the integral part of $\log_3 108$ and q be the integral part of $\log_5 375$ then $p + q - pq$ has the value equal to
(A) -5 (B) 5
(C) 7 (D) None

17. The equation $\log_{x^2} 16 + \log_{2x} 64 = 3$ has
(A) one irrational solution
(B) no prime solution
(C) two real solutions
(D) one integral solution

18. The number of ordered pair(s) satisfying simultaneously, the system of equations,
$2^{\sqrt{x}+\sqrt{y}} = 256$ & $\log_{10}\sqrt{xy} - \log_{10} 1.5 = 1$,
is
(A) zero (B) exactly one
(C) exactly two (D) more than two

19. Let, $N = \dfrac{81^{\frac{1}{\log_5 9}} + 3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}$.

$\left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25} 7}} - 125^{\log_{25} 6}\right)$ then $\log_2 N$ has the value =
(A) 0 (B) 1 (C) -1 (D) None

20. The equation, $\log_2(2x^2) + \log_2 x \cdot x^{\log_x(\log_2 x + 1)} + \dfrac{1}{2}\log_4 x^4 + 2^{-3\log_{1/2}(\log_2 x)} = 1$
has
(A) exactly one real solution
(B) two real solutions
(C) 3 real solutions
(D) no solution

21. The number of zeros after decimal before a significant figure in $(75)^{-10}$ is
(use $\log_{10} 2 = 0.301$ & $\log_{10} 3 = 0.477$)
(A) 20 (B) 19
(C) 18 (D) none

22. If $a = \log_x yz$, $b = \log_y zx$ & $c = \log_z xy$ where x, y, z are positive reals not equal to unity, then $abc - a - b - c$ is equal to
(A) 2 (B) 1
(C) -1 (D) zero

23. The number of real solutions of the equation $\log_{10}(7x-9)^2 + \log_{10}(3x-4)^2 = 2$ is
(A) 1 (B) 2 (C) 3 (D) 4

24. If $5x^{\log_2 3} + 3^{\log_2 x} = 162$ then logarithm of x to the base 4 has the value equal to
(A) 2 (B) 1
(C) -1 (D) 3/2

25. Assuming all logarithms to be well defined, the value of
$\dfrac{1}{\log_{bc^2} abc} + \dfrac{1}{\log_{ca^2} abc} + \dfrac{1}{\log_{ab^2} abc}$ equals
(A) 3 (B) 2
(C) 1/2 (D) 3/2

26. The solution set of the equation $\log_{10}(3x^2 + 12x + 19) - \log_{10}(3x + 4) = 1$ is
(A) a null set (B) a singleton
(C) a set consisting of exactly two elements
(D) a set consisting of more than two elements

27. If x_1, x_2 & x_3 are the three real solutions of the equation
$x^{\log^2_{10} x + \log_{10} x^3 + 3} = \dfrac{2}{\frac{1}{\sqrt{x+1}-1} - \frac{1}{\sqrt{x+1}+1}}$
where $x_1 > x_2 > x_3$, then
(A) $x_1 + x_3 = 2x_2$ (B) $x_1 \cdot x_3 = x_2^2$
(C) $x_2 = \dfrac{2x_1x_2}{x_1 + x_2}$ (D) $x_1^{-1} + x_2^{-1} = x_3^{-1}$

28. The solution set of the inequality,
$2 - \log_2(x^2 + 3x) \geq 0$ is
(A) $[-4, 1]$
(B) $[-4, -3) \cup (0, 1]$

www.pdfworld.net

www.pdfworld.net

(C) $(-\infty, -3) \cup (1, \infty)$
(D) $(-\infty, -4) \cup [1, \infty)$

29. The set of values of x satisfying the inequality $\frac{1}{\log_4 \frac{x+1}{x+2}} \leq \frac{1}{\log_4 (x+3)}$ is

(A) $(-3, -2)$
(B) $(-3, -2) \cup (-1, \infty)$
(C) $(-1, \infty)$
(D) None

30. If $x_n > x_{n-1} > \ldots\ldots > x_2 > x_1 > 1$, then the value of $\log_{x_1} \log_{x_2} \log_{x_3} \ldots\ldots \log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{\cdot^{x_1}}}}}$ is

(A) 1 (B) 0
(C) 2 (D) None

31. If $\log_e 2 \log_b 625 = \log_{10} 16 \log_e 10$, then b is
(A) 4 (B) 5 (C) 6 (D) 7

32. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$, $z = 1 + \log_c ab$, then $\frac{xyz}{xy + yz + zx}$ is equal to

(A) 0 (B) 1 (C) – 1 (D) 2

33. If $(\log_5 x)(\log_x 3x)(\log_{3x} y) = \log_x x^3$, then y equals
(A) 125 (B) 25
(C) $\frac{5}{3}$ (D) 243

34. If $\frac{\log_e a}{y - z} = \frac{\log_e b}{z - x} = \frac{\log_e c}{x - y}$, then $a^{y^2+yz+z^2}\ b^{z^2+zx+y^2}\ c^{x^2+xy+y^2}$ is equal to
(A) 2 (B) 0 (C) 1 (D) None

35. The solution set of the equation $\log_x 2 \log_{2x} 2 = \log_{4x} 2$ is

(A) $\left\{2^{-\sqrt{2}}, 2^{\sqrt{2}}\right\}$ (B) $\left\{\frac{1}{2}, 2\right\}$
(C) $\left\{\frac{1}{4}, 2^2\right\}$ (D) None of these

36. The number of solutions of equation $\log_3 x \log_4 x \log_5 x = \log_3 x \log_4 x + \log_5 x$ are
(A) 2 (B) 3
(C) 1 (D) None

37. If $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, then the number of integers in 5^{200} are
(A) 120 (B) 145
(C) 160 (D) None

38. The set of real values of x for which $\log_{0.2} \frac{x+2}{x} \leq 1$ is

(A) $\left(-\infty, \frac{-5}{2}\right] \cup (0, \infty)$
(B) $\left[\frac{5}{2}, +\infty\right)$
(C) $(-\infty, -2) \cup (0, \infty)$
(D) None of these

39. If $\log_{1/2} \frac{x^2 + 6x + 9}{2(x+1)} < -\log_2(x+1)$, then x belongs to the interval
(A) $(-1, 1 + 2\sqrt{2})$ (B) $(2, 1 - 2\sqrt{2})$
(C) $(-1, \infty)$ (D) None of these

40. If $\log_{(x+5/2)} \left(\frac{x+5}{2x-3}\right)^2 < 0$, then range of x is
(A) $(-2, -3/2) \cup (8/3, \infty) - \{5\}$
(B) $(-2, 5)$

www.pdfworld.net

www.pdfworld.net

(C) $\left(\frac{-1}{5}, \frac{2}{3}\right) \cup (8/3, \infty)$

(D) None of these

41. If $\log_{\left(\frac{x^2-12x+30}{10}\right)}\left(\log_2 \frac{2x}{5}\right) > 0$, then x belongs to interval

(A) $(2, 6 + \sqrt{6}) \cup (10, \infty)$

(B) $(2, 6 - \sqrt{6}) \cup (10, \infty)$

(C) $(6\sqrt{6} + \sqrt{6}) \cup (10, \infty)$

(D) None of these

42. The set of value of x for which

$\log_{-4x^2+12x-8}|4x - 5| > 0$ is

(A) (1, 5/4) ∪ (5/4, 3/2)

(B) $\left(1, \frac{3}{2}\right) \cup \left(\frac{3}{2}, 2\right)$

(C) (1, 5/4)

(D) None of these

43. The solution set of the inequality

$\log_3(x^2 - 2) < \log_3\left(\frac{3}{2}|x| - 1\right)$ is

(A) $(-2, -\sqrt{2}) \cup (\sqrt{2}, 2)$

(B) $(2, \sqrt{2}) \cup (\sqrt{3}, 3)$

(C) (1, 5/4)

(D) None of these

44. Let a function f satisfy $x - f(x) = 8\left[\frac{x}{8}\right] - 5\left[\frac{f(x)}{5}\right]$ for all x, where $11 < f(10) < 14$, then the value of f(10) is

(A) 10 (B) 12

(C) 14 (D) None

45. If $[x]^2 - 5[x] + 6 = 0$, where [.] denotes the greatest integer function, then -

(A) $x \in [3, 4)$ (B) $x \in [2, 3]$

(C) $x \in \{2, 3\}$ (D) None of these

46. The solutions of the equation $2x - 2[x] = 1$, where [x] is the greatest integer less than or equal to x, are

(A) $x = n + \frac{1}{2}, n \in N$ (B) $x = n - \frac{1}{2}, n \in N$

(C) $x = n + \frac{1}{2}, n \in I$ (D) $x = n - \frac{1}{2}, n \in I$

47. The complete solution set of the equation $(x + 1)^2 + [x - 1]^2 = (x - 1)^2 + [x + 1]^2$, where [x] and (x) are the greatest integer and nearest integer to x, is

(A) $x \in R$ (B) $x \in N$

(C) $x \in I$ (D) $x \in Q$

48. If $g(x) = [x^2] - [x]^2$, where [.] denotes the greatest integer function, and $x \in [0, 2]$ then the set of values of g(x) is

(A) {–1, 0} (B) {–1, 0, 1}

(C) {0} (D) {0, 1, 2}

49. If a, b, $c \in R$ and $a^2 + b^2 + c^2 = 1$, then the possible values of [ab + bc + ca] are

(A) {–1, 0, 1} (B) {0, 1}

(C) {1, 2} (D) None of these

50. The positive values of x satisfying the equations

$[x]\left[x + \frac{1}{2}\right] = 1$ are

(A) $\frac{1}{2} \le x < \frac{3}{2}$ (B) $1 \le x < 2$

(C) $1 \le x < \frac{3}{2}$ (D) none of these

www.pdfworld.net

www.pdfworld.net

Multiple Correct Answer Type for JEE Advanced

51. Which one of the following is/ are false ?
(A) There are at least two whole numbers that are not natural numbers.
(B) Using interval notation, the set $\{x| x < 6\}$ can be expressed as $[-\infty, 6)$
(C) If a is a positive number and b is a negative number, then $|a + b| = |a| + |b|$.
(D) The least natural number can be expressed as $\left|\frac{27-5}{5-27}\right|$

52. Which of the following is/are false ?
(A) The solution set to $x^2 > 25$ is $(5, \infty)$.
(B) The inequality $\frac{x-2}{x+3} < 2$ can be solved by multiplying both sides by $x + 3$, resulting in the equivalent inequality $x - 2 < 2(x + 3)$.
(C) $(x + 3)(x - 1) > 0$ and $\frac{x+3}{x-1} > 0$ have the same solution set.
(D) $\log(x + 2y) = \log x + \log y^2$

53. Which of the following fractions donot change in value if x and y are doubled?
(A) $\frac{2x+3y}{3y-x}$ (B) $\frac{2x^2-3y^2}{x+y}$
(C) $\frac{2x^2-3y^2}{x^2+y^2}$ (D) None of these

54. If $\left(a^{\log_b x}\right)^2 - 5\, x^{\log_b a} + 6 = 0$, where $a > 0$, $b > 0$ & $ab \neq 1$, then the value of x can be equal to
(A) $2^{\log_b a}$ (B) $3^{\log_a b}$
(C) $b^{\log_a 2}$ (D) $a^{\log_b 3}$

55. The polynomial $x^5 - x^4 - 2x^3 + 2x^2 + x - 1$ is divisible by
(A) $x^2 - 1$ (B) $(x + 1)^2$
(C) $(x - 1)^3$ (D) $x^3 + 1$

56. The roots of the equation
$(a + \sqrt{b})^{x^2-15} + (a - \sqrt{b})^{x^2-15} = 2a$,
where $a^2 - b = 1$ are
(A) ± 2 (B) ± 4
(C) $\pm\sqrt{14}$ (D) $\pm\sqrt{3}$

57. Which of the following numbers are non positive ?
(A) $5^{\log_{11} 7} - 7^{\log_{11} 5}$
(B) $\log_3\left(\sqrt{7} - 2\right)$
(C) $\log_7\left(\frac{1}{2}\right)^{-1/2}$
(D) $\log_{\sqrt{2}-1}\frac{\sqrt{2}+1}{\sqrt{2}-1}$

58. The number $N = \frac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \log_6^2 2$ when simplified reduces to :
(A) a prime number
(B) an irrational number
(C) a real which is less than $\log_3 \pi$
(D) a real which is greater than $\log_7 6$

59. If $p, q \in N$ satisfy the equation $x^{\sqrt{x}} = \left(\sqrt{x}\right)^x$ then p & q are :
(A) relatively prime
(B) twin prime
(C) coprime
(D) if $\log_q^p$ is defined then $\log_p^q$ is not & vice versa

60. Which of the following when simplified, reduces to unity ?
(A) $\log_{10} 5 \cdot \log_{10} 20 + \log_{10}^2 2$
(B) $\frac{2\log 2 + \log 3}{\log 48 - \log 4}$
(C) $-\log_5 \log_3 \sqrt{\sqrt[5]{9}}$
(D) $1/6 \log_{\frac{\sqrt{3}}{2}}\left(\frac{64}{27}\right)$

www.pdfworld.net

www.pdfworld.net

61. The equation $\frac{\log_8\left(\frac{8}{x^2}\right)}{\left(\log_8 x\right)^2} = 3$ has

(A) no integral solution
(B) one natural solution
(C) two real solutions
(D) one irrational solution

62. Which of the following statement(s) is/are true?

(A) $\log_{10} 2$ lies between $\frac{1}{3}$ & $\frac{1}{4}$

(B) $\log_{10} 1 + \frac{1}{2}\log_{10} 3 + \log_{10}\left(2+\sqrt{3}\right) = \log\left(1+\sqrt{3}+\left(2+\sqrt{3}\right)\right)$

(C) $e^{\ln(\ln 3)}$ is smaller than 1

(D) $\log_{\text{cosec}\,(5\pi/6)} \cos\frac{5\pi}{3} = -1$

63. If $\ln(x+z)+\ln(x-2y+z)=2\ln(x-z)$, then:

(A) $y=\frac{2xz}{x+z}$ (B) $y^2=xz$

(C) $2y=x+z$ (D) $\frac{x}{z}=\frac{x-y}{y-z}$

64. Let $N=\frac{\log_3 135}{\log_{15} 3}-\frac{\log_3 5}{\log_{405} 3}$. Then N is

(A) a natural number
(B) a prime number
(C) a rational number
(D) an integer

65. Which of the following when simplified reduces to an integer?

(A) $\frac{2\log 6}{\log 12+\log 3}$ (B) $\frac{\log 32}{\log 4}$

(C) $\frac{\log_5 16-\log_5 4}{\log_5 128}$ (D) $\log_{1/4}\left(\frac{1}{16}\right)^{-2}$

66. The expression, $\log_p \log_p \underbrace{\sqrt[p]{\sqrt[p]{\sqrt[p]{.....\sqrt[p]{p}}}}}_{\text{n radical sign}}$ where $p \geq 2$, $p \in N$, when simplified is :

(A) independent of p, but dependent on n
(B) independent of n, but dependent of p
(C) dependent on both p & n
(D) negative .

67. If $\left(\sqrt{5\sqrt{2}-7}\right)^x + 6\left(\sqrt{5\sqrt{2}+7}\right)^x = 7$, then the value of x can be equal to

(A) 0

(B) $\log_{\left(5\sqrt{2}-7\right)} 36$

(C) $-\left(\log_6\sqrt{5\sqrt{2}+7}\right)^{-1}$

(D) $\log_{\sqrt{5\sqrt{2}-7}} 6$

68. It is given that x = 9 is a solution of the equation

$\log_e(x^2+15a^2)-\log_e(a-2)=\log_e\left(\frac{8ax}{a-2}\right)$,

then

(A) a = 3
(B) a = 3/5
(C) other solution is x = 15
(D) other solution is x = 3/5

69. The interval of x in which the inequality $5^{1/4\log_5^2 x} > 5x^{1/5\log_5 x}$ holds true

(A) $(0, 5^{-2\sqrt{5}}]$ (B) $[5^{2\sqrt{5}}, \infty)$

(C) $(5^{-2\sqrt{5}}, 5^{2\sqrt{5}}]$ (D) None of these

70. The number of integers satisfying the inequality $[|x-1|]+[|x-4|]\leq 3$ is

(A) less than 2 (B) more than 3
(C) 4 (D) more than 5

Comprehension - 1

Consider the number N = 7 7 4 9 5 8 P 9 6 Q

71. If P = 2 and the number N is divisible by 3, then the number of possible values of Q is

(A) 0 (B) 2 (C) 3 (D) 4

www.pdfworld.net

www.pdfworld.net

72. If N is divisible by 4, then
(A) P can be any integer and Q = 0, 2, 4, 6, 8
(B) P can be any rational number and Q = 0, 4, 8
(C) P can be any single digit whole number and Q = 0, 4, 8
(D) P can be any real number and Q = 0, 4, 8

73. If N is divisible by 8 and 9 both, then the number of possible ordered pairs (P, Q) is
(A) 3 (B) 2 (C) 1 (D) 4

Comprehension - 2

The Cantor Set is constructed as follows. We start with the closed interval [0, 1] and remove the open interval (1/3, 2/3). That leaves the two intervals [0, 1/3] and [2/3, 1] and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding set. The Cantor Set consists of the numbers that remain in [0, 1] after all those intervals have been removed.

74. A number not lying in the Cantor Set is
(A) $\frac{2}{9}$ (B) $\frac{8}{9}$ (C) $\frac{2}{27}$ (D) $\frac{1}{6}$

75. The series representing the sum 'S' of the total lengths of all intervals that are removed is
(A) A.P. (B) G.P.
(C) A.G.P (D) None

76. The value of S is
(A) $\frac{8}{9}$ (B) $\frac{1}{3}$
(C) 1 (D) None

Comprehension - 3

Consider the equation $|2x| - |x - 4| = x + 4$

77. The least integer satisfying the equation, is
(A) –4 (B) 4 (C) 5 (D) –5

78. The total number of prime numbers less than 20, satisfying the equation, is
(A) 3 (B) 4 (C) 5 (D) 6

79. If P is the greatest composite number less than 34 satisfying the given equation, then P^{2007} has the digit in its units place
(A) 8 (B) 1 (C) 7 (D) 0

Comprehension - 4

Let $x^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ and let

$x^- = \begin{cases} -x & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$

80. The incorrect statement is
(A) $x = x^+ - x^-$ (B) $x^+ = \frac{|x| + x}{2}$
(C) $|x| = x^+ + x^-$ (D) None of these

81. The number of solutions of the equation $(x-1)^+ + (x-1)^- = 2x^+ - 2x^-$ is
(A) zero (B) one
(C) two (D) infinite

82. The complete solution set of the inequality $(x+1)^+ - 2(x-2)^- \geq 0$ is
(A) $[1, \infty)$ (B) $[2, \infty)$
(C) $(-\infty, -1]$ (D) None of these

Comprehension - 5

A function f from the positive integers to the positive integers has the properties

(i) $f(1) = 1$
(ii) $f(n) = 2$ if $n \geq 100$
(iii) $f(x) = f\left(\frac{n}{2}\right)$ if n is even and $n < 100$
(iv) $f(n) = f(n^2 + 7)$ if n is odd and $n > 1$.

83. The value of f(12) is
(A) 1
(B) 2
(C) cannot be determined
(D) None of these

www.pdfworld.net

www.pdfworld.net

84. The number of positive integers for which the stated properties require that f(n) = 1 is
(A) 12 (B) 18
(C) 22 (D) None

85. The number of positive integers for which the stated properties do not determine f(n) is
(A) 81 (B) 77
(C) 63 (D) None

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

86. **Assertion (A):** There are three natural numbers x less than 100 so that $x^2 - x$ is divisible by 100.
Reason (R): Since $x^2 - x = x(x - 1)$ and the numbers x and x – 1 are relatively prime, one of them must be divisible by 4 and one of them must be divisible by 25 for $x^2 - x$ to be divisible by 100.

87. **Assertion (A):** Let $x, y, z \in N$ such that $S = x^4 + y^4 + x^4$ is divisible by 3 then S is also divisible by 81.
Reason (R): if x, y, z each are divisible by 3 then $S = x^4 + y^4 + z^4$ is clearly divisible by 3^4.

88. **Assertion (A):** Let f be a polynomial of degree 2 with integer coefficients. Suppose that f(k) is divisible by 5 for every integer k, then all coefficients of f are divisible by 5.
Reason (R): Let $f(x) = ax^2 + bx + c$. If f(0), f(1), f(–1) are divisible by 5 then c, a + b + c, a – b + c are divisible by 5. This means that 2a, 2b, c are divisible by 5.

89. **Assertion (A):** The rational numbers x and y written in lowest terms have denominators 60 and 70 respectively. The smallest possible denominator of x + y is 84
Reason (R): Let $x + y = \frac{a}{60} + \frac{b}{70} = \frac{7a + 6b}{420}$.
Since (a, 60) and (b, 70) are co-prime pairs, none of the primes 2, 3, 7 can divide 7a + 6b. Since $420 = 2^2 . 3 . 7 . 5$, then denominator will be atleast $2^2 . 3 . 7 = 84$.

90. **Assertion (A):** The number $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$ simplifies to 1.
Reason (R): Observe that $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$ a real root of the cubic equation $x^3 + 3x - 4 = 0$ and the given equation has only one real root.

91. **Assertion (A):** The smallest possible value of $|x - 1| + |y - x| + |y - 2010|$ for any real number x and y is 2009.
Reason (R): $|p - s| \le |p - q| + |q - r| + |r - s|$

92. **Assertion (A):** There are exactly 5 integer in the range of the function $f(x) = \frac{6}{1 + 7e^x}$.
Reason (R): If the range of a function is [m, M] then the number of integers in the range is [M – m] where [.] is the G.I.F.

93. **Assertion (A):** The system of equation $|x + y| = 99$, $|x - y| = c$ has four real solutions (x, y) if $c \ge 0$.
Reason (R): If c is positive, we have $x + y = \pm 99$, $x - y = \pm c$ which gives four solutions (x, y).

94. **Assertion (A):** The smallest integer n such that $\sqrt{n+99} - \sqrt{n} < 1$ is 2402
Reason (R): $\sqrt{n+99} < \sqrt{n} + 1 \Rightarrow n > 49^2 \Rightarrow n \ge 49^2 + 1$

www.pdfworld.net

www.pdfworld.net

95. **Assertion (A):** Let $g(x) = |x + 1| + |x - 1|$ and $f(x) = \dfrac{\sqrt{x^2 + 3xg(x) + g^2(x) - 2x - 2g(x) + 4}}{x.g(x) + 4}$.

Then f(x) is constant function for $-1 \le x \le 1$.
Reason (R): f(x) is dependent on x in $-1 \le x \le 1$ even if $g(x) = 2$ over this interval.

Match the Columns for JEE Advanced

96. Consider the equation $\big||x - 1| - 1\big| = px + c$. Match the number of solutions with the values of p and c.

Column-I	**Column-II**
(A) One solution	(P) $p = \frac{1}{2}, c = 1$
(B) Two solutions	(Q) $p = -\frac{1}{2}, c = 1$
(C) Three solutions	(R) $\frac{-3}{4} < p < \frac{-1}{2}, c = \frac{3}{2}$
(D) More than three solutions	(S) $p = 1, -2 < c < 0$
	(T) $-2 < p < -1, -1 < c < 0$

97. Match the Statements/Expressions in **Column-I** with the Statements/Expressions in **Column-II.**

Column - I	**Column - II**
(A) If $a^2 - 4a + 1 = 4$, then the value of $\dfrac{a^3 - a^2 + a - 1}{a^2 - 1}$ $(a^2 \ne 1)$ is equal to	(P) 1
(B) The value(s) of x satisfying the equation $\sqrt[4]{|x - 3|^{x+1}} = \sqrt[3]{|x - 3|^{x-2}}$ is	(Q) 2
(C) The value(s) of x satisfying the equation $3^x + 1 - |3^x - 1| = 2\log_5 |6 - x|$ is	(R) 4
(D) An element in the solution set of $\sqrt{x^2 - 4x} + 4 < 3$ and $\frac{1}{4} \le \frac{1}{3 - x} \le \frac{1}{2}$, is	(S) 11
	(T) 0

www.pdfworld.net

www.pdfworld.net

98.

	Column-I	Column-II
(A)	The integral solutions of the equation $(4-x)^{4-x} + (5-x)^{5-x} + 10 = 4^x + 5^x$ lie in	(P) [0, 1]
(B)	If [.] represents the G.I.F. then all the values of $[2x] - 2[x]$ lies in	(Q) [1, 3)
(C)	If a polynomial function f satisfies $f(x + 2f(y)) = f(x) + y + f(y)$ for all real x and y then the value of f(0) lies in	(R) [–1, 1]
		(S) [–1, 0]
(D)	All the values of 'a' for which $\frac{x-2a-3}{x-a+2} < 0$ for all $x \in [1, 2]$ lies in	(T) $\left(-\frac{1}{2}, 3\right)$

99.

	Column - I	Column - II
(A)	When the repeating decimal 0.363636.....is written as a rational fraction in the simplest form, the sum of the numerator and denominator is	(P) 4
(B)	Given positive integer p, q and r with $p = 3^q \cdot 2^r$ and $100 < p < 1000$. The difference between maximum and minimum values of (q + r), is	(Q) 8
(C)	If $\log_8 a + \log_8 b = (\log_8 a)(\log_8 b)$ and $\log_a b = 3$, then the value of 'a' is	(R) 15
(D)	Let $N = (2+1)(2^2+1)(2^4+1).....(2^{32}+1)+1$ then $\log_{256} N$ is	(S) 16

100.

	Column-I	Column-II
(A)	If x_1 and x_2 satisfy the equation $x^{\log_{10} x} = 100\,x$ then the value of $x_1 x_2$ equals	(P) irrational
(B)	Sum of the squares of the roots of the equation $\log_2(9 - 2^x) = 3 - x$ is	(Q) rational
(C)	If $\log_{1/8}(\log_{1/4}(\log_{1/2} x)) = 1/3$ then x is	(R) prime
(D)	If $\log_b a = 3$, $\log_b c = -4$ and if the value of x satisfying the equation $a^{3x} = c^{x-1}$ is expressed in the form p/q, where p and q are relatively prime then p + q is	(R) composite

Review Exercises for JEE Advanced

1. Solve the following equations for x :

(i) $x + \sqrt{a + \sqrt{x}} = a$.

(ii) $x^2 - \sqrt{a - x} = a$.

2. Solve the following equations for x :

(i) $a\sqrt{x} - \sqrt{x + 2ax\sqrt{x^2 + 7a^2}} = 0$

(ii) $x + \sqrt{x^2 - x} = a$

3. Solve the following equations :

(i) $|x - 1 - x^2| \le |x^2 - 3x + 4|$

(ii) $|x^2 - 2x - 3| < 3x - 3$

(iii) $|3x - 1| + |2x - 3| - |x + 5| < 2$

(iv) $|2x - |3 - x| - 2| \le 4$

4. Solve the following equations :

(i) $\frac{x^2 - 7|x| + 10}{x^2 - 6x + 9} < 0$

(ii) $\frac{|x+3| + x}{x+2} > 1$

www.pdfworld.net

www.pdfworld.net

(iii) $\left|\dfrac{x^2-5x+4}{x^2-4}\right| \le 1$

(iv) $\dfrac{|x-3|}{x^2-5x+6} \ge 2$

5. Solve the following equations :

(i) $\dfrac{9}{|x-5|-3} \ge |x-2|$

(ii) $\left|\dfrac{x^2-2x+1}{x^2-4x+4}\right| + \left|\dfrac{x-1}{x-2}\right| - 12 < 0$

(iii) $\dfrac{|x+2|-|x|}{\sqrt{8-x^3}} \ge 0$

6. Solve the following equations :

(i) $|x^2-5|x|-4| \ge |2x^2-3|x|+1|$

(ii) $\left|-\dfrac{5}{x+2}\right| < \left|\dfrac{10}{x-1}\right|$

(iii) $\dfrac{|x^2-4x|}{x^2+|x-4|} \le 1$

(iv) $\sqrt{4-x^2} + \dfrac{|x|}{x} \ge 0$

7. Find the integral solutions of the equation $|2x-y|+|x-y|=3$

8. For every value of a determine the number of solutions of the following equations :

(i) $\sqrt{2|x|-x^2} = a$

(ii) $|x^2-2x-3| = a$

9. (i) Find the number of real solutions of $e^x = |x-1|$. Also find the solution if any exists.

(ii) Find all solutions of the equation $(2|x|-1)^2 = |x|$ belonging to the domain of definition of the function $y = \log(4x-1)$.

10. Solve the following equations :

(i) $4^{x-\sqrt{x^2-5}} - 12 \cdot 2^{x-1-\sqrt{x^2-5}} + 8 = 0.$

(ii) $3^{2x-3} - 9^{x-1} + 27^{2x/3} = 675.$

(iii) $5^{2x} - 7^x - 5^{2x}.35 + 7^x \cdot 35 = 0.$

(iv) $7^{\log x} - 5^{\log x+1} = 3 \cdot 5^{\log x-1} - 13 \cdot 7^{\log x-1}$

11. (i) What is larger, $\log_3 108$ or $\log_5 375$?

(ii) Prove that $(1/\log_2\pi) + (1/\log_{4.5}\pi) < 2$.

(iii) If $a > b > 0$, prove that

$$\log b/a < \log\left(\frac{1+b}{1+a}\right)$$

12. Solve for x :

$$\frac{6}{5}a^{\log_a x \log_{10} a \log_a 5 - 3^{\log_{10}(x/10)}} = 9^{\log_{100} x + \log_4 2}$$

13. Solve

$$2 - \log_{a^2}(1+x) = 3\log_a\sqrt{x-1} - \log_{a^2}(x^2-1)^2$$

14. Solve the following equations :

(i) $\sqrt[x-4]{5^{\frac{x}{\sqrt{x}+2}}.(0.2)^{\frac{4}{\sqrt{x}+2}}} = 125.\ (0.04)^{\frac{x-2}{x-4}}.$

(ii) $27 \times 2^{-3x} + 9 \times 2^x - 2^{3x} - 27 \times 2^{-x} = 8$

(iii) $|x|^{x^2-2x} = 1$

(iv) $(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \dfrac{101}{10(2-\sqrt{3})}$

15. Solve the following equations :

(i) $\sqrt{1+\log_2 x} + \sqrt{4\log_4 x - 2} = 4$

(ii) $\log_{5-x}(x^2-2x+65) = 2.$

(iii) $3^{\log_3 \log\sqrt{x}} - \log x + \log^2 x - 3 = 0.$

(iv) $x^{\frac{\log x+5}{3}} = 10^{5+\log x}.$

16. Solve the following equations :

(i) $\log_{1/3} x - 3\sqrt{\log_{1/3} x} + 2 = 0$

(ii) $2(\log_x\sqrt{5})^2 - 3\log_x\sqrt{5} + 1 = 0.$

(iii) $2\log_3\dfrac{x-3}{x-7} + 1 = \log_3\dfrac{x-3}{x-1}.$

(iv) $\log_3(\log^2_{1/2} x - 3\log_{1/2} x + 5) = 2.$

www.pdfworld.net

www.pdfworld.net

17. Solve the following equations :

(i) $\dfrac{1-2(\log x^2)^2}{\log x - 2(\log x)^2} = 1.$

(ii) $\log_x \sqrt{5} + \log_x (5x) - 2.25 = (\log_x \sqrt{5})^2.$

(iii) $\sqrt{\log_2(2x^2).\log_4(16x)} = \log_4 x^3.$

(iv) $\log_3 (9^x + 9) = x + \log_3 (28 - 2 . 3^x)$

18. Solve $\log_x (ax) . \log_a x = 1 + \log_x \sqrt{a}$.

19. Find the solution set of the system of equations,

$\log_3 x + \log_3 y = 2 + \log_3 2$ and $\log_{27}(x + y) = \dfrac{2}{3}$.

20. Prove that $\log_2 17 . \log_{1/5} 2 . \log_3 \dfrac{1}{5} > 2$

21. Solve the following inequalities :

(i) $\log_{5/8}\left(2x^2 - x - \dfrac{3}{8}\right) \geq 1$

(ii) $\log_{1/4}(2x + 3) > \log_9 27$

(iii) $\log_4 (3^x - 1). \log_{1/4}\left(\dfrac{3^x - 1}{16}\right) \leq \dfrac{3}{4}$

22. Solve the following inequalities :

(i) $x^{1/\log x} \log x < 1$

(ii) $\log_x(x^3 + 1). \log_{x+1} x > 2$

(iii) $(\log_2 x)^4 - \left(\log_{1/2} \dfrac{x^5}{4}\right)^2 - 20 \log_2 x + 148 < 0$

23. Solve the following inequalities :

(i) $\log_x \dfrac{4x+5}{6-5x} < -1$

(ii) $\log_{x+\frac{5}{2}}\left(\dfrac{x-5}{2x-3}\right)^2 > 0$

(iii) $\log_{\sqrt{2x^2-7x+6}}\left(\dfrac{x}{3}\right) > 0$

(iv) $\log_{x/2} 8 + \log_{x/4} 8 < \dfrac{\log_2 x^4}{\log_2 x^2 - 4}$

24. Solve $[2x] - 2x = [x + 1]$, where $[x]$ is the greatest integer $\leq x$.

Target Exercises for JEE Advanced

1. Solve the following equations for x :

(i) $\sqrt{x^2 + 3a^2} - \sqrt{x^2 - 3a^2} = x\sqrt{2}$

(ii) $2\sqrt{x + a} > x + 1$

2. Solve $\sqrt{x^2 + x} < a - x$

3. Solve the inequality $2|x - a| < 2ax - x^2 - 2$ for every value of the parameter a.

4. Knowing that $b = 8^{1/1-\log_8 a}$ and $c = 8^{1/(1-\log_8 b)}$, express $\log_8 a$ in terms of $\log_8 c$.

5. Prove that if $(ac)^{\log_a b} = c^2$,then the numbers $\log_a N, \log_b N, \log_c N$ are three successive terms of an arithmetic progression for any positive value of N.

6. If $\log_7 12 = a$ & $\log_{12} 24 = b$, then show that, $\log_{54} 168 = \dfrac{1+ab}{a(8-5b)}$.

7. Solve $3^x + 1 - |3^x - 1| = 2 \log_5 |6 - x|$.

8. Solve the equation $x^{\log_a x} = (a^{\pi})^{\log_a x}$ (where $a > 0$ & $a \neq 1$).

9. Solve $\log_{\sqrt{x}}(x + |x - 2|) = \log_x (5x - 6 + 5|x - 2|)$.

www.pdfworld.net

www.pdfworld.net

10. Solve $3x^2 - 2x^3 = \log_2 (x^2 + 1) - \log_2 x$.

11. Solve the following equations :

(i) $\dfrac{1+\log_2(x-4)}{\log_{\sqrt{2}}(\sqrt{x+3}-\sqrt{x-3}} = 1$

(ii) $\log_5 [(2+\sqrt{5})^x - (\sqrt{5}-2)^x]$

$= \dfrac{1}{2} - 3\log_{1/5} 2$

12. Solve the following equations:

(i) Solve $\log_3 (\sqrt{x} + |\sqrt{x} - 1|)$
$= \log_9 (4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$

(ii) $\log_3(-x^2 - 8x - 14) \cdot \log_{x^2+4+4x} 9 = 1$.

13. Solve the following equations:

(i) $3^{\log_a x} + 3 . x^{\log_a 3} = 2$

(ii) $\log_a(1 - \sqrt{1+x}) = \log_{a^2} (3 - \sqrt{1+x})$.

14. Solve the following equations:

(i) $\log_{\sqrt{x}} a \cdot \log_{a^2}\left(\dfrac{a^2-4}{2a-x}\right) = 1$

(ii) $\dfrac{\log_x(2a-x)}{\log_x 2} + \dfrac{\log_a x}{\log_a 2} = \dfrac{1}{\log_{(a^2-1)} 2}$.

15. Solve $x^2 2^{2x} + 9(x+2)2^x + 8x^2$
$\le (x+2)2^{2x} + 9x^2 2^x + 8x + 16$

16. Solve the following inequations :

(i) $\log_{x/2} 8 + \log_{x/4} 8 < \dfrac{\log_2 x^4}{\log_2 x^2 - 4}$

(ii) Solve $\log_x 2 . \log_{2x} 2 . \log_2 4x > 1$.

17. Solve the following inequations :

(i) $\log_3 \dfrac{|x^2 - 4x| + 3}{x^2 + |x-5|} \ge 0$.

(ii) $\log_{\log_2(0.5x)} (x^2 - 10x + 22) > 0$.

(iii) $\log_{x^2+2x-3} \dfrac{|x+4| - |x|}{x-1} > 0$

(iv) $\log_{|x|} (\sqrt{9-x^2} - x - 1) \ge 1$.

18. Solve

(i) $\dfrac{3\log_a x + 6}{\log_a^2 x + 2} > 1$

(ii) $\log_a (\sqrt{25-x^2} - 1) \ge \log_a (|x| + 1)$

19. The inequality $\log_a(x^2 - x - 2) > \log_a(-x^2 + 2x + 3)$ is known to be satisfied for $x = -14/9$. Find all solutions of this inequality.

20. Solve $\dfrac{1}{\log_{0.5}\sqrt{x+3}} \le \dfrac{1}{\log_{0.5}(x+1)}$

21. Solve the system of inequalities :

(i) $\begin{cases} \dfrac{\sqrt{(x-8)(2-x)}}{\log_{0.3}\left(\dfrac{10}{7}(\log_2 5 - 1)\right)} \ge 0 \\ 2^{x-3} - 31 > 0 \end{cases}$

(ii) $\begin{cases} \dfrac{\sqrt{\log_2^2 x - 3\log_2 x + 2}}{\log_5\left(\dfrac{1}{3}(\log_3 5 - 1)\right)} \ge 0 \\ x - \sqrt{x} - 2 \ge 0 \end{cases}$

22. Solve the following inequalities :

(i) $2^{\log^2_{0.5} x} + x^{\log_{0.5} x} > 2.5$

(ii) $\log_3(\log_2(2 - \log_4 x) - 1) < 1$

(iii) $\dfrac{3x^2 - 16x + 21}{\log_{0.3}(x^2+4)} < 0$

(iv) $\log_2 x . \log_3 2x + \log_3 x . \log_2 3x \ge 0$

23. Solve the following inequalities :

(i) $\log_{1/3}\sqrt{x+1} < \log_{1/3}\sqrt{4-x^2} + 1$

(ii) $2^{\log_3(x^2-6x+9)} \le 3^{2\log_x \sqrt{x}-1}$

www.pdfworld.net

www.pdfworld.net

(iii) $\log_2(x+1)^2 + \log_2 \sqrt{x^2+2x+1} > 6$

(iv) $\frac{1}{\log_2 x} \le \frac{1}{\log_2 \sqrt{x+2}}$

24. Solve $[x^2] + 2[x] = 3x,\ 0 \le x \le 2$

25. If $y = 4 - [x]^2$ where x is real then find x for which $[y] + y = 6$ where $[x]$ is the greatest integer $\le x$.

Previous Years' Questions (JEE Advanced)

A. Fill in the blanks

1. If the product of the roots of the equation $x^2 + 3kx + 2e^{2\ln k} - 1 = 0$ is 7, then the roots are real for k = ... **[IIT - 1984]**

2. The solution of equation $\log_7 \log_5(\sqrt{x+5} + \sqrt{x}) = 0$ is ..... **[IIT - 1986]**

3. The sum of all the real roots of the equation $|x-2|^2 + |x-2| - 2 = 0$ is..... **[IIT - 1997]**

B. True / False

4. If $n_1, n_2, \ldots\ldots n_p$ are p positive integers, whose sum is an even number, then the number of odd integers among them is odd. **[IIT - 1985]**

C. Multiple Choice Questions with ONE correct answer

5. The least value of the expression $2\log_{10}x - \log_x(0.01)$, for $x > 1$, is **[IIT - 1980]**

(A) 10 (B) 2
(C) –0.01 (D) None

6. The expression $\frac{12}{3+\sqrt{5}+2\sqrt{2}}$ is equal to **[IIT - 1980]**

(A) $1 - \sqrt{5} + \sqrt{2} + \sqrt{10}$
(B) $1 + \sqrt{5} + \sqrt{2} - \sqrt{10}$
(C) $1 + \sqrt{5} - \sqrt{2} + \sqrt{10}$
(D) $1 - \sqrt{5} - \sqrt{2} + \sqrt{10}$

7. If p, q, r are any real numbers, then **[IIT - 1982]**

(A) max (p, q) < max (p, q, r)
(B) min (p, q) = 1/2(p + q – |p – q|)
(C) max (p, q) < min (p, q, r)
(D) None of these

8. If x satisfies $|x-1| + |x-2| + |x-3| \ge 6$, then **[IIT - 1983]**

(A) $0 \le x \le 4$ (B) $x \le -2$ or $x \ge 4$
(C) $x \le 0$ or $x \ge 4$ (D) None of these

9. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has **[IIT - 1984]**

(A) no root
(B) one root
(C) two equal roots
(D) infinitely many roots

10. The number of solution of the equation $\sin(e)^x = 5^x + 5^{-x}$ is **[IIT - 1990]**

(A) 0 (B) 1
(C) 2 (D) Infinitely many

11. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has **[IIT - 1997]**

(A) no solution
(B) one solution
(C) two solutions
(D) more than two solutions

12. The number of solution of $\log_4(x-1) = \log_2(x-3)$ is **[IIT - 2001]**

(A) 3 (B) 1 (C) 2 (D) 0

13. The set of all real numbers x for which $x^2|x+2| + x > 0$, is **[IIT - 2002]**

(A) $(-\infty, 2) \cup (2, \infty)$

www.pdfworld.net

www.pdfworld.net

(B) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(C) $(-\infty, -1) \cup (1, \infty)$

(D) $(\sqrt{2}, \infty)$

14. Let (x_0, y_0) be the solution of the following equations **[IIT - 2011]**

$(2x)^{\ln 2} = (3y)^{\ln 3}$

$3^{\ln x} = 2^{\ln 3}$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$

(C) $\frac{1}{2}$ (D) 6

D. Multiple Choice Questions with ONE or MORE THAN ONE correct answer

15. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains **[IIT - 1984]**

(A) $\left(-\infty, -\frac{3}{2}\right)$ (B) $\left(-\frac{3}{2}, -\frac{1}{4}\right)$

(C) $\left(-\frac{1}{4}, \frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, 3\right)$

(E) None of these

16. The equation $x^{3/4(\log_2 x)^2 + \log_2 x - 5/4} = \sqrt{2}$ has **[IIT - 1989]**

(A) at least one real solution

(B) exactly three solutions

(C) exactly one irrational solution

(D) complex roots

17. Number of divisors of the form $4n + 2$ $(n \geq 0)$ of the integer 240 is **[IIT - 1998]**

(A) 4 (B) 8 (C) 10 (D) 3

E. Subjective Problems

18. Solve for x : $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$ **[IIT - 1978]**

19. Solve for x : $\sqrt{x+1} - \sqrt{x-1} = 1$. **[IIT - 1978]**

20. Solve the following equation for x:

$2 \log_x a + \log_{ax} a + 3 \log_{a^2 x} a = 0, a > 0$ **[IIT - 1978]**

21. Show that the square of $\frac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$ is a rational number. **[IIT - 1978]**

22. Find all integers x for which $(5x - 1) < (x + 1)^2 < (7x - 3)$. **[IIT - 1978]**

23. Draw the graph of $y = |x|^{1/2}$ for $-1 \leq x \leq 1$. **[IIT - 1978]**

24. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, find f(6). **[IIT - 1979]**

25. If $x > y > 0$, then show that the expression $(\sqrt{2}(2x + \sqrt{x^2 - y^2})(\sqrt{x - \sqrt{x^2 - y^2}}))$ can be simplified to $\sqrt{(x+y)^3} - \sqrt{(x-y)^3}$ **[IIT -1980]**

26. Let $y = \sqrt{\frac{(x+1)(x-3)}{(x-2)}}$. Find all real values of x for which y takes real values. **[IIT - 1980]**

27. For what values of m, does the system of equations

$3x + my = m$

$2x - 5y = 20$

has solution satisfying the conditions $x > 0$, $y > 0$. **[IIT - 1980]**

28. Solve for x;

$(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$ **[IIT -1985]**

29. For $a \leq 0$, determine all real roots of the equation

$x^2 - 2a|x - a| - 3a^2 = 0$ **[IIT - 1986]**

30. Find the set of all x for which

$\frac{2x}{(2x^2 + 5x + 2)} > \frac{1}{(x+1)}$ **[IIT - 1987]**

pdfworld.net

www.pdfworld.net

www.pdfworld.net

31. Solve $|x^2 + 4x + 3| + 2x + 5 = 0$ **[IIT - 1988]**

32. Let $\{x\}$ & $[x]$ denote the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$ **[IIT - 1994]**

33. Find the set of all solutions of the equation $2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1$. **[IIT - 1997]**

ANSWERS

Practice Problems—A

1. (i) (a) $\{\sqrt{4}, 7, \frac{18}{2}, 100\}$

(b) $\{0, \sqrt{4}, 7, \frac{18}{2}, 100\}$

(c) $\{-10, 0, \sqrt{4}, 7, \frac{18}{2}, 100\}$

(d) $\left\{-10, -\frac{3}{4}, 0, \frac{4}{5}, \sqrt{4}, 7, \frac{18}{2}, 100\right\}$

(e) $\{-\sqrt{2}, \pi\}$

(ii) (a) none (b) $\left\{\frac{0}{3}\right\}$ (c) $\left\{-\sqrt[3]{8}, \frac{0}{3}\right\}$

(d) $\left\{-\sqrt[3]{8}, \frac{0}{3}, \sqrt{\frac{4}{9}}, 1.\overline{126}\right\}$ (e) $\{\sqrt[3]{7}\}$

3. $n = 5k-3, k \in N$ **4.** $n \geq 2$

5. irrational **8.** 173/55

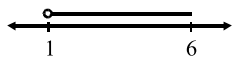
11. $2-\sqrt{3}$ **12.** $\frac{\sqrt{6}}{3}$

13. $p = 3, q = 2, r = 5$ **14.** (0, 0), (2, 2)

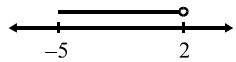
15. 1074

Practice Problems—B

1. (i) $1 < x \leq 6$

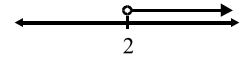


(ii) $-5 \leq x < 2$

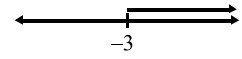


(iii) $-3 \leq x \leq 1$

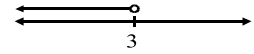
(iv) $2 < x < \infty$



(v) $-3 \leq x < \infty$



(vi) $-\infty < x < 3$



2. (i) $\{x \mid x < 6\}$; $(-\infty, 6)$

(ii) $\{x \mid x \geq -1\}$; $[-1, \infty)$

(iii) $\{x \mid 5 < x < 12\}$; $(5, 12)$

(iv) $\{x \mid 2 < x \geq 13\}$; $(2, 13]$

(v) $\{x \mid x \leq 6\}$; $(-\infty, 6]$

(vi) $\{x \mid 2 \leq x \leq 5\}$; $[2, 5]$

3. (i) $[2, 5)$ (ii) $(5, \infty)$ (iii) Φ

(iv) $(-1, \infty)$ (v) $(-\infty, 4]$ (vi) $(-3, -1) \cup [2, 4]$

Practice Problems—D

12. 34056, 34452, 34956.

Practice Problems—E

3. $x = 2, y = 3, z = 7$

4. (i) $(a - b)(a + b - 1)(a^2 + ab + b^2)$

(ii) $x(x - 1)(3x + 1)(x^2 - x + 1)$

(iii) $(a + b)(a - b)(a + 1)(a - 1)$

5. $3x^3 + 4x^2 - 5x - 2$

6. $\left\{\frac{a^2 - 3a + 9}{a - 3}\right\}$ for $a \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$; $\{c \mid c \in R\}$ for $a = -3$; ϕ for $a = 3$

7. ϕ for $a = -2$, $a = -1$; $-2a^2 - 3a$ for $\begin{cases} a \neq -2 \\ a \neq -1 \end{cases}$

8. (7, 0), (0, 7), (3, 4) and (4, 3)

9. (m, –m, 1) $m \in I$; (0, 1, 0), (–2, –3, 6), (1, 0, 0), (0, –2, 3), (–2, 0, 3), (–3, –2, 6)

Practice Problems—F

1. $x \in \{1, 2, 3, 4\}$ **2.** $x = -1$

3. (–7/2, 0) **4.** $x = 1$

5. $[1/\alpha, \infty)$ for $\alpha \in (-\infty, 0)$, $(-\infty, \infty)$ for $\alpha = 0$, $(-\infty, 1/\alpha]$ for $\alpha \in (0, \infty)$

6. ϕ for $a = -1$; $(-\infty, 1 - a)$ for $a > -1$; $(1 - a, \infty)$ for $a < -1$

www.pdfworld.net

www.pdfworld.net

Practice Problems—G

1. $x \in (-\infty, -1) \cup (4, \infty)$ **2.** $[-1, 4]$

3. $x \in R - \{3\}$ **4.** $x \in \left\{-\frac{1}{2}\right\}$

5. $x \in R$ **6.** $x \in \left\{\frac{1}{2}\right\}$

7. $[-4, -1) \cup (3, \infty)$ **8.** $\{3\}$

9. $\{4\}$ **10.** $(-\infty, -3)$

11. $[-4, -2] \cup \{1\}$ **12.** $(-\infty, -1] \cup \{1\} \cup [2, \infty)$

13. $(-5, -1)$ **14.** $(-\infty, -2] \cup [0, +\infty)$

15. $x = 2$

Practice Problems—H

1. (i) $x \in R$ (ii) $\begin{matrix} x = a, & a \neq 0 \\ x \in R, & a = 0 \end{matrix}$

(iii) $\begin{cases} x = a + 2, & a \neq 2 \\ x = 2, & a = 2 \end{cases}$

$x = \dfrac{a^2 - 3a + 9}{a - 3}$ if $a \neq \pm 3$

(iv) $x \in R$ if $a = -3$

ϕ if $a = 3$

(v) -2

(vi) $\begin{matrix} x = \dfrac{4 - 2a}{a}, & a \neq 0 \\ \phi, & a = 0 \end{matrix}$

2. (i) $(-\infty, 6) \cup [-2, 0) \cup (3, +\infty)$

(ii) $(-5, 1) \cup \{3\}$

(iii) $(-\infty, -4) \cup (-3, 3) \cup (6, +\infty)$

(iv) $(-\sqrt{2}, -1) \cup (-1, \sqrt{2}) \cup [3, 4)$

(v) $(-\infty, -5) \cup (1, 2) \cup (6, +\infty)$

(vi) $(-\infty, -3) \cup (-2, 3)$

3. (i) $[1, 4) \in (4, \infty)$ (ii) $(-\infty, -3) \cup (2, 4)$

(iii) $\left\{-\frac{5}{2}, \frac{3}{2}\right\}$ (iv) $(-1, 0)$

(v) $(-\infty, -2) \cup (-1, 1) \cup (2, 3) \cup (4, 6) \cup (7, \infty)$

(vi) $\left(-2, \frac{5}{2}\right) \cup \left(\frac{8}{3}, \infty\right)$

(vii) $[-2, \infty)$

(viii) $(-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$

4. $\{3, 4, 5, 6, 7\}$ **5.** $\frac{1}{2} < a < 1$

Practice Problems—I

1. $\dfrac{4}{x-2} + \dfrac{3}{x-1}$

2. $\dfrac{3}{x} + \dfrac{2}{x-1} - \dfrac{1}{x+3}$ **3.** $\dfrac{6}{x-1} - \dfrac{5}{(x-1)^2}$

4. $\dfrac{7}{x} - \dfrac{6}{x-1} + \dfrac{10}{(x-1)^2}$ **5.** $\dfrac{3}{x-1} + \dfrac{2x-4}{x^2+1}$

6. $\dfrac{4}{x+1} + \dfrac{2x-3}{x^2+1}$

7. $\dfrac{-1}{x} + \dfrac{2x}{x^2+1} + \dfrac{2x-1}{(x^2+1)^2}$

8. $\dfrac{1}{x-1} + \dfrac{2x+1}{x^2-2x+2} + \dfrac{x}{(x^2-2x+2)^2}$

Practice Problems—J

1. $x^{11/6} - x^{10/6} a^{1/4} + x^{9/6} a^{1/2} - \ldots\ldots + x^{1/6} a^{10/4} - a^{11/4}$

2. $\dfrac{x - 1 + \sqrt{x^2 - 1} + \sqrt{2x(x-1)}}{x - 1}$

3. (i) $\dfrac{1}{\sqrt{2}} \left| \sqrt{2x+1} + \sqrt{x-4} \right|$

(ii) $\dfrac{1}{\sqrt{2}} \left| \sqrt{3a+x} - \sqrt{a-x} \right|$

4. (i) $3 - \sqrt{5}$ (ii) $3 - 2\sqrt{2}$

5. (i) $\begin{matrix} x = a^2 - 1, & a \geq 0 \\ \phi, & a < 0 \end{matrix}$

(ii) $\begin{matrix} x = \dfrac{a-3}{2} & \text{if} \quad a \geq -3 \\ \phi & \text{if} \quad a < -3 \end{matrix}$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

(iii) No solution (iv) –7, 2

(v) 5 (vi) $1, -\frac{1}{3}$

6. (i) {3} (ii) $\left\{3+\frac{2\sqrt{5}}{5}\right\}$

(iii) {5} (iv) {2}

(v) {20} (vi) {–15, 13})

(vii) {16/25} (viii) {8}

(ix) {[2, + ∞)}

(x) {(–1 – $\sqrt{5}$)/2, ($\sqrt{5}$ –1)/2, 1}

Practice Problems—K

1. (i) $(-\infty, -1] \cup [2, +\infty)$

(ii) (1/2, 2]

(iii) [3, +∞)

2. (i) (6/5, – 1] ∪ [2, 3)

(ii) (–∞, 1)

(iii) (6, 8]

3. (i) [0, 2] (ii) (–∞, –2) ∪ [5, 74/13]

(iii) (–1, 0) ∪ (3/5, 1]

4. (i). (1, 4] (ii) (3, 5]

(iii) [3, 4) ∪ (7, ∞)

5. (i) [1/2, 2) ∪ (5, ∞)

(ii) (–2 –2 $\sqrt{6}$, – 1) ∪ [–2 + 2 $\sqrt{6}$, 3]

(iii) [–1, ∞)

6. (i) {–3, 1) (ii) [(16 + $\sqrt{7}$)/2, 10]

7. ϕ for a = 2; $\left[\frac{1}{3}, \frac{a-1}{3}\right)$ for a > 2;

$\left(\frac{a-1}{3}, \frac{1}{3}\right]$ for a < 2

Practice Problems—L

1. (i) (– ∞ , – 5] ∪ [4 , ∞)

(ii) [– 4 , –2]

2. (i) $\frac{3}{2}$ (ii) $\frac{1 \pm \sqrt{17}}{2}$

(iii) 4

3. (i) 2, 5 (ii) $-2, -4, -1-\sqrt{3}$

(iii) 1 (iv) ±1

4. (i) $\frac{-7-\sqrt{17}}{2}$ (ii) x = –2

(iii) –4 ≤ x ≤ 4 (iv) –6, 2

(v) $-4, -1, -\sqrt{3}$

5. (i) 0 (ii) $-\frac{2}{5}, 2$

6. (i) –2 (ii) x = –3, x ≥ –1

(iii) x = –1, 0 ≤ x ≤ 1

7. $(1-\sqrt{2})a, (\sqrt{6}-1)a$

8. (a) (3 , 1) ; (–3/2 , 5/2) ; (–9 , –5) ; (9/2 , – 1/2)};

(b) (–3 , –2) ; (–2 , –3) ; (3 , 2); (2 , 3)

9. $\pm\frac{a+\sqrt{a^2-16a+60}}{15-4a}$ for $a < \frac{15}{4}$; ϕ for $a \geq \frac{15}{4}$

10. –2α **11.** {–2, –1/9}

12. {–1}

Practice Problems—M

1. (i) $(-\infty, -1) \cup (3, \infty)$

(ii) $\left(\frac{3}{4}, 1\right) \cup (1, \infty)$

(iii) $(-\infty, -2) \cup (-1, \infty)$ (iv) $(-\infty, -4) \cup [0, \infty)$

2. (i) (–(1 + $\sqrt{21}$)/2, ($\sqrt{21}$ – 1)/2)

(ii) (– 3, 0) ∪ (1, 2)

(iii) $(-\infty, -4) \cup \left(\frac{8}{7}, \infty\right) - \{2\}$

(iv) (–∞, (4 – $\sqrt{2}$)/2] ∪ [(5 + $\sqrt{3}$)/2, ∞)

3. (i) R (ii) $(-\infty, -2) \cup (3, \infty)$

(iii) (–1, 3)

4. (i) (9/2, ∞) (ii) (–7, –2) ∪ (3, 4)

(iii) $\left[\frac{2}{7}, \frac{2}{3}\right]$

5. (i) [1 – $\sqrt{17}$, $\sqrt{5}$ – 1]

www.pdfworld.net

www.pdfworld.net

(ii) $x \le -2-\sqrt{2}, x \ge 1+\sqrt{3}$

(iii) $1 \le x \le 3, x = 4$

6. (i) $[0, \sqrt{2}]$ (ii) $(-\infty, 1] \cup [5, \infty)$

(iii) R (iv) No solution

Practice Problems—N

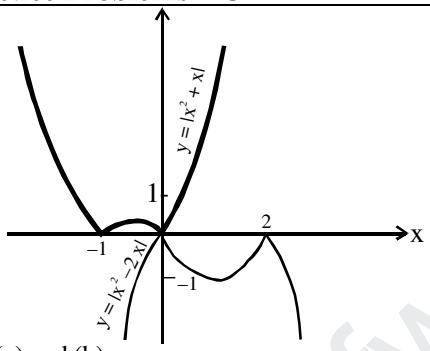
4. $[0, \infty)$ **5.** $\{0\} \cup (1, \infty)$

6. $[-2, -1] \cup [0, \infty)$

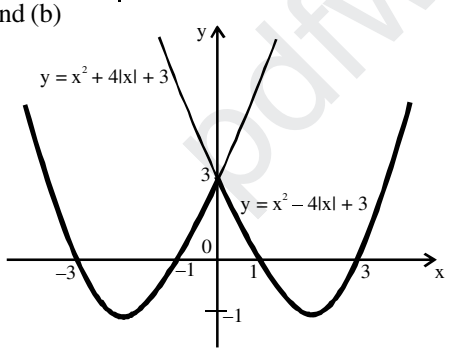
7. $\left[-2, \frac{5-\sqrt{21}}{2}\right] \cup \left[\frac{5+\sqrt{21}}{2}, \infty\right)$

Practice Problems—O

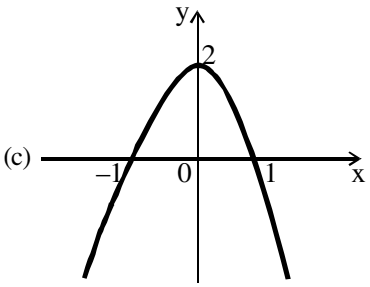
1.



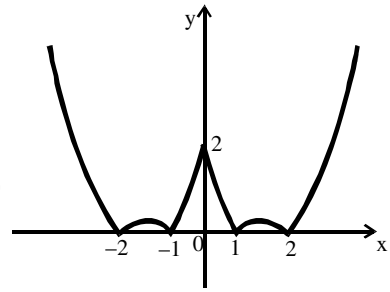
2. (a) and (b)



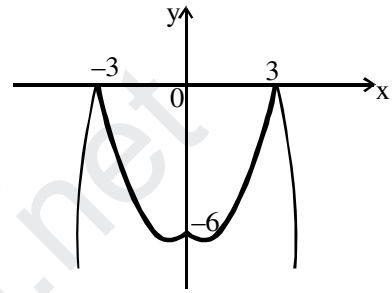
(c)



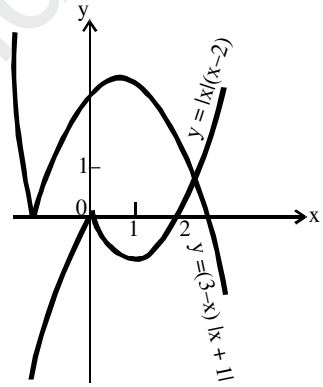
3. (a)



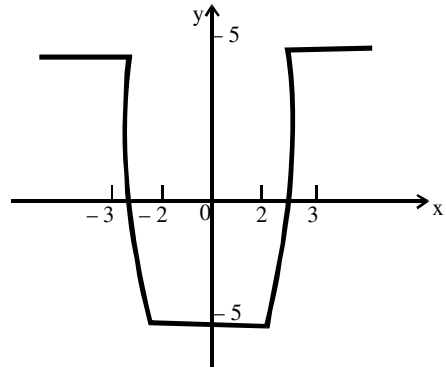
(b)



4.



5.

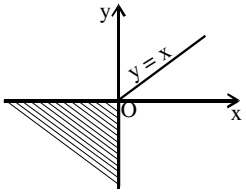


pdfworld.net

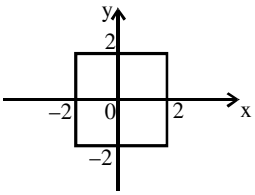
www.pdfworld.net

www.pdfworld.net

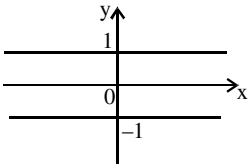
6. (i)



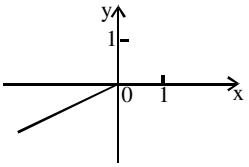
(ii)



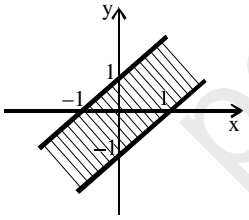
(iii)



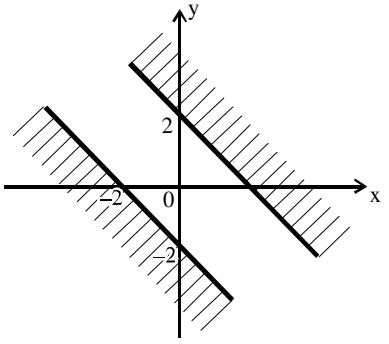
(iv)



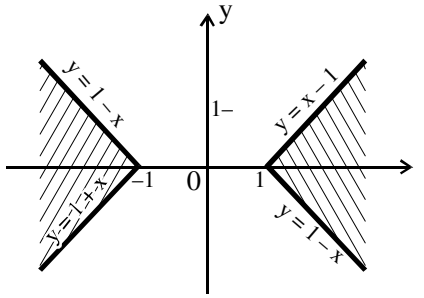
7. (a)



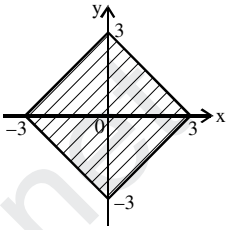
(b)



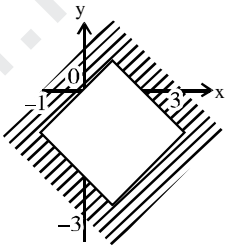
(c)



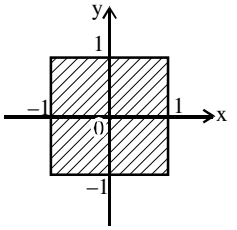
(d)



(e)



(f)



8. ϕ if $|a| < \sqrt{2}/2$, or $|a| > 1$; four if $|a| = \sqrt{2}/2$, or $|a| = 1$; eight if $\sqrt{2}/2 < |a| < 1$.

9. $(0, 1)\ (1, 0), \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$

10. $\{2+\sqrt{4-a}\}$ if $a < -4$; $\{2-\sqrt{4-a}, 2+\sqrt{4-a}, 2+\sqrt{4-a}\}$ if $-4 \le a < 0$ $\{0, 4\}$ if $a = 0$; $\{2-\sqrt{4-a}\}$ if $a > 0$.

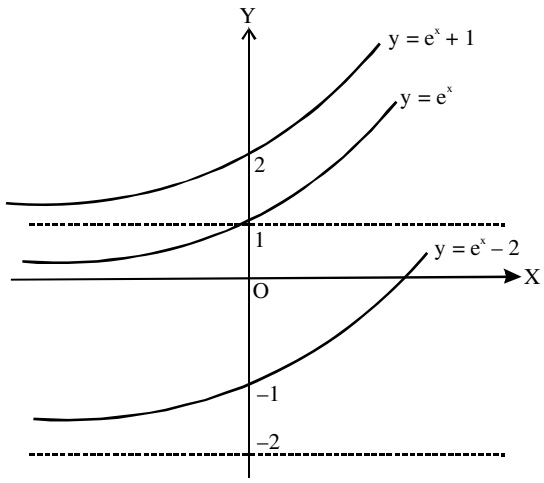
pdfworld.net

www.pdfworld.net

www.pdfworld.net

Practice Problems—P

1.



2. $\frac{1-\sqrt{5}}{2} \le x < 0,\ x \ge \frac{1+\sqrt{5}}{2}$

3. 2 **4.** $7\frac{1}{196}$

6. (i) ln3 (ii) $ln|a|$
(iii) 1 (iv) 2

7. 9

8. (i) 2 (ii) $\frac{1}{3}\log_5 6$
(iii) 1 (iv) 1

12. 1 **13.** 2

14. $\frac{1}{\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1}}$

15. $ab(a-b)^2$

16. (i) 1 (ii) a^2+a+1
(iii) $\frac{1}{\log_a b - 1}$ (iv) 6

17. (i) 1, –2 (ii) 5, 2

18. 1.948445 **20.** 9

21. (i) 0.4409 (ii) 301

Practice Problems—Q

1. $2 \pm \sqrt{7/2}$ **2.** $-\frac{5}{2}, 3$

3. 2 **4.** 0

5. $\log_{3/2} 2, 2\log_{3/2} 2$ **6.** $1 \pm \sqrt{3}$

7. –2 **8.** {–2, 2}

9. $3; -\frac{1}{5}$ **10.** $-1; 1; \sqrt{2}; -\sqrt{2}$

11. 1 **12.** –1; 1

13. $1;\ 1+\sqrt{2};\ 1-\sqrt{2}$

14. $2; -1-\log_5 2$ **15.** –1; 1; 2

16. –3; 1; 2; 3; 4 **17.** 1

Practice Problems—R

1. $x \in R$ **2.** (–5, 5))

3. (2, 6)

4. $(-\infty,\ \log_2(-1+\sqrt{7})]$

5. $[-4, 2) \cup (0, \infty)$ **6.** $(-\infty, 0) \cup [1, \infty)$

7. $(-\sqrt{7}, -\sqrt{3}) \cup (\sqrt{3}, \sqrt{7})$

8. (2, ∞)

9. $x \in (a^2 \log_3^2 2, \infty)$ for $a \in [0, \infty)$, $x \in [0, \infty)$ for $a \in (-\infty, 0)$

10. $(-\infty, \log_2(\sqrt{2}-1] \cup [1/2, \infty)$

11. (1, 2)

Practice Problems—S

1. $-\frac{8}{3}$ **2.** 3, 2

3. 2 **4.** 2

5. –1, 7 **6.** x = 1

7. 8 **8.** 1, 3

9. $\left[\frac{1}{5}, \infty\right)$ **10.** $10^{-5}, 10^3$

11. $2^{-2\sqrt{2}}, 2^{2\sqrt{2}}$ **12.** 2

13. {9, 81} **14.** $\frac{1}{20}, \frac{1}{5}$

15. 100 **16.** $\pm\frac{1}{e}, \pm e^3, 2$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

17. 1 **18.** $\pm\frac{1}{2}$

19. $1, 4, \frac{1}{4}\sqrt[5]{8}$ **20.** $\frac{1}{81}, 3$

Practice Problems—T

1. $(-1, 2) \cup (3, 4)$ **2.** $[-1, 1) \cup (3, 5]$

3. $[-7, -\sqrt{35}) \cup [5, \sqrt{35})$

4. $(0, 10)$ **5.** $[\log_3 0.9, 2)$

6. $(-1, 0) \cup (1, \infty)$ **7.** $(3, 4) \cup (6, \infty)$

8. $(0, 1/2) \cup (2, 3)$ **9.** $(2, \infty)$

10. $(10-\sqrt{43}, 4) \cup (10+\sqrt{43}, \infty)$

11. $(0, 1/4] \cup [1, 4)$

12. $x \in (1, (1+\sqrt{1+4a^2}\ /2))$ for $a \in (0, 1)$;

$x \in ((1+\sqrt{1+4a^2})/2, \infty)$ for $a \in (1, \infty)$.

13. $(0, 2^{-1/3}] \cup [2, \infty)$ **14.** $\left(\frac{1}{2^{\sqrt{2}}}, \frac{1}{2}\right) \cup (1, 2^{\sqrt{2}})$

15. $(-\infty, -2/3] \cup [1/2, 2]$

16. $(-1, 0)$

17. $[\log_a(4\sqrt{a^2+16}),\ 3\log_a 2),\ 0<a<1$;

$[\log_a(4+\sqrt{16+a^2}), \infty),\ a>1$

18. $[-1, 1/2) \cup (1, 2) \cup (2, 7/2)$

19. $(-\infty, -2) \cup (-\sqrt{2}, -1) \cup (1, \sqrt{2}) \cup (2, \infty)$

20. $[\sqrt{6}-1, 2) \cup (2, 5]$

Practice Problems—U

1. 75 **3.** 15

4. 99 **6.** $0, n+\frac{1}{2}$ where $n \in I$

7. No real solution. **8.** 2

Objective Exercises

1. C	**2.** D	**3.** A
4. B	**5.** A	**6.** C
7. B	**8.** C	**9.** B
10. A	**11.** B	**12.** B
13. A	**14.** A	**15.** B
16. A	**17.** C	**18.** C
19. A	**20.** D	**21.** C
22. A	**23.** B	**24.** D
25. A	**26.** C	**27.** B
28. B	**29.** C	**30.** A
31. B	**32.** B	**33.** A
34. C	**35.** A	**36.** A
37. D	**38.** A	**39.** A
40. A	**41.** B	**42.** A
43. A	**44.** B	**45.** ABC
46. C	**47.** C	**48.** D
49. A	**50.** C	**51.** ABC
52. ABD	**53.** AC	**54.** BC
55. ABC	**56.** BC	**57.** ABD
58. CD	**59.** ACD	**60.** ABC
61. BC	**62.** ABD	**63.** AD
64. ABCD	**65.** AD	**66.** AD
67. ABCD	**68.** AC	**69.** AB
70. BC	**71.** D	**72.** C
73. A	**74.** D	**75.** B
76. C	**77.** A	**78.** D
79. C	**80.** D	**81.** B
82. A	**83.** A	**84.** B
85. A	**86.** D	**87.** D
88. A	**89.** A	**90.** A
91. B	**92.** C	**93.** D
94. A	**95.** C	

96. (A) – T (B) – PS (C) – Q (D) – R

97. (A) – R (B) – QRS (C) – PS (D) – PT

98. (A) – QT (B) – PRT (C) – PRST (D) – T

99. (A) – R, (B) – P, (C) – S, (D) – Q

100. (A) – QS, (B) – QS, (C) – P, (D) – QR

Review Exercises for JEE Advanced

1. (i) $x \in \phi$ if $a \in (-\infty, 0) \cup (0, 1)$; $x = \{0\}$ if $a = 0$; $x = \{(2a-1-\sqrt{4a-3})/2\}$ if $a \in [1, \infty)$

(ii) $x \in \phi$ if $a \in (-\infty, -1/4)$, $x = \{(-1 \pm \sqrt{4a+1})/2\}$ if $a \in [-1/4, 0]$; $x = \{(-1-\sqrt{4a+1})/2\}$ if $a \in (0, 1)$; $x = \{(-1-\sqrt{4a+1})/2, (1+\sqrt{4a-3})/2$ if $a \in [1, \infty)$

2. (i) $x = \{0\}$

www.pdfworld.net

www.pdfworld.net

(ii) ϕ for $a<0$, $\frac{1}{2} \le a < 1$; $\frac{a^2}{2a-1}$ for $0 \le a < \frac{1}{2}$, $a \ge 1$

3. (i) $x \le 3/2$ (ii) $2 < x < 5$

(iii) $\left(-\frac{1}{2}, \frac{11}{4}\right)$ (iv) $\left[\frac{1}{3}, 3\right]$

4. (i) $(-5, -2) \cup (2, 3) \cup (3, 5)$
(ii) $(-5, -2) \cup (-1, \infty)$
(iii) $[0, 8/5] \cup [5/2, \infty]$(iv) $[3/2, 2]$

5. (i) $[-1, 2) \cup (8, 5 + \sqrt{18}]$

(ii) $\left(-\infty, \frac{7}{4}\right) \cup \left(\frac{5}{2}, \infty\right)$

(iii) $[-1, 2)$

6. (i) $[-3, 3]$
(ii) $(-\infty, -5] \cup (-1, 1) \cup \cup (1, \infty)$

(iii) $[-4/3, \infty)$ (iv) $[-\sqrt{3}, 0) \cup (0, 2]$

7. (3,6), (–3, –6), (1,3), (–1, –3), (3, 5), (–3, –5), (1,0), (–1,0), (3,4), (–3, –4), (3,3), (–3, –3)

8. (i) For $a < 0$ there are no solutions, for $a = 0$ there are three solutions, for $0 < a < 1$ there are four solutions, for $a = 1$ there are two solutions, for $a > 1$ there are no solutions.
(ii) for $a < 0$ there are no solutions, for $a = 0$ there are two solutions, for $0 < a < 4$ there are four solutions, for $a = 4$ there are three solutions, for $a > 4$ there are two solutions.

9. (i) one solution, $x = 0$
(ii) $\{1\}$

10. (i) $\{9/4, 3\}$ (ii) $\{3\}$
(iii) $\{0\}$ (iv) $\{100\}$

11. (i) $\log_3 108$
12. $x = 100$
13. ϕ for $1 < a \le 0$, $a = 1$, $a \ge 2\sqrt{2}$;
$4 - a^2$ for $0 < a < 1$, $1 < a < 2\sqrt{2}$
14. (i) $\{9\}$ (ii) 0
(iii) $\{-1, 1, 2\}$

(iv) $x_{1,2} = 1 \pm \sqrt{1 + \log_{2+\sqrt{3}} 10}$

15. (i) 8 (ii) $\{-5\}$
(iii) $\{100\}$ (iv) $\{10^{-5}, 10^3\}$

16. (i) $\{1/81, 1/3\}$ (ii) $\{\sqrt{5}; 5\}$
(iii) $\{-5\}$ (iv) $\{1/16, 2\}$

17. (i) $\{1/\sqrt{10}, \sqrt[3]{10}\}$

(ii) $\{\sqrt[5]{5}, 5\}$
(iii) $\{16\}$
(iv) $\{-1, 2\}$

18. $x_1 = a^{1/\sqrt{2}}$, $x_2 = a^{-1/\sqrt{2}}$

21. (i) $\left[-\frac{1}{2}, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$

(ii) $\left(-\frac{3}{2}, -\frac{23}{16}\right)$ (iii) $(0, 1] \cup [2, \infty)$

22. (i) $(0, 1) \cup ((1, \sqrt[10]{10})$
(ii) $(2, \infty)$
(iii) $1/16 < x < 1/8$, $8 < x < 16$.

23. (i) $1/2 < x < 1$
(ii) $-5/2 < x < -2, -3/2 < x < 3/2, 3/2 < x < 8/3$.
(iii) $1 < x < 3/2, 2 < x < 5/2, x > 3$.
(iv) $0 < x < 2, x > 4$

24. $-1, -\frac{1}{2}$

Target Exercises for JEE Advanced

1. (i) $-a\sqrt{3}$ for $a < 0$; $a\sqrt{3}$ for $a \ge 0$

(ii) ϕ for $a \le 0$; $(1 - 2\sqrt{a}\,;\, 1 + 2\sqrt{a})$ for $0 < a \le 1$; $[a, 1+2\sqrt{a})$ for $a > 1$.

2. ϕ for $a \le -1$; $\left(\frac{a^2}{1+2a}, -1\right]$ for $-1 < a < -\frac{1}{2}$;

$(-\infty; -1]$ for $-\frac{1}{2} \le a \le 0$; $(-\infty, -1] \cup \left[0, \frac{a^2}{1+2a}\right)$ for $a > 0$.

www.pdfworld.net

www.pdfworld.net

3. For $|a| \le \sqrt{2}$ the inequality has no solutions, for $|a| < \sqrt{2}$ the solution is $a - \sqrt{a^2-1} + 1 < x < a + \sqrt{a^2-1} - 1$

4. $\log_8 a = \dfrac{1}{1 - \log_8 c}$

7. $\{1, 11\}$, for $x < 0$, no solution

8. $\{1, a^{\pi}\}$

9. 2, 7/3

10. 1

11. (i) 5 (ii) 2

12. (i) $\{4\} \cup [0, 1]$ (ii) $x = -4$

13. (i) $x = 2^{-\log_3 a}$
(ii) $x \in \phi$ for $a \in (-\infty, 0] \cup \{1, 2\}$, $x = a + 2$
for $a \in (0, 1) \cup (1, 2) \cup \{3\}$, $x = a \pm 2$
for $a \in (2, 3) \cup (3, \infty)$

14. (i) $x \in \phi$ for $a \in (-\infty, 0] \cup \{1, 2\}$, $x = a + 2$
for $a \in (0, 1) \cup (1, 2) \cup \{3\}$, $x = a \pm 2$
for $a \in (2, 3) \cup (3, \infty)$
(ii) $x \in \phi$ for $a \in (-\infty, 1]$; $x = a \pm 1$
for $a \in (1, \sqrt{2}) \cup (\sqrt{2}, 2) \cup (2, +\infty)$, $x = a + 1 = 3$
for $a = 2$.

15. $[-1, 0] \cup [2, 3]$

16. (i) $0 < x < 2$, $x > 4$ (ii) $\left(\dfrac{1}{2^{\sqrt{2}}}, \dfrac{1}{2}\right) \cup (1, 2^{\sqrt{2}})$

17. (i) $(-\infty, -2/3] \cup [1/2, 2]$
(ii) $(3, 5 - \sqrt{3}) \cup (7, \infty)$
(iii) $(-1 - \sqrt{5}, -3) \cup (\sqrt{5} - 1, 5)$
(iv) $[-\sqrt{8}, -1) \cup (1, \sqrt{41} - 1)/5]$

18. (i) $x \in (a^4, a^{-1})$ for $a \in (0, 1)$; $x \in (a^{-1}, a^4)$ for $x \in (1, \infty)$
(ii) $-1 - \sqrt{23/2} \le x \le \sqrt{23/2} - 1$ if $a > 1$;
$-\sqrt{24} < x \le 1 - \sqrt{23/2}$,
$\sqrt{23/2} - 1 \le x < \sqrt{24}$ if $0 < a < 1$

19. $(-3, -1)$ **20.** $(-1, 0) \cup [1, \infty)$

21. (i) $\{8\}$ (ii) $\{4\}$

22. (i) $(0, 0.5) \cup (0.5, \infty)$
(ii) $(4^{-14}, 1)$
(iii) $\left(-\infty, \dfrac{7}{3}\right) \cup (3, \infty)$ (iv) $\left(0, \dfrac{1}{\sqrt{6}}\right] \cup [1, \infty)$

23. (i) $(\sqrt{61} - 9)/2 < x < 2$
(ii) $[2, 3) \cup (3, 4]$
(iii) $(-\infty, -5) \cup (3, \infty)$ (iv) $(0, 1) \cup [2, \infty)$

24. 0, 1

25. 1, –1, ± 1 + k where k is any positive proper fraction

Previous Years' Questions (JEE Advanced)

1. 2 **2.** 4 **3.** 4
4. F **5.** D **6.** B
7. B **8.** C **9.** A
10. A **11.** A **12.** B
13. B **14.** C **15.** A, D
16. A,B,C **17.** A **18.** 3/2
19. 5/4 **20.** $a^{-1/2}$, $a^{-4/3}$ **22.** 3
24. 3 **26.** $[-1, 2) \cup [3, \infty)$

27. $m \in \left(-\infty, \dfrac{-15}{2}\right) \cup (30, \infty)$

28. $\pm 2, \pm\sqrt{2}$ **29.** $\{a - a\sqrt{2}, -a + a\sqrt{6}\}$

30. $(-2, -1) \cup \left(-\dfrac{2}{3}, -\dfrac{1}{2}\right)$

31. $-4, -1 - \sqrt{3}$

32. $x = 0$ or $5/3$

33. $\{-1\} \cup [1, \infty)$

www.pdfworld.net

www.pdfworld.net

2 Quadratic Equations

2.1 INTRODUCTION

A polynomial is an algebraic expression consisting of a finite number of terms with whole number exponents on the variables. Some examples of polynomials are

$3x + 2$, $x^4 - 5x^2 + 3x + 4$, and $3x^2y - 6xy - 7$.

The first two examples are called polynomials in x and the third is a polynomial in x and y. A polynomial containing exactly one term is called a monomial, one with exactly two terms is a binomial, and one with exactly three terms is a trinomial. Thus, $2x^2$ is a monomial, $2x^2 + 11x$ a binomial, and $2x^2 + 11x + 13$ a trinomial.

Algebraic expressions whose variables do not contain whole number exponents, such as $3x^{-2} + 7$ and $5x^{3/2} + 9x^{1/2} + 2$ are not polynomials.

Equation and Solution Sets

An equation is a statement indicating that two algebraic expressions are equal.

Examples of equations in x :
$2x - 3 = 5$ $x^2 - 3x = 2$ $2x + 3x = 5x$.

The third equation in the list is true no matter what value the variable x represents. An equation that is true for all real numbers for which both sides are defined is called an identity.

The first two equations in our list are true for only some numbers.

Solving an equation in x means determining all values of x which when substituted into the equation result in a true statement. Such values are called solutions or roots of the equation. These values are said to satisfy the equation. The set of all such solutions is called the equation's solution set.

For example, the equation $2x - 3 = 5$. becomes a true statement if we substitute 4 for x, resulting in $2(4) - 3 = 5$ (or $8 - 3 = 5$), a true statement. Since 4 is the only number that results in a true statement for this equation, the equation's solution set is {4}.

Linear Equation

A linear or first-degree equation in one variable x is an equation that can be written in the standard form $ax + b = 0$ where a and b are real numbers and $a \neq 0$.

An example of a linear equation in one variable is $4x + 12 = 0$.

The solution is $x = -3$.

www.pdfworld.net

www.pdfworld.net

2.2 QUADRATIC EQUATION

An equation that contains a variable with an exponent of 2, but no higher power, is called a quadratic equation.

A quadratic equation in x is an equation that can be written in the standard form

$ax^2 + bx + c = 0$

where a, b and c are real numbers with $a \neq 0$. A quadratic equation in x is also called a second-degree polynomial equation in x.

In the definition of a quadratic equation, we must state $a \neq 0$. If we allowed a to become 0 in $ax^2 + bx + c = 0$ the resulting equation would be $bx + c = 0$ which is linear, and not quadratic.

Solving Quadratic Equations by Factoring

Factoring is the process of writing a polynomial as the product of two or more polynomials. We generally factor over the set of reals, meaning that the numerical coefficients in the factors are reals. For example,

$$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$$

and $$\frac{1}{9}x^2 - 25 = \left(\frac{1}{3}x + 5\right)\left(\frac{1}{3}x - 5\right)$$

We say that $x^2 + 4$ cannot be factored using real coefficients. Such polynomials are irreducible over reals.

The goal in factoring a polynomial is to use one or more factoring techniques until each of the polynomial's factors is irreducible. In this situation, the polynomial is said to be a factored completely.

If the trinomial $ax^2 + bx + c$ can be factored, then $ax^2 + bx + c = 0$ can be solved by using the zero product principle.

The zero product principle

Let A and B be real numbers, variables, or algebraic expressions. If AB = 0, then A = 0 or B = 0 or A and B are both 0. In words, this says that if a product is zero, then at least one of the factors is equal to zero.

Factoring $ax^2 + bx + c$ by Grouping

An Example : $6x^2 + 19x - 7$

1. Multiply a and c $\quad 6(-7) = -42$
2. Find the factors of ac whose sum is b.
   21 and –2 are factors of –42 whose sum is 19.
3. Rewrite the middle term, bx as a sum or difference using the factors from step 2.
   $6x^2 + 21x - 2x - 7$
4. Factor by grouping.
   $= 3x(2x + 7) - 1(2x + 7)$
   $= (2x + 7)(3x - 1)$ or $(3x - 1)(2x + 7)$

The quadratic formula

The solution of the quadratic equation, $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Proof :

$ax + bx + c = 0$

Dividing by a we get $\quad x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If we set b/2 = k, then the formula will take the form

$$x = \frac{-k \pm \sqrt{k^2 - ac}}{a}$$

www.pdfworld.net

www.pdfworld.net

This formula is advisable to be used in the cases when the coefficient b is an even number.

The Discriminant

The expression under the radical in the quadratic formula, $b^2 - 4ac = D$, is called the **discriminant**. The value of the discriminant determines whether the quadratic equation has two real solutions, one real solution, or no real solutions depending on whether D is positive, zero or negative. In the latter case, there will be two imaginary roots.

If D = 0, both roots reduce to $-\frac{b}{2a}$, and are thus equal to one another. In this case we do not say that the equation has only one root, but that it has two equal roots. It is clear that the roots will be unequal unless D = 0. When D = 0, the expression $ax^2 + bx + c$ is a perfect square of a linear expression in x.

Special Forms of Quadratic Equation

We will now consider some special forms of quadratic equations, in which one or more of the coefficients vanish.

A. If c = 0, the equation reduces to

$$ax^2 + bc = 0,$$

or $$x(ax + b) = 0,$$

the roots of which are 0 and $-\frac{b}{a}$.

B. If c = 0 and also b = 0, the equation reduces to $ax^2 = 0$, both roots of which are zero.

C. If b = 0, the equation reduces to $ax^2 + c = 0$ (called as **pure** quadratic equation), the roots of which are $\pm\sqrt{-\frac{c}{a}}$. The roots are therefore equal and opposite when b = 0, that is when the coefficient of x is zero.

D. If a, b and c are all zero, the equation is clearly satisfied for all values of x.

E. If a and b are zero but c is not zero,

put $x = \frac{1}{y}$ in the equation $ax^2 + bx + c = 0$;

then we have, after multiplying by y^2,

$cy^2 + by + a = 0$.

Now one root of this quadratic in y is zero if a = 0, and both roots are zero if a = 0 and also b = 0.

But since $x = \frac{1}{y}$, x is infinity when y is zero. Thus one root of $ax^2 + bx + c = 0$ is infinite if a = 0; also both roots are infinite if a = 0 and also b = 0.

Thus the quadratic equation

$$(a - a')x^2 + (b - b')x + c - c' = 0$$

has one root infinite, if a = a′; it has two roots infinite, if a = a′ and also b = b′; and the equation is satisfied for all values of x, if a = a′, b = b′ and c = c′.

Again, the equation

$$a(x + b)(x + c) + b(x + c)(x + a) = c(x + a)(x + b),$$

is a quadratic equation for all values of c except only when c = a + b, in which case the coefficient of x^2 in the quadratic equation is zero. When c = a + b we may still however consider that the equation is a quadratic equation, but with one of its roots infinite.

It is however to be remarked that since infinite roots are not often of practical importance in Algebra, they are generally neglected unless specially required.

METHODS OF SOLVING A QUADRATIC EQUATION

Description and Form of the Quadratic Equation	*Most Efficient Solution Method*	*Example*
$ax^2 + c = 0$ The quadratic equation has no linear (x) term.	Solving for x^2 and the square root method	$4x^2 - 7 = 0$ $4x^2 = 7$ $x^2 = \frac{7}{4}$

www.pdfworld.net

www.pdfworld.net

		$x = \pm \frac{\sqrt{7}}{2}$
$(ax + b)^2 = d$	The square root method	$(x + 4)^2 = 5$ $x + 4 = \pm \sqrt{5}$ $x = -4 \pm \sqrt{5}$
$ax^2 + bx + c = 0$ where $ax^2 + bx + c$ can be easily factored.	Factoring and the zero product principle	$3x^2 + 5x - 2 = 0$ $(3x - 1)(x + 2) = 0$ $3x - 1 = 0$ or $x + 2 = 0$ $x = \frac{1}{3}$ $\quad x = -2$
$ax^2 + bx + c = 0$ and $ax^2 + bx + c$ cannot be easily factored	The quadratic formula : $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x^2 - 2x - 6 = 0$ $x = \frac{2 \pm \sqrt{4 - 4(1)(-6)}}{2(1)}$ $= \frac{2 \pm \sqrt{28}}{2}$ $= 1 \pm \sqrt{7}$

A quadratic equation cannot have more than two roots.

For, if possible, let the equation $ax^2 + bx + c = 0$ have three different roots α, β, γ. Then since each of these values must satisfy the equation, we have

$a\alpha^2 + b\alpha + c = 0,$ ...(1)

$a\beta^2 + b\beta + c = 0,$ ...(2)

$a\gamma^2 + b\gamma + c = 0.$ ...(3)

From (1) and (2), by substraction,

$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$;

dividing by $\alpha - \beta$ which is not zero, we get ;

$a(\alpha + \beta) + b = 0$

Similarly from (2) and (3)

$a(\beta + \gamma) + b = 0$;

$\therefore$ by subtraction

$a(\alpha - \gamma) = 0$;

which is impossible, since a is not zero.

➤ **Example 1.** Solve the equation $x^2 - 7x + 12 = 0$

➤ ***Solution*** Let x_1 and x_2 be roots of the given equation . Then $x_1x_2 = 12$, that is x_1 and x_2 are numbers having the same sign. Further, $x_1 + x_2 = 7$ and, hence, x_1 and x_2 are positive numbers. Thus, we need to find two positive numbers whose sum is equal to 7, and the product to 12. It is easy to guess that these numbers are 3 and 4. Thus, the roots of the given equation are $x_1 = 3, x_2 = 4$.

➤ **Example 2.** Solve the equation $(x - 1)(x-2)(x-3)(x-4) = 3$.

➤ ***Solution*** We have $(x - 1)(x - 4) = x^2 - 5x + 4$ and $(x - 2)(x - 3) = x^2 - 5x + 6$. The given equation can be rewritten as follows:

$$(x^2 - 5x + 4)(x^2 - 5x + 6) = 3.$$

We now introduce a new variable setting

$$x^2 - 5x + 4 = y;$$

note that $x^2 - 5x + 6 = (x^2 - 5x + 4) + 2 = y + 2$.

www.pdfworld.net

www.pdfworld.net

We obtain

$$y(y+2)=3 \Rightarrow y^2+2y-3=0 \Rightarrow y_1=-3,\ y_2=1.$$

It remains now to solve the following two equations:

$$x^2-5x+4=-3,\ x^2-5x+4=1.$$

The first equation has no solution, while from the second equation we find $x_1 = (5+\sqrt{13})/2$, $x_2 = (5-\sqrt{13})/2$.

STUDY TIP Whenever an equation contains fractions in whose denominators the unknown quantity occurs, the equation may be multiplied by the lowest common multiple of the denominators.

➤ **Example 3.** Solve the equation $\dfrac{3}{x-5}+\dfrac{2x}{x-3}=5.$

➤ ***Solution*** Multiply by $(x-5)(x-3)$, the L.C.M. of the denominators; then we have

$$3(x-3)+2x(x-5)=5(x-5)(x-3),$$

$$\therefore \quad 3x^2-33x+84=0,$$

$$x=4 \text{ or } x=7.$$

➤ **Example 4.** Solve the equation

$$\frac{2}{2-x}+\frac{1}{2}-\frac{4}{x(2-x)}=0$$

➤ ***Solution*** This equation contains the variable in the denominator. Let us transpose all the terms of the given equation into a common fraction.

$$\frac{2}{2-x}+\frac{1}{2}-\frac{4}{x(2-x)}=0;$$

$$\frac{4x+x(2-x)-8}{x(2-x)}=0; \qquad \frac{x^2-6x+8}{x(2-x)}=0.$$

Since the fraction is equal to zero, we get

$$\begin{cases} x^2-6x+8=0 \\ x(2-x)\neq 0 \end{cases}.$$

From the equation $x^2-6x+8=0$, we find : $x=2$, $x=4$. The statement $2(2-2)\neq 0$ is false; hence, 2 is not a root of the given equation (since for $x=2$ the denominator vanishes). The statement $2(4-2)\neq 0$ is true; hence, 4 is the only root of the given equation.

➤ **Example 5.** Solve the equation

$$\frac{x^2-3x}{x^2-1}+2+\frac{1}{x-1}=0.$$

➤ ***Solution*** Multiply by x^2-1, the L.C.M. of the denominators; then we have

$$x^2-3x+2(x^2-1)+x+1=0,$$

which reduces to $3x^2-2x-1=0$,

that is $(3x+1)(x-1)=0.$

Thus the roots appear to be $-\dfrac{1}{3}$ and 1; the latter root is however due to the multiplication by x^2-1. Since

$$\frac{x^2-3x}{x^2-1}+\frac{1}{x-1}=\frac{x^2-3x+x+1}{x^2-1}=\frac{(x-1)^2}{x^2-1}=\frac{x-1}{x+1},$$

the equation is equivalent to

$$\frac{x-1}{x+1}+2=0,$$

which has only one root, namely $x=-\dfrac{1}{3}$.

CAUTION From the above example it will be seen that when an equation has been made integral by multiplication, some of the roots of the resulting equation may have to be rejected.

Practice Problems [A]

1. Solve for x in the equation
   $2m(1+x^2)-(1+m^2)(x+m)=0.$
2. Find the roots of the equation
   $(b-c)x^2+(c-a)x+(a-b)=0.$
3. Solve the equation
   $(x^2-3x)^2+3x^2-9x-28=0$ for real roots.
4. Solve the equation
   $$\left(\frac{x-1}{x}\right)^2-3\left(\frac{x-1}{x}\right)-4=0.$$

www.pdfworld.net

www.pdfworld.net

5. Solve the equation $1-\frac{3-2x}{5-x}=\frac{3}{3-x}-\frac{x+3}{x+1}$.

2.3 SUM AND PRODUCT OF ROOTS

Vieta's Theorem

If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

(i) $\alpha+\beta=-b/a$ (ii) $\alpha\beta=c/a$

Proof :

Let the quadratic equation $ax^2 + bx + c = 0$ be written as

$$\frac{1}{a}(ax^2+bx+c)=0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad ...(1)$$

If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$ then

$$(x-\alpha)(x-\beta)=0$$

$$x^2-(\alpha+\beta)x+\alpha\beta=0 \qquad ...(2)$$

Comparing the coefficients of similar powers of x in (1) & (2) we have

$$\text{sum of roots}=\alpha+\beta=-\frac{b}{a}$$

$$\text{product of roots}=\alpha\beta=\frac{c}{a}$$

Note

$$|\alpha-\beta|=\sqrt{(\alpha-\beta)^2}=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

$$=\sqrt{\frac{b^2-4ac}{a^2}}=\frac{\sqrt{D}}{|a|}$$

Formation of Quadratic Equation

Although we cannot in all cases find the roots of a given equation, it is very easy to solve the converse problem, namely the problem of finding an equation which has given roots.

For example, to find the equation whose roots are 4 and 5.

We want to find an equation which is satisfied when x = 4, or when x = 5; that is when x – 4 = 0, or when x – 5 = 0; and in no other cases. The equation required must be

$$(x-4)(x-5)=0,$$

that is,

$$x^2-9x+20=0.$$

Suppose α and β are the roots of a quadratic equation

$$ax^2+bx+c=0$$

Then, we have

$$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$$

$$=x^2-(\text{sum of roots})\,x+(\text{product of roots})$$

$$=x^2+\frac{b}{a}x+\frac{c}{a}=\frac{1}{a}(ax^2+bx+c).$$

This leads to the factorization,

$$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

Thus, if we know the roots of a quadratic equation, then we can write down a factorization of the corresponding quadratic function.

Conversely, if α and β are two numbers (real or complex), then the most general quadratic equation having α and β as its roots is given by

$$a(x-\alpha)(x-\beta)=0,$$

where a is a non-zero, real or complex number. If we restrict the coefficient of x^2 to be 1 then we get a unique quadratic equation

$$(x-\alpha)(x-\beta)=0.$$

This is the same as $x^2-(\alpha+\beta)x+\alpha\beta=0$.

STUDY TIP A quadratic polynomial in which the coefficient of x^2 is 1 is called a monic quadratic polynomial (or a reduced quadratic).

➤ **Example 1.** Set a quadratic equation with the given numbers as its roots 7 and –3.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** We shall seek the desired equation in the form $x^2 - px + q = 0$. We have $p = (7 - 3) = 4$, $q = 7(-3) = -21$; then $x^2 - 4x - 21 = 0$.

➤ **Example 2.** A quadratic polynomial p(x) has $1 + \sqrt{5}$ and $1 - \sqrt{5}$ as roots and it satisfies $p(1) = 2$. Find the quadratic polynomial.

➤ ***Solution*** sum of the roots = 2

product of the roots = – 4

$\therefore$ let $p(x) = a(x^2 - 2x - 4)$

$p(1) = 2$

$\Rightarrow$ $2 = a(1^2 - 2 \cdot 1 - 4)$

$\Rightarrow$ $a = -2/5$

$\therefore$ $p(x) = -2/5\,(x^2 - 2x - 4)$.

➤ **Example 3.** Let α and β be roots of the equation $2x^2 - 3x - 7 = 0$. Without computing α and β, set a quadratic equation with α/β and β/α as its roots.

➤ ***Solution*** We shall seek the required equation in the form $x^2 - px + q = 0$, then

$$p = \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$q = \frac{\alpha}{\beta}\,\frac{\beta}{\alpha} = 1.$$

Since α and β are roots of the equation

$2x^2 - 3x - 7 = 0$,

$$\alpha\beta = -\frac{7}{2},\ \alpha + \beta = \frac{3}{2}.$$

$$\alpha^2 + \beta^2 = \left(\frac{3}{2}\right)^2 - 2\left(-\frac{7}{2}\right) = \frac{9}{4} + 7 = \frac{37}{4}$$

hence, $$p = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{37/4}{-7/2} = -\frac{37.2}{4.7} = -\frac{37}{14}$$

Thus, the coefficients p and q of the desired quadratic equation have been found : $p = -37/14$, $q = 1$. The equation has the form

$$x^2 + \frac{37}{14}x + 1 = 0, \text{ or } 14x^2 + 37x + 14 = 0.$$

➤ **Example 4.** If α and β are the roots of the equation $x^2 + \sqrt{2}\,x + 3 = 0$, find the monic quadratic having zeros

$\alpha^2 + \beta^2$ and $2\,\alpha\beta$.

➤ ***Solution*** The monic quadratic having $\alpha^2 + \beta^2$ and $2\alpha\beta$ as its roots is given by

$$x^2 - (\alpha^2 + \beta^2 + 2\alpha\beta)\,x + 2\alpha\beta(\alpha^2 + \beta^2). \quad ...(1)$$

However, we know that

$$\alpha + \beta = -\sqrt{2},\ \alpha\beta = 3.$$

Hence, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 2 - 6 = -4$.

This gives, $\alpha^2 + \beta^2 + 2\alpha\beta = 2$

and $2\alpha\beta\,(\alpha^2 + \beta^2) = -24$.

Hence, from (1) the required quadratic is $x^2 - 2x - 24 = 0$.

➤ **Example 5.** The quadratic equation $x^2 + mx + n = 0$ has roots which are twice those of $x^2 + px + m = 0$ and m, n and $p \neq 0$. Find the value of $\frac{n}{p}$.

➤ ***Solution*** $x^2 + mx + n = 0$ $\begin{cases} 2\alpha \\ 2\beta \end{cases}$

and $x^2 + px + n = 0$ $\begin{cases} \alpha \\ \beta \end{cases}$

$2(\alpha + \beta) = -m$ ....(1)

$4\,\alpha\beta = n$ ....(2)

and $\alpha + \beta = -p$ ....(3)

$\alpha\beta = m$ ....(4)

$\therefore$ (1) and (3) $\Rightarrow$ $2p = m$ and (2) and (4) $\Rightarrow$ $4m = n$

$$\Rightarrow \quad \frac{n}{p} = \frac{4m}{m/2} = 8.$$

➤ **Example 6.** Find the values of a for which one of the roots of $x^2 + (2a + 1)\,x + (a^2 + 2) = 0$ is twice the other root. Find also the roots of this equation for these values of a.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** $\alpha + 2\alpha = -(2a + 1)$

and $2\alpha^2 = a^2 + 2$.

The first relation gives $\alpha = \dfrac{-(2a+1)}{3}$

Substituting this value of α in the second relation, we get

$2(2a + 1)^2 = 9(a^2 + 2)$.

This is the same as

$a^2 - 8a + 16 = 0$.

Thus we get a quadratic equation for a and this equation has coincident roots a = 4. Thus there is a unique value of a for which the conditions of the problem are fulfilled. Corresponding to this value of a, the given equation reduces to

$x^2 + 9x + 18 = 0$

Now $x^2 + 9x + 18 = (x + 6)(x + 3)$

so that the roots are $\alpha = -6, \beta = -3$.

➤ **Example 7.** Let α, β be the roots of $ax^2 + bx + c = 0$, $\alpha_1, -\beta$ be the roots of $a_1x^2 + b_1x + c_1 = 0$ then find a quadratic equation whose roots are α, α_1.

➤ ***Solution*** Given $\alpha + \beta = -b/a$ ...(1)

$\alpha\beta = c/a$ ...(2)

Also $\alpha_1 - \beta = -b_1/a_1$ ...(3)

$-\alpha_1\beta = c_1/a_1$ ...(4)

$$\text{Now } (\alpha + \beta) + (\alpha_1 - \beta_1) = -\frac{b}{a} - \frac{b_1}{a_1}$$

$$\Rightarrow \quad \alpha + \alpha_1 = -\frac{b}{a} - \frac{b_1}{a_1} \qquad ...(5)$$

Form (1) & (2),

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-b/a}{c/a} = -\frac{b}{c} \qquad ...(6)$$

Similarly from (3) & (4),

$$\frac{1}{\alpha_1} - \frac{1}{\beta} = \frac{-b_1}{c_1} \qquad ...(7)$$

From (6) + (7),

$$\frac{1}{\alpha} + \frac{1}{\alpha_1} = -\frac{b}{c} - \frac{b_1}{c_1}$$

$$\Rightarrow \quad \frac{\alpha + \alpha_1}{\alpha\alpha_1} = \frac{-b}{c} - \frac{b_1}{c_1} \qquad ...(8)$$

Now the equation whose roots are α, α_1 is $x^2 - (\alpha + \alpha_1)x + \alpha\alpha_1 = 0$

$$\Rightarrow \quad \frac{x^2}{\alpha + \alpha_1} - x + \frac{\alpha\alpha_1}{\alpha + \alpha_1} = 0$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{b}{a} + \frac{b_1}{a_1}\right)} + x + \frac{1}{\left(\frac{b}{c} + \frac{b_1}{c_1}\right)} = 0$$

➤ **Example 8.** If α is a root of $4x^2 + 2x - 1 = 0$ prove that $4\alpha^3 - 3\alpha$ is the other root.

➤ ***Solution*** Let the other root be β then

$$\alpha + \beta = -\frac{2}{4} = -\frac{1}{2}$$

$$\Rightarrow \quad \beta = -\frac{1}{2} - \alpha \qquad ...(1)$$

and $4\alpha^2 + 2\alpha - 1 = 0$, because α is a root of $4x^2 + 2x - 1 = 0$.

Now $4\alpha^3 - 3\alpha = \alpha(4\alpha^2 - 3)$

$= \alpha(1 - 2\alpha - 3)$ {since $4\alpha^2 + 2\alpha - 1 = 0$}

$$= -2\alpha^2 - 2\alpha = -\frac{1}{2}(4\alpha^2) - 2\alpha = -\frac{1}{2}(1 - 2\alpha) - 2\alpha$$

$$= -\frac{1}{2} - \alpha = \beta \qquad \{\text{from (1)}\}$$

Practice Problems [B]

1. Suppose the sum of the roots of $ax^2 - 6x + c = 0$ is -3 and their product is 2, find the values of a and c.
2. Given that α and β are the roots of $6x^2 - 5x - 3 = 0$ find the reduced quadratic equation whose roots are

   $\alpha - \beta^2$ and $\beta - \alpha^2$.

www.pdfworld.net

www.pdfworld.net

3. Find a quadratic in x which is divisible by $x - 2$ and assumes value -7 and 3 for $x = 1$ and -1 respectively.

4. Find the values of m, for which the equation $5x^2 - 24x + 2 + m(4x^2 - 2x - 1) = 0$ has
(a) equal roots
(b) the product of the roots is 2
(c) the sum of the roots is 6

5. Suppose x_1 and x_2 are roots of the equation $x^2 + x - 7 = 0$. Find
(a) $x_1^2 + x_2^2$
(b) $x_1^3 + x_2^3$
(c) $x_1^4 + x_2^4$ without solving the equation.

6. If one root of $k(x - 1)^2 = 5x - 7$ is double the other root, show that k is either 2 or -25.

7. For what values of a is the ratio of the roots of the equation $x^2 + ax + a + 2 = 0$ equal to 2 ?

8. For what value of a is the difference between the roots of the equation $(a - 2)x^2 - (a - 4)x - 2 - 0$ equal to 3 ?

9. If the difference of roots of the equation $2x^2 - (a + 1)x + a - 1 = 0$ is equal to their product, then prove that $a = 2$.

10. Two candidates attempt to solve a quadratic equations of the form, $x^2 + px + q = 0$. One starts with a wrong value of 'p' and finds the roots to be 2 & 6. The other starts with a wrong value of 'q' and finds the roots to be 2 & -9. Find the correct roots.

11. If α and β are the roots of the equation $x^2 + px + q = 0$, then show that α/β is a root of the equation $qx^2 - (p^2 - 2q)x + q = 0$.

12. Find $b \in I$ for the equation $5x^2 + bx - 28 = 0$ if the roots α and β satisfy $5\alpha + 2\beta = 1$.

13. If the roots of the equations $ax^2 + bx + c = 0$ are $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, prove that $(a + b + c)^2 = b^2 - 4ac$.

14. If α, β be the roots of the equation $ax^2 + bx + c = 0$ and γ, δ those of equation $lx^2 + mx + n = 0$, then find the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$.

15. Let p(x) be a quadratic polynomial such that for distinct reals α and β, $p(\alpha) = \alpha$, $p(\beta) = \beta$ show that α and β are roots of $p[p(x)] - x = 0$.

2.4 IDENTITY

An **identity** in x is satisfied by all permissible values of x, where as an **equation** in x is satisfied by some particular value of x. For example, $(x + 1)^2 = x^2 + 2x + 1$ is an identity in x. It is satisfied for all values of x.

A quadratic equation is satisfied exactly two values of 'x' which may be real or imaginary. We should note that the quadratic equation $ax^2 + bx + c = 0$ cannot have more than two different roots, unless $a = b = c = 0$; and when a, b, c are all zero it is clear that the equation $ax^2 + bx + c = 0$ will be satisfied for all values of x, that is to say the equation is an identity.If a quadratic equation is satisfied by three or more distinct values of 'x', then it is an identity.

1. Two equations are said to be **equivalent** if they have the same roots.

2. Two equations in x are **identical** if and only if the coefficient of similar powers of x in the two equations are proportional. Identical equations have the same roots.

3. An equation remains unchanged if it is multiplied or divided by a non-zero number.
Thus, $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ are identical, if $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$
$x^2 - 5x + 4 = 0$ and $2x^2 - 10x + 8 = 0$ are identical equations.
Both these equations have same roots 1 and 4.

www.pdfworld.net

www.pdfworld.net

STUDY TIP The equation $ax^2 + bx + c = 0$ is :

a quadratic equation if	$a \neq 0$	Two Roots
a linear equation if	$a = 0, b \neq 0$	One Root
a contradiction if	$a = b = 0, c \neq 0$	No Root
an identity if	$a = b = c = 0$	Infinite Roots

➤ **Example 1.** Solve the equation

➤ ***Solution***

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^2$$

The equation is satisfied is satisfied by $x = a$, by $x = b$, or by $x = c$. Since it is only of the second degree in x, it must be an identity.

➤ **Example 2.** Solve the equation a^2

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}=x^2.$$

➤ ***Solution*** The equation is clearly satisfied by $x = a$, and also by $x = b$; hence a, b are roots of the equation, and these are the only roots of the quadratic equation. The equation is not an identity, for it is not satisfied by $x = c$.

➤ **Example 3.** If $a+b+c=0$; $an^2+bn+c=0$ and $a+bn+cn^2=0$ where $n \neq 0, 1$, then prove that $a = b = c = 0$.

➤ ***Solution*** Note that $ax^2 + bx + c = 0$ is satisfied by

$$x = 1 \; ; \; x = n \; \& \; x = \frac{1}{n} \quad \text{where} \quad n \neq \frac{1}{n}$$

$\Rightarrow$ The quadratic equation has 3 distinct real roots which implies that it must be an identity.

➤ **Example 4.** Show that

$$\frac{(x+b)(x+c)}{(b-c)(c-a)}+\frac{(x+c)(x+a)}{(c-b)(a-b)}+\frac{(x+a)(x+b)}{(a-c)(b-c)}=1$$

is an identity.

➤ ***Solution*** The given equation is quadratic equation in x.

put $x = -a$ in the given equation

$$\Rightarrow \quad \text{L.H.S.} = \frac{(b-a)(c-a)}{(b-a)(c-a)} = 1 = \text{R.H.S.}$$

Put $x = -b$ in given equation

$$\Rightarrow \quad \text{L.H.S.} = \frac{(c-b)(a-b)}{(c-b)(a-b)} = 1 = \text{R.H.S.}$$

Put $x = -c$ in given equation

$$\Rightarrow \quad \text{L.H.S.} = \frac{(a-c)(b-c)}{(a-c)(b-c)} = 1 = \text{R.H.S.}$$

So, given equation is satisfied by 3 values. So it is an identity.

➤ **Example 5.** Show that values of a, b, c exist such that $2x^2 - 5x - 1 \equiv a(x+1)(x-2) + b(x-2)(x-1) + c(x-1)(x+1)$ and find these values.

➤ ***Solution*** The right-hand side

$= a(x^2 - x - 2) + b(x^2 - 3x + 2) + c(x^2 - 1)$

$= (a + b + c)x^2 - (a + 3b)x - (2a - 2b + c)$.

This is equal to the left-hand side for all values of x, if values for a, b, c can be found such that

$a + b + c = 2$, $a + 3b = 5$, $2a - 2b - c = 1$.

These equation have one solution, namely,

$a = 2$, $b = 1$, $c = -1$.

STUDY TIP In a question of this kind if it has been shown that an identity of the specified form exists, the values of the constants may be found by giving special values to the variables.

Thus, assuming that an identity of the specified form exists, we can find a, b, c, by putting $x = 1, -1, 2$ in succession.

2.5 TRANSFORMATION OF EQUATION

From the equation $f(x) = 0$ we may form an equation whose roots are connected with those of the given equation by some assigned relation.

www.pdfworld.net

www.pdfworld.net

Let y be a root of the required equation and let $\phi(x, y) = 0$ denote the assigned relation ; then the transformed equation can be obtained either by expressing x as a function of y by means of the equation $\phi(x, y) = 0$ and substituting this value of x in $f(x) = 0$; or by eliminating x between the equations $f(x) = 0$ and $\phi(x, y) = 0$.

STUDY TIP Let α, β be the roots of $ax^2 + bx + c = 0$, and suppose that we require the equation whose roots are a uniform function of α, β say $\phi(\alpha), \phi(\beta)$, where $f(x)$ is a given function of x.

Let $y = \phi(x)$ and suppose that from this equation we can find x in terms of y,

which we denote by $x = \phi^{-1}(y)$.

Transforming the equation $ax^2 + bx + c = 0$ by the substitution $x = \phi^{-1}(y)$, we obtain a new equation in y, which contains the roots $\phi(\alpha), \phi(\beta)$.

To find the equation whose roots are the squares of those of a proposed equation.

Let $f(x) = 0$ be the given equation ; putting $y = x^2$, we have $x = \sqrt{y}$; hence the required equation is $f(\sqrt{y}) = 0$.

➤ **Example 1.** Find the equation whose roots are the squares of those of the equation $x^2 + ax + b = 0$.

➤ ***Solution*** Putting $x = \sqrt{y}$, and transposing, we have

$$(y + b) = -a\sqrt{y} \ ;$$

whence $y^2 + 2by + b^2 = a^2y$,

or $y^2 + (2b - a^2)y + b^2 = 0$

To transform an equation into another whose roots exceed those of the proposed equation by a given quantity.

Let $f(x) = 0$ be the proposed equation, and let h be the given quantity ; put $y = x + h$, so that $x = y - h$; then the required equation is $f(y - h) = 0$.

Similarly $f(y + h) = 0$ is an equation whose roots are less by h than those of $f(x) = 0$.

➤ **Example 2.** Remove the second term from the equation $px^2 + qx + r = 0$.

➤ ***Solution*** Let α, β be the roots, so that $\alpha + \beta = -\frac{q}{p}$. The if we increase each of the roots by $\frac{q}{2p}$, in the transformed equation the sum of the roots will be equal to $-\frac{q}{p} + \frac{q}{p}$; that is, the coefficient of the second term will be zero.

Hence the required transformation will be effected by substituting $x - \frac{q}{2p}$ for x in the given equation.

$$p\left(x - \frac{q}{2p}\right)^2 + q\left(x - \frac{q}{2p}\right) + r = 0$$

➤ **Example 3.** Let α and β be the roots of $x^2 - 3x + 1 = 0$. Find a quadratic equation whose roots are $\frac{\alpha}{\alpha - 2}$ and $\frac{\beta}{\beta - 2}$. Hence or otherwise find the value of $\frac{\alpha}{\alpha - 2} \cdot \frac{\beta}{\beta - 2}$.

➤ ***Solution*** Let $y = \frac{x}{x - 2}$ where x can take values α, β

$$\therefore \quad xy - 2y = x \quad \Rightarrow \quad x = \frac{2y}{y - 1} \qquad \text{....(1)}$$

substituting the value of x in $x^2 - 3x + 1 = 0$ and simplifying we get the required quadratic as

$$y^2 - 4y - 1 = 0 \qquad \text{....(2)}$$

now from (2) $\frac{\alpha}{\alpha - 2} \cdot \frac{\beta}{\beta - 2} = -1$ using product of roots.

www.pdfworld.net

www.pdfworld.net

➤ **Example 4.** If $\tan\alpha, \tan\beta$ are the roots of $x^2 - px + q = 0$ and $\cot\alpha, \cot\beta$ are the roots of $x^2 - rx + s = 0$ then find the value of rs in terms of p and q.

➤ ***Solution*** $x^2 - px + q = 0 \begin{cases} \tan\alpha \\ \tan\beta \end{cases}$ ....(1)

$x^2 - rx + s = 0 \begin{cases} \cot\alpha \\ \cot\beta \end{cases}$ ....(2)

Here roots of (2) are reciprocal of (1)

$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

$\therefore$ put x as $\frac{1}{y}$ in (2) $\Rightarrow \frac{1}{y^2} - \frac{r}{y} + s = 0$

$\Rightarrow sy^2 - ry + 1 = 0$

We can replace y by x to get an equation in x,

$sx^2 - rx + 1 = 0$ ....(3)

Comparing (1) and (3)

$$\frac{1}{s} = \frac{p}{r} = q \; s = \frac{1}{q}; \; r = \frac{p}{q}$$

hence $rs = \frac{p}{q^2}$.

Practice Problems [C]

1. If $(p^2 - 1)x^2 + (p - 1)x + p^2 - 4p + 3 = 0$ be an identity in x, then find the value of p.
2. Find all polynomials p satisfying $p(x + 1) = p(x) + 2x + 1$.
3. If n is any number, show that n^2 can be expressed in the form $a(n-1)^2 + b(n-2)^2 + c(n-3)^2$, and find the values of a, b, c.
4. Show that constants a, b, c can be found such that $3x^2 + 16x + 23 \equiv a(x + 2)(x + 3) + b(x + 3)(x + 1) + (x + 1)(x + 2)$ and find the values of a, b, c.
5. If a, b, c are distinct real numbers, then solve for x:

$$\frac{(a+x)^2}{(a-b)(a-c)} + \frac{(b+x)^2}{(b-c)(b-a)} + \frac{(c+x)^2}{(c-a)(c-b)} = 1.$$

6. Solve the equation, $\frac{x-ab}{a+b} + \frac{x-bc}{b+c} + \frac{x-ca}{c+a} = a + b + c$. What happens if $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 0$.
7. If α, β are the roots of $2x^2 - 5x - 4 = 0$, find the simplest quadratic equation whose roots are $\alpha + 1/\alpha, \beta + 1/\beta$.
8. Prove that the equation whose roots are squares of the roots of the equation $ax^2 + bx + c = 0$ is $a^2x^2 + (2ac - b^2)x + c^2 = 0$
9. If α and β are roots of the equation $x^2 - 2x + 3 = 0$, find the equation whose roots are
   (i) $\alpha + 2, \beta + 2$ (ii) $\frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}$
10. If α and β are roots of the equation $ax^2 + bx + c = 0$, find the roots of the equation
    (i) $ac\,x^2 - (b^2 - 2ac)x + ac = 0$
    (ii) $acx^2 + b(a + c)x + (a + c)^2 = 0$ in terms of α and β.

2.6 SYMMETRIC FUNCTIONS OF ROOTS

By the symmetric function of roots we mean those functions which remain unaltered in value when any two of the roots are interchanged. Without knowing the separate values of the roots in terms of the coefficients, we can calculate the values of symmetric functions of roots in terms of the coefficients. For example, if α, β are roots of quadratic equation $ax^2 + bx + c = 0$, we can find out the value of

(i) $\alpha + \beta + \alpha\beta$, (ii) $\alpha^2\beta + \beta^2\alpha$

in terms of a, b and c since each of these relations are symmetric functions of the roots, ase there will be no change in them if α and β are interchanged.

To find out the value of the above symmetric functions, we take the help of the following relations, $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$ and some important formulae which are given as follows:

www.pdfworld.net

www.pdfworld.net

STUDY TIP In order to find the value of symmetric function of α and β, express the given function in terms of α + β and αβ. For this use the following results whichever is applicable.

(i) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha.\beta$

(ii) $\alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

(iii) $\alpha^4+\beta^4 = (\alpha^3+\beta^3)(\alpha+\beta)-(\alpha\beta^3+\alpha^3\beta)$

(iv) $(\alpha+\beta)^4 = (\alpha^4+6\alpha^3\beta+4\alpha^2\beta^2+6\alpha\beta^3+\beta^4$
$= \alpha^4+\beta^4+6\alpha\beta(\alpha^2+\beta^2)+4\alpha^2\beta^2$

(v) $\alpha^5+\beta^5 = (\alpha^3+\beta^3)(\alpha^2+\beta^2)-(\alpha^2\beta^3+\alpha^3\beta^2)$
$= (\alpha^3+\beta^3)(\alpha^2+\beta^2)-\alpha^2\beta^2(\alpha+\beta)$

➤ **Example 1.** If α and β are the roots of $x^2+4x+6=0$, then find the value of

(i) $1/\alpha+1/\beta$ (ii) $\alpha^3+\beta^3$ (iii) $\alpha/\beta+\beta/\alpha$

we have $\alpha+\beta=-4$, $\alpha\beta=6$.

➤ ***Solution*** Hence $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{-4}{6}=\frac{-2}{3}$.

We can write

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=-64+72=8.$$

And lastly, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

$$=\frac{(-4)^2-(2\times6)}{6}=4/6=2/3.$$

➤ **Example 2.** Let α and β be the roots of the quadratic equation $(x-2)(x-3)+(x-3)(x+1)+(x+1)(x-2)=0$. Find the value of

$$\frac{1}{(\alpha+1)(\beta+1)}+\frac{1}{(\alpha-2)(\beta-2)}+\frac{1}{(\alpha-3)(\beta-3)}.$$

➤ ***Solution*** Quadratic equation is

$3x^2-8x+1=0 \Rightarrow \alpha, \beta$

$\alpha+\beta=\frac{8}{3}$ and $\alpha\beta=\frac{1}{3}$

to find the value of

$$\frac{1}{\alpha\beta+(\alpha+\beta)+1}+\frac{1}{\alpha\beta-2(\alpha+\beta)+4}$$
$$+\frac{1}{\alpha\beta-3(\alpha+\beta)+9}$$

$$=\frac{1}{4}-1+\frac{3}{4}=-\frac{3}{4}+\frac{3}{4}=0$$

➤ **Example 3.** Let p & q be the two roots of the equation, $mx^2+x(2-m)+3=0$. Let m_1, m_2 be the two values of m satisfying $\frac{p}{q}+\frac{q}{p}=\frac{2}{3}$. Determine the numerical value of $\frac{m_1}{m_2^2}+\frac{m_2}{m_1^2}$.

➤ ***Solution*** $mx^2+(2-m)x+3=0 \Rightarrow p, q$

$$p+q=\frac{m-2}{m}\ ;\ pq=\frac{3}{m}$$

now m_1 and m_2 satisfies $\frac{p}{q}+\frac{q}{p}=\frac{2}{3}$

$$\Rightarrow \frac{p^2+q^2}{pq}=\frac{2}{3}$$

$$\frac{(p+q)^2-2pq}{pq}=\frac{2}{3}$$

$$\left(\frac{m-2}{m}\right)^2-\frac{6}{m}=\frac{2}{3}.\frac{3}{m}=\frac{2}{m}$$

$$\Rightarrow \frac{(m-2)^2}{m^2}=\frac{8}{m}$$

$$m^2-4m+4=8m \Rightarrow m^2-12m+4=0$$

$\therefore$ $m_1+m_2=12$ and $m_1m_2=4$

now $\frac{m_1}{m_2^2}+\frac{m_2}{m_1^2}=\frac{m_1^3+m_2^3}{(m_1m_2)^2}$

$$=\frac{(m_1+m_2)^3-3m_1m_2(m_1+m_2)}{(m_1m_2)^2}$$

$$=\frac{12^3-12.12}{16}=\frac{12^2.11}{16}=99$$

www.pdfworld.net

www.pdfworld.net

Newton's Theorem

If α, β are roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ then for $n > 2$, $n \in N$, we have $aS_n + bS_{n-1} + cS_{n-2} = 0$

Proof :

$a(\alpha^n + \beta^n) + b(\alpha^{n-1} + \beta^{n-1}) + c(\alpha^{n-2} + \beta^{n-2})$

$= \alpha^{n-2}(a\alpha^2 + b\alpha + c) + \beta^{n-2}(a\beta^2 + b\beta + c)$

$= 0$ since α, β satisfy the equation.

➤ **Example 4.** If α, β are roots of $x^2 - 3x + 1 = 0$, find the value of $\alpha^4 + \beta^4$.

➤ ***Solution*** For the equation

$S_1 = \alpha + \beta = 3, \qquad \alpha\beta = 1.$

$S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 3^2 - 2 \times 1 = 7$

For $n = 3$, Newton's theorem gives

$S_3 - 3S_2 + S_1 = 0$

$\Rightarrow \quad S_3 = 3S_2 - S_1 = 3 \times 7 - 3 = 18$

For $n = 4$,

$S_4 - 3S_3 + S_2 = 0$

$\Rightarrow \quad S_4 = 3S_3 - S_2 = 3 \times 18 - 7 = 47$

$\therefore \quad \alpha^4 + \beta^4 = 47$

Alternative :

$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$

$= ((\alpha + \beta)^2 - 2\alpha\beta)^2 - 2(\alpha\beta)^2$

$= (3^2 - 2 \times 1)^2 - 2(1)^2$

$= 7^2 - 2 = 47$

➤ **Example 5.** If α & β are the roots of the equation $x^2 - ax + b = 0$ and $v_n = \alpha^n + \beta^n$, show that $v_{n+1} = av_n - bv_{n-1}$ and hence obtain the value of $\alpha^5 + \beta^5$.

➤ ***Solution*** $\alpha + \beta = a$; $\alpha\beta = b$; $v_n = \alpha^n + \beta^n$

$v_{n+1} = \alpha^{n+1} + \beta^{n+1}$

$= (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta^n - \beta\alpha^n$

$= a\, v_n - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

Now $\alpha^5 + \beta^5 = v_5 = a\, v_4 - b\, v_3 = a(a\, v_3 - b\, v_2) - b\, v_3$

$= (a^2 - b)\, v_3 - ab\, v_2$

$= (a^2 - b)[a\, v_2 - b\, v_1] - ab\, v_2$

$= (a(a^2 - b) - ab)\, v_2 - b(a^2 - b)\, v_1$

$= (a^3 - 2ab)(a^2 - 2b) - ab(a^2 - b)$

$= a^5 - 2a^3b - 2a^3b + 4ab^2 - a^3b + ab^2$

$= a^5 - 5a^3b + 5ab^2$

$\Rightarrow \quad v_{n+1} = a\, v_n - b\, v_{n-1}$

2.7 POLYNOMIAL FUNCTION OF ROOT

Let α be a root of the equation $ax^2 + bx + c = 0$. Let $g(x)$ be a polynomial in x, then $g(\alpha)$ can be reduced to a linear expression in α using $a\alpha^2 + b\alpha + c = 0$.

$g(x) = (ax^2 + bx + c).\, h(x) + px + q$

$\therefore \quad g(\alpha) = p\alpha + q$

➤ **Example 1.** When $x = \dfrac{3 + 5\sqrt{-1}}{2}$, find the value of $2x^3 + 2x^2 - 7x + 70$ and show that it will be unaltered if $\dfrac{3 - 5\sqrt{-1}}{2}$ be substituted for x.

➤ ***Solution*** Let us form a quadratic equation whose roots are $\dfrac{3 \pm 5\sqrt{-1}}{2}$;

the sum of the roots = 3 ;

the product of the roots = $\dfrac{17}{2}$;

hence the equation is $2x^2 - 6x + 17 = 0$;

$\therefore \quad 2x^2 - 6x + 17$ is a quadratic expression which vanishes for either of the values $\dfrac{3 \pm 5\sqrt{-1}}{2}$.

Now $2x^3 + 2x^2 - 7x + 70$

$= x(2x^2 - 6x + 17) + 4(2x^2 - 6x + 17) + 2$

$= x \times 0 + 4 \times 0 + 2 = 2$;

which is the numerical value of the expression in each of the supposed cases.

www.pdfworld.net

www.pdfworld.net

Practice Problems [D]

1. If α and β are the roots of $x^2 + 3x - 2(x + 7) = 0$, compute $\alpha^3 + \beta^3$, $\alpha/\beta + \beta/\alpha$ and $\alpha^2\beta + \alpha\beta^2$.
2. If α, β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $\alpha^2 + \beta^2$ and $\alpha^{-2} + \beta^{-2}$.
3. Form the equation whose roots are the squares of the sum and of the difference of the roots of $2x^2 + 2(m + n)x + m^2 + n^2 = 0$
4. If β, β are roots of the equation $x^2 - px + q = 0$, find the value of
   (i) $\alpha^2(\alpha^2\beta^{-1} - \beta) + \beta^2(\beta^2\alpha^{-1} - \alpha)$
   (ii) $(\alpha - p)^{-4} + (\beta - p)^{-4}$
5. If α, β are roots of $ax^2 + bx + c = 0$,then find the value of $(a\alpha + b)^{-3} + (a\beta + b)^{-3}$
6. Let α, β be the roots of the equation
   $x^2 + ax - \dfrac{1}{2a^2} = 0$, being a real parameter,
   prove that $\alpha^4 + \beta^4 \geq 2 + \sqrt{2}$.
7. If $x = -5 + 2\sqrt{-4}$ then find the value of the expression $x^4 + 9x^3 + 35x^2 - x + 4$.
8. If α, β are the roots of equation, $x^2 - 2x + 3 = 0$, find the equation whose roots are $\alpha^3 - 3\alpha^2 + 5\alpha - 2$ and $\beta^3 - \beta^2 + \beta + 5$
9. If $x^2 - 3x + 4 = 0$, prove that $x^4 = 3x - 20$. If the roots of the equation $x^2 - 3x + 4 = 0$ are α and β, construct the equation whose roots are α^4 and β^4.

2.8 NATURE OF ROOTS

Nature of roots of a quadratic equation $ax^2 + bx + c = 0$ implies whether the roots are real or imaginary. By analysing the expression $b^2 - 4ac$, called the discriminant D, we get an idea about the nature of roots.

Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in R$ and $a \neq 0$ then

(i) $D > 0 \Leftrightarrow$ roots are real and distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real and coincident (equal).

(iii) $D < 0 \Leftrightarrow$ roots are imaginary. If $p + iq$ is one root of a quadratic equation with real coefficients, then the other must be its conjugate $p - iq$ and vice versa. $(p, q \in R$ & $i = \sqrt{-1})$.

STUDY TIP Difference between Root and Solution:

A root of a polynomial equation may be real or imaginary while a solution has to be real. A quadratic equation having two distinct real roots is said to have two solutions. If it has two equal real roots then it has one solution. If it has two imaginary roots then we say that the equation has no solution.

CAUTION By complex numbers, we mean numbers which can be written in the form a + i b where a, b are real numbers and $i = \sqrt{-1}$. This includes both real and imaginary numbers. Hence by saying that a quadratic equation has complex roots, it does not imply that the roots are imaginary.

Factorization

If the equation $ax^2 + bx + c = 0$ has real roots α and β, then we have the factorization

$$ax^2 + bx + c = a(x - \alpha)(x - \beta). \quad ...(i)$$

Thus the quadratic polynomial $ax^2 + bx + c$ can be written as a product of linear factors, each linear factor having only real coefficients. If the given equation has no real roots, then a factorization of the form (i) with α and β real is impossible. Nevertheless, we can find a factorization of the form (i) with imaginary α and β. If the quadratic expression $ax^2 + bx + c$ is such that it has no factorization of the form (i) with real α and β, then we say the polynomial $ax^2 + bx + c$ is irreducible over R.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

Since the given equation has real roots if and only if the discriminant is non-negative, we conclude that $ax^2 + bx + c$ is irreducible over R is and only if the discriminant $D = b^2 - 4ac$ is negative. A quadratic expression will be a perfect square of a linear expression if the discriminant of its corresponding equation is zero.

1. If D_1 and D_2 are discriminant of two quadratic equations and $D_1 + D_2 \geq 0$ then atleast one of D_1 & $D_2 \geq 0$.
   $\Rightarrow$ Atleast one of the equations has real roots.
2. $D_1 + D_2 < 0$
   $\Rightarrow$ At least one of D_1 and $D_2 < 0$.
   $\Rightarrow$ At least one of the equations has imaginary root.
3. If D_1 is the discriminant of the equation $a_1x^2 + b_1x + c_1 = 0$ and D_2 is of $a_2x^2 + b_2x + c_2 = 0$ and $(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x\, c_2) = 0$ ...(1)
   (i) $D_1 D_2 < 0$
   $\Rightarrow D_1 > 0$ & $D_2 < 0$ or $D_1 < 0$ & $D_2 > 0$
   Then equation (1) has two real roots.
   (ii) $D_1 D_2 > 0$
   Case I : $D_1 > 0$ & $D_2 > 0$
   Then equation (1) has four real roots.
   Case II : $D_1 < 0$ & $D_2 < 0$
   Then equation (1) has no real roots.
   (iii) $D_1 D_2 = 0$
   Case I : $D_1 > 0$ & $D_2 = 0$ or $D_1 = 0$ & $D_2 > 0$
   Then equation (1) has two equal real roots & two distinct roots.
   Case II : $D_1 < 0$ & $D_2 = 0$ or $D_1 = 0$ & $D_2 < 0$
   Then equation (1) has two equal real roots.
   Case III : $D_1 = 0$ & $D_2 = 0$
   Then equation (1) has two real roots each repeated twice.
4. If sum of the coefficients of equation $ax^2 + bx + c = 0$ vanishes i.e. $a + b + c = 0$, then $x = 1$ is a root of the equation. Similarly if $4a + 4b + c = 0$ then $x = 2$ is a root of the equation $ax^2 + bx + c = 0$.

➤ **Example 1.** If the equation $x^2 + 2(k + 2)x + 9k = 0$ has equal roots, find k.

➤ ***Solution*** The condition for equal roots gives
$(k + 2)^2 = 9k$,
$k^2 - 5k + 4 = 0$,
$(k - 4)(k - 1) = 0$;
$k = 4$ or 1.

➤ **Example 2.** Find all values of the parameter a for which the quadratic equation
$(a+1)x^2 + 2(a + 1)x + a - 2 = 0)$ has
(a) two distinct roots, (b) no real roots,
(c) two equal roots.

➤ ***Solution*** For the quadratic equation $a \neq -1$ and the discriminant $D = 4(a+1)^2 - 4(a + 1)(a - 2)$
$= 4(a + 1)(a + 1 - a + 2) = 12(a + 1)$.
If $a > -1$, then $D > 0$ and this equation has two distinct roots.
If $a < -1$ then $D < 0$, and this equation has no real roots.
This equation cannot have two equal roots since $D = 0$ only for $a = -1$, and this is not acceptable.

➤ **Example 3.** If the equation $ax^2 + 2bx + c = 0$ has real roots, a, b, c being real numbers and if m and n are real numbers such that $m^2 > n > 0$ then prove that the equation $ax^2 + 2mbx + nc = 0$ has real roots.

➤ ***Solution*** Since roots of the equation
$ax^2 + 2bx + c = 0$ are real
$\therefore$ $(2b)^2 - 4ac \geq 0$
$\therefore$ $b^2 - ac \geq 0$ ...(1)
and discriminant of $ax^2 + 2mbx + nc = 0$
is $D = (2mb)^2 - 4anc$
$D = 4m^2b^2 - 4anc$ ...(2)
from (1) $b^2 \geq ac$ ...(3)
and given $m^2 > n$ ...(4)
$\therefore$ $b^2 m^2 \geq anc$
$\Rightarrow$ $4b^2m^2 - 4\,anc \geq 0$
$\Rightarrow$ $D \geq 0$ {from (2)}
Hence roots of equation $ax^2 + 2mbx + nc = 0$ are real.

www.pdfworld.net

www.pdfworld.net

➤ **Example 4.** Show that the expression $x^2 + 2(a + b + c)\,x + 3(bc + ca + ab)$ will be a perfect square if $a = b = c$.

➤ ***Solution*** Given quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero.

i.e. $4(a + b + c)^2 - 4.3\,(bc + ca + ab) = 0$

or $(a + b + c)^2 - 3\,(bc + ca + ab) = 0$

or $\frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2) = 0$

which is possible only when $a = b = c$.

➤ **Example 5.** If $p, q, r, s \in R$ and α & β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has two real roots.

➤ ***Solution*** $\alpha + \beta = -p\,;\ \alpha\beta = q\,;\ \alpha^4 + \beta^4 = r\,;\ \alpha^4\beta^4 = s$

Now $D = 16\,q^2 - 4\,(2\,q^2 - r)$

$= 8\,q^2 + 4\,r = 4\,(2\,q^2 + r)$

$= 2\,[\,2\,\alpha^2\beta^2 + \alpha^4 + \beta^4\,]$

$= 2\,[\,(\alpha^2 + \beta^2)^2 + 2\,\alpha^2\,\beta^2\,]$

$= 2\,[\,\{\alpha + \beta)^2 - 2\,\alpha\,\beta\}^2 + 2\,\alpha^2\,\beta^2\,]$

$= 2\,(p^2 - 2\,q)^2 + 2\,q^2\,] > 0$

Hence the equation $x^2 - 4qx + 2q^2 - r = 0$ has two real roots.

➤ **Example 6.** If $a, b, c \in R$ and $x^2 + bx + c = 0$ has no real roots. Prove that equation $x^2 + bx + c(x + a)\,(2x + b) = 0$ has real roots for every a.

➤ ***Solution*** The second equation is

$(1 + 2c)\,x^2 + [b + c(b + 2a)]x + abc = 0$

whose discriminant $D = [b + c(b + 2a)]^2 - 4abc\,(1 + 2c)$

Arranging D as a quadratic in a,

$D = 4a^2c^2 - 4abc^2 + b^2(c + 1)^2$ ...(1)

Now roots of the second equation will be real for all a if the last expression is non-negative for all a. The necessary and sufficient condition for this is that the discriminant D* of the last expression is non-positive.

Indeed $D^* = 16b^2c^4 - 16b^2c^2(c + 1)^2$

$= 16b^2c^2(c^2 - (c + 1)^2) = 16b^2c^2(-2c - 1)$ ...(2)

Now the original equation has no real roots

$\Rightarrow$ $b^2 - 4c < 0 \Rightarrow c > 0 \Rightarrow$ The expression in (2) is ≤ 0

$\Rightarrow$ The expression (1) is positive for all a.

Practice Problems [E]

1. Find the values of a for which the roots of the equation $(2a - 5)\,x^2 - 2\,(a - 1)\,x + 3 = 0$ are equal.
2. Find the values of a for which the equation $a^2x^2 + 2(a + 1)x + 4 = 0$ has coincident roots.
3. If $a + b + c = 0$ and a, b, c are real, then prove that equation $(b - x)^2 - 4(a - x)\,(c - x) = 0$ has real roots and the roots will not be equal unless $a = b = c$.
4. For each of the following equations find the set of values of a for which the equation has real roots.
   (a) $ax^2 + 9x - 1 = 0$
   (b) $2x^2 + 4x + a = 0$
   (c) $x^2 + (a + 2)x + a = 0$
5. If $a \neq b$, prove that the roots of $2(a^2 + b^2)\,x^2 + 2\,(a + b)\,x + 1 = 0$ are imaginary.
6. If the roots of the equation $x^2 - 2cx + ab = 0$ are real and unequal, prove that the roots of $x^2 - 2\,(a + b)\,x + a^2 + b^2 + 2c^2 = 0$ will be imaginary.
7. Prove that the equation $(bx + c)^2 - a^2x^4 = 0$ has atleast two real roots for all real values of a, b, c.
8. If the equation $x^2 - 2px + q = 0$ has two equal roots, then prove that the equation $(1 + y)x^2 - 2(p + y)x + (q + y) = 0$ will have its roots real and distinct only when y is negative and p is not unity.
9. Find the nature of roots of $(b - x)^2 - 4\,(a - x)\,(c - x) = 0$ where a, b, c are distinct real numbers.
10. Suppose p and q are real numbers which do not take simultaneously the values $p = 0$, $q = 1$. Suppose the equation $(1 - q + p^2/2)x^2 + p(1 + q)\,x + q(q - 1) + p^2/2 = 0$ has two equal roots. Prove that $p^2 = 4q$.
11. If the roots of the equation $x^2 + bx + c = 0$ are real, show that the roots of the equation $x^2 + bx + c(x + a)\,(2x + b) = 0$ are again real for every real number a.
12. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real root.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

2.9 RATIONAL ROOTS

Consider the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$ then ;

(i) If $-\frac{b}{a} \in Q, \frac{c}{a} \in Q$ and $D = b^2 - 4ac$ is a perfect square of a rational number then both roots are rational.

(ii) If $-\frac{b}{a} \in Q, \frac{c}{a} \in Q$ and $D = b^2 - 4ac > 0$ is not a perfect square of a rational number then both roots are irrational.

If $p + \sqrt{q}$ is one root in this case, (where p and q are rational) then the other root must be its conjugate i.e. $p - \sqrt{q}$ and vice versa.

(iii) If $-\frac{b}{a} \in I, \frac{c}{a} \in I$ and $\frac{b^2 - 4ac}{4a^2}$ is a perfect square of an integer, then both the roots are integers. In particular when $a = 1, b, c \in I$ and $b^2 - 4ac$ is a perfect square then both roots are integers.

Proof :

Let the equation $ax^2 + bx + c = 0$ have two integral roots, say α and β.

Since $\alpha, \beta \in I, \alpha + \beta = -\frac{b}{a} \in I$

$$\alpha\beta = \frac{c}{a} \in I$$

Also $\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{1}{2}\left(-\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 4.\frac{c}{a}}\right)$$

Let as assume $\frac{b}{a} = p, \frac{c}{a} = q, p, q \in I.$

Then $\alpha, \beta = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4q}\right)$

For integral roots, $p^2 - 4q$ must be a perfect square. $\sqrt{p^2 - 4q}$ is even or odd depending on p being even or odd. In both situations α, β come out to be integers since $-p \pm \sqrt{p^2 - 4q}$ is always even.

Hence if $-\frac{b}{a} \in I, \frac{c}{a} \in I$ and $\frac{b^2 - 4ac}{4a^2}$ is a perfect square of an integer, then both the roots are integers. In the special case when $a = 1, b, c \in I$ and $b^2 - 4ac$ is a perfect square then both roots are integers.

STUDY TIP It should be remarked that both roots are rational or both irrational, if the coefficients are rational, and that both roots are real or both imaginary, if the coefficients are real. If the coefficients are real, then a rational root and an irrational root may occur together and if the coefficients are complex, then a real root and an imaginary root may occur together.

➤ **Example 1.** Show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational, where p, q, r are rational numbers.

➤ ***Solution*** The roots will be rational provided the coefficients are rational (which is obvious) and

$D = (-2p)^2 - 4(p^2 - q^2 + 2qr - r^2) = 4(q^2 - 2qr + r^2)$
$= 4(q - r)^2$ is a perfect square of a rational number. Hence the roots are rational.

➤ **Example 2.** Form a quadratic equation with rational coefficients if one of its root is $\cot^2 18°$.

➤ ***Solution*** $\cot^2 18° = \frac{1 + \cos 36°}{1 - \cos 36°} = \frac{1 + \frac{\sqrt{5}+1}{4}}{1 - \frac{\sqrt{5}+1}{4}}$

$$= \frac{(5 + \sqrt{5})(3 + \sqrt{5})}{(3 - \sqrt{5})(3 + \sqrt{5})}$$

$$= \frac{20 + 5\sqrt{5} + 3\sqrt{5}}{9 - 5} = \frac{20 + 8\sqrt{5}}{4} = 5 + 2\sqrt{5}$$

www.pdfworld.net

www.pdfworld.net

Hence if $\alpha = 5+2\sqrt{5}$ then $\beta = 5-2\sqrt{5}$

$\therefore$ $\alpha+\beta = 10;\ \alpha\beta = 25-20 = 5$

$\therefore$ The quadratic equation is $x^2 - 10x + 5 = 0$.

➤ **Example 3.** If both the roots of the quadratic equation $x^2 - (2n + 18)x - n - 11 = 0$, $n \in I$, are rational, then find the value(s) of n.

➤ ***Solution*** Here the coefficients are integers, hence rational.

Now the discriminant of given equation must be a perfect square of a rational number.

i.e. $4[(n+9)^2 + n + 11]$ must be perfect square

$\Rightarrow$ $n^2 + 19n + 92$ must be perfect square of a whole number since $n \in I$

$\Rightarrow$ $n^2 + 19n + 92 = m^2$ where $m \in W$

$$\Rightarrow \quad n = \frac{-19 \pm \sqrt{4m^2 - 7}}{2}$$

$\Rightarrow$ $4m^2 - 7$ is a perfect square of a whole number

$\Rightarrow$ $4m^2 - 7 = p^2$ where $p \in W$

$\Rightarrow$ $4m^2 - p^2 = 7$

$\Rightarrow$ $(2m + p)(2m - p) = 7$

$\Rightarrow$ either $2m + p = \pm 1,\ 2m - p = \pm 7$

or $2m + p = \pm 7,\ 2m - p = \pm 1$

But $m, p \in W$

either $2m + p = 1,\ 2m - p = 7$

or $2m + p = 7,\ 2m - p = 1$

$2p = -6$ (not acceptable as $p \in W$)

$2p = 6,\ 2m = 4$

$\Rightarrow$ $m = 2$

$\Rightarrow$ $n^2 + 19n + 92 = 4$

$\Rightarrow$ $(n + 8)(n + 11) = 0 \Rightarrow n = -8$ or -11

➤ **Example 4.** Find $a \in I$ so that both the roots of the equation $ax^2 + (3a + 1)x - 5 = 0$ are integers.

➤ ***Solution*** We write the equation as

$$x^2 + \left(3 + \frac{1}{a}\right)x - \frac{5}{a} = 0.$$

$$3 + \frac{1}{a} \in I \text{ when } a = \pm 1.$$

$$\frac{5}{a} \in I \text{ when } a = \pm 1, \pm 5.$$

We try $a = \pm 1$ to check whether D is a perfect square or not. With $a = 1$, D is a perfect square. Hence the value of a is 1.

➤ **Example 5.** Find a, if $ax^2 - 4x + 9 = 0$ has integral roots.

➤ ***Solution*** Let $a = \frac{1}{b}$, so that the given equation becomes $x^2 - 4bx + 9b = 0$.

This equation has integral roots if b is an integer and $16b^2 - 36b$ is a perfect square

Let $b(4b - 9) = k^2 \Rightarrow 4b^2 - 9b - k^2 = 0$

$$\Rightarrow \quad \left(2b - \frac{9}{4}\right)^2 - k^2 = \frac{81}{16}$$

$\Rightarrow$ $(8b - 9)^2 - 16k^2 = 81$

$\Rightarrow$ $(8b - 9 - 4k)(8b - 9 + 4k) = 81 = 3 \times 27.$

Since b and k are integers, $8b - 9 - 4k = 3$ and $8b - 9 + 4k = 27$

$$\Rightarrow \quad 16b - 18\ \ 30 \Rightarrow b = 3 \Rightarrow a = \frac{1}{3}.$$

For any other factorization of 81, b will not be an integer.

➤ **Example 6.** Find all the integral values of a for which the quadratic expression $(x - a)(x - 10) + 1$ can be factored as a product of two factors $(x + \alpha)(x + \beta)$ where $\alpha, \beta \in I$.

➤ ***Solution*** We have $(x-a)(x-10)+1 = (x+\alpha)(x+\beta)$

Putting $x = -\alpha$ on both sides, we obtain

$(-\alpha - a)(-\alpha - 10) + 1 = 0$

$(\alpha + a)(\alpha + 10) = -1$

$\alpha + a$ and $\alpha + 10$ are integers

(since $a, \alpha \in I$)

$\therefore$ $\alpha + a = -1$ and $\alpha + 10 = 1$

or $\alpha + a = 1$ and $\alpha + 10 = -1$

(i) If $\alpha + 10 = -1$ then $\alpha = -9$ and $a = 8$. Similarly $\beta = -9$

Here $(x - 8)(x - 10) + 1 = (x - 9)^2$

(ii) If $a + 10 = -1$ then $\alpha = -11$ and $a = 12$. Similarly $\beta = 12$

Here $(x - 12)(x - 10) + 1 = (x - 11)^2$.

www.pdfworld.net

www.pdfworld.net

➤ **Example 7.** If $a \neq 1, -2$ and $a \in Q$, show that the roots of the equation $(a^2 + a - 2)x^2 + (2a^2 + a - 3)x + a^2 - 1 = 0$ are rational.

➤ ***Solution*** Notice that $x = -1$ satisfies this equation since

$$(a^2 + a - 2)(-1)^2 + (2a^2 + a - 3)(-1) + a^2 - 1 = 0.$$

Using product of roots, the other root is $\frac{1-a^2}{a^2+a-2}$ which is a rational number since $a \in Q$.
Hence both the roots are rational.

Practice Problems [F]

1. Find all integral values of m for which the roots of the equation $mx^2 + (2m-1)x + m - 2 = 0$ are rational.
2. If $a, b, c \in Q$ & $a + c \neq b$, then prove that the roots of the quadratic equation $(a + c - b)x^2 + 2cx + (b + c - a) = 0$ are rational.
3. If a, b c are rational numbers, show that the roots of the equation $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$ are rational.
4. Prove that the roots of the quadratic equation $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0 (a, b, c \in Q)$ are rational.
5. For what integral values of 'a' the equation $x^2 - x(1-a) - (a+2) = 0$ has integral roots.
6. Find the integral values of x, for which $x^2 + 7x + 13$ is a perfect square.
7. Suppose α, a, b are integers and $b \neq -1$. Show that if α satisfies the equation $x^2 + ax + b + 1 = 0$, then $a^2 + b^2$ is composite.

2.10 COMMON ROOTS OF QUADRATIC EQUATIONS

Two Common Roots

Let us find the condition that the equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ may have two common roots.

The above equations have both roots common if they are identical. They are identical if and only if the coefficient of similar powers of x in the two equations are proportional.
Thus, $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have both roots common if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

STUDY TIP If it is known that an imaginary root is common in the above equations (where the coefficients are real) then the other root being its conjugate must also be common.
Similarly if it is known that an irrational root is common in the above equations (where the coefficients are rational) then the other root being its conjugate must also be common.
In these cases the condition for both common roots must be applied.

Atleast one Common Root

To find the condition that the equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ may have atleast one common root.
Suppose these equations are both satistied by $x = \alpha$; then

$$a\alpha^2 + b\alpha + c = 0,$$
$$a'\alpha^2 + b'\alpha + c' = 0;$$
$\therefore$ by cross multiplication
$$\frac{\alpha^2}{(ca' - b'c)} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}$$

To eliminate α, square the second of these equal ratios and equate it to the product of the other two; thus

$$\frac{\alpha^2}{(ca' - c'a)^2} = \frac{\alpha^2}{(bc' - b'c)} \cdot \frac{1}{ab' - a'b};$$
$\therefore$ $(ca' - c'a)^2 = (bc' - b'c)(ab' - a'b)$,
which is the condition required.
Also note that the common root is

$$\alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}.$$

STUDY TIP The above condition is also satisfied if the coefficients of the two equations are proportional. Hence this con-

www.pdfworld.net

www.pdfworld.net

dition includes the situation of two common roots. By applying this condition we get atleast one root common.

1. We can prove that this is also the condition that the two quadratic functions $ax^2 + bxy + cy^2$ and $a'x^2 + b'xy + c'y^2$ may have a common linear factor.
2. If $f(x) = 0$ & $g(x) = 0$ are two polynomial equations having some common root(s) then those common root(s) is/are also the root(s) of $h(x) = a\,f(x) + b\,g(x) = 0$, but not all roots of $h(x)$ are necessarily common roots.
3. To find the common root between the equations $ax^2 + bx + c = 0$, $a'x^2 + b'x + c' = 0$ make the coefficient of x^2 same by multiplying the equations by a' and a respectively and substract the resulting equations.

➤ **Example 1.** If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 5 = 0$ have a common root, show that $a : b : c = 1 : 2 : 5$.

➤ ***Solution*** Given equations are : $x^2 + 2x + 5 = 0$ .... (i)

and $ax^2 + bx + c = 0$ .... (ii)

Clearly roots of equation (i) are imaginary. Since equations (i) and (ii) have a common root, the common root must be imaginary and hence both roots will be common.

Threfore equations (i) and (ii) are identical

$$\therefore \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{5} \qquad \Rightarrow \qquad a : b : c = 1 : 2 : 5$$

➤ **Example 2.** If equations $ax^2 + bx + c = 0$ and $x^2 + 6x + 4 = 0$ have common root, show that $a : b : c = 1 : 6 : 4$.

➤ ***Solution*** Since roots of $x^2 + 6x + 4 = 0$ are irrational and equations $x^2 + 6x + 4 = 0$ & $ax^2 + bx + c = 0$ have a common root, both roots have to be common. Hence the equations have proportional coefficients.

$$\therefore \quad \frac{a}{1} = \frac{b}{6} = \frac{c}{4} \Rightarrow \qquad a : b : c = 1 : 6 : 4$$

➤ **Example 3.** Find the value of p if the equation $3x^2 - 2x + p = 0$ and $6x^2 - 17x + 12 = 0$ have a common root.

➤ ***Solution*** Given equations are

$3x^2 - 2x + p = 0$ ...(i)

and $6x^2 - 17x + 12 = 0$ ...(ii)

Let α be the common root, then

$3\alpha^2 - 2\alpha + p = 0$ ...(iii)

and $6\alpha^2 - 17\alpha + 12 = 0$ ...(iv)

$$\therefore \quad \underset{(1)}{\frac{\alpha^2}{-24 + 17p}} = \underset{(2)}{\frac{\alpha}{6p - 36}} = \underset{(3)}{\frac{1}{-51 + 12}} = \frac{1}{-39}$$

from (1) and (2), $\alpha = \dfrac{17p - 24}{6p - 36}$ ...(A)

from (2) and (3)

$$\alpha = \frac{6p - 36}{-39} = -\frac{2p - 12}{13} \qquad \text{...(B)}$$

from (A) and (B), $\dfrac{17p - 24}{6p - 36} = -\dfrac{2p - 12}{13}$

or, $12p^2 - 144p + 432 = -221p + 312$

or, $12p^2 + 77p + 120 = 0$

or, $4p(3p + 8) + 15(3p + 8) = 0$

or, $(4p + 15)(3p + 8) = 0$

$$\therefore \quad p = -\frac{15}{4}, -\frac{8}{3}.$$

➤ **Example 4.** Find the values of 'k' so that the equations

$x^2 + kx + (k + 2) = 0$ and

$x^2 + (1 - k)x + 3 - k = 0$

have exactly one common root.

➤ ***Solution*** Using the condition of common root :

$(ca' - ac')^2 = (bc' - cb')(ab' - ba')$

We have $[(k + 2) \cdot 1 - 1 \cdot (3 - k)]^2 = [k(3 - k) - (k + 2)(1 - k)] \cdot [(1 - k) \cdot 1 - k \cdot 1]$

$\Rightarrow \quad (2k - 1)^2 = 2(1 - 2k)(2k - 1)$

$$\Rightarrow \quad (2k - 1)^2 = 0 \Rightarrow \qquad k = \frac{1}{2}$$

www.pdfworld.net

www.pdfworld.net

CAUTION

We do not immediately produce the answer as $k = \frac{1}{2}$. Here we need to check whether this value is associated with exactly one common root or both common roots.

We place the value of k is the given equations.

We get $x^2 + \frac{1}{x}x + \frac{5}{2} = 0$

$$x^2 + \frac{1}{x}x + \frac{5}{2} = 0$$

Since all the coefficients are equal with $k = \frac{1}{2}$, we get both common roots.

Hence, we donot have any value of k for which exactly one root is common.

➤ **Example 5.** If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

➤ ***Solution*** $\alpha^2 + b\alpha + c = 0$ ....(1)

$b\alpha^2 + c\alpha + 1 = 0$ ....(2)

$$\frac{\alpha^2}{b - c^2} = \frac{\alpha}{bc - 1} = \frac{1}{c - b^2}$$

$\Rightarrow \quad \alpha = \frac{b - c^2}{bc - 1}$ or $\alpha = \frac{bc - 1}{c - b^2}$

$\Rightarrow \quad (bc - 1)^2 = (b - c^2)(c - b^2)$

$\Rightarrow \quad b^3 + c^3 + 1 - 3bc = 0$

$\Rightarrow \quad (b + c + 1)(b^2 + c^2 + 1 - bc - c - b) = 0$

$\Rightarrow \quad b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$

➤ **Example 6.** Determine all possible value(s) of 'p' for which the equation $ax^2 - px + ab = 0$ and $x^2 - ax - bx + ab = 0$ may have a common root, given a, b are non zero real numbers.

➤ ***Solution*** $x^2 - (a + b)x + ab = 0$

or $(x - a)(x - b) = 0 \quad \Rightarrow \quad x = a$ or b

if $x = a$ is the root of other equation,

$a^3 - ap + ab = 0 \quad \Rightarrow \quad p = a^2 + b$

if $x = b$ is the root of the other equation, then $ab^2 - pb + ab = 0; \quad p = a(1 + b)$

Hence $p = a^2 + b$ or $a(1 + b)$

Practice Problems [G]

1. If the equations $3x^2 + px + 1 = 0$ and $2x^2 + qx + 1 = 0$ have a common root, show that $2p^2 + 3q^2 - 5pq + 1 = 0$.
2. Prove that the equations $ax^2 + bx + c = 0$ and $2x^2 - 3x + 4 = 0$ cannot have a common root unless $6a = -4b = 3c$.
3. Prove that the equations $(q - r)x^2 + (r - p)x + p - q = 0$ and $(r - p)x^2 + (p - q)x + q - r = 0$ have a common root.
4. Let f(x) and g(x) be two quadratic polynomials all of whose coefficients are rational numbers. Suppose f(x) and g(x) have a common irrational root, then show that $g(x) = r f(x)$ for some rational number r.
5. If the equations $x^2 + abx + c = 0$ and $x^2 + acx + b = 0$ have a common root, prove that their other roots satisfy the equation $x^2 - a(b + c)x + a^2 bc = 0$.
6. If the equation $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and the other root of the second equation is the reciprocal of the first, then prove that $(q - b)^2 - bq(p - a)^2 = 0$.
7. If the equation $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$ has equal roots, and $4x^2 \sin^2\theta - 4 \sin\theta . x + 1 = 0$ and the previous equation have a common root, find the value of θ.
8. Find λ, μ so that the equation $4x^2 - 8x + 3 = 0$, $x^2 + \lambda x - 1 = 0$ and $2x^2 + x + \mu = 0$ may have a common root for each pair of equations.

www.pdfworld.net

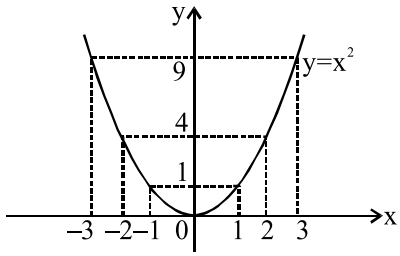
www.pdfworld.net

2.11 GRAPH OF QUADRATIC FUNCTION

Graph of $y = x^2$

Let us compile a table of its values :

x	–3	–2	–1	0	1	2	3
y	9	4	1	0	1	4	9

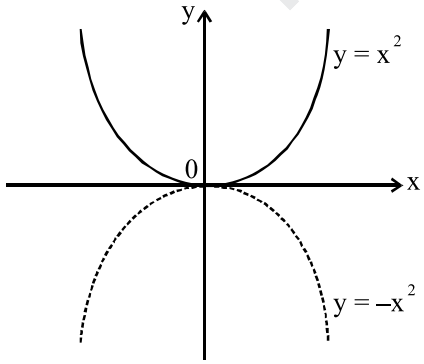


The graph of $y = x^2$ is called a parabola. Its graphs is symmetric about the y-axis. The point of intersection of the parabola with its axis of symmetry is called the vertex of the parabola. The vertex of the parabola $y = x^2$ is origin (0, 0).

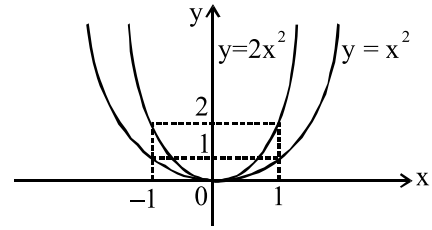
Also note that the function $y = x^2$ increases on the interval $(0, \infty)$ and decreases on the interval $(-\infty, 0)$.

Let us compare the function $y = -x^2$ and $y = x^2$. For the same value of x, the values of these functions are equal in magnitude but opposite in sign.

Consequently, the graph of the function $y = -x^2$ can be obtained by reflection of $y = x^2$ about the x-axis. The parabola $y = x^2$ opens upwards while $y = -x^2$ opens downward.

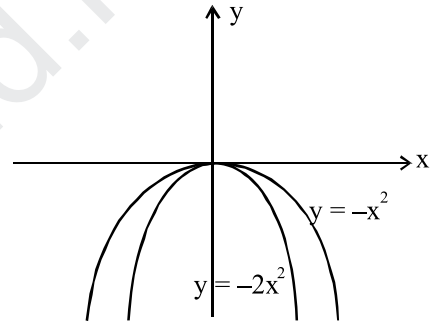


Let us now compare the functions $y = 2x^2$ and $y = x^2$. For the same value of x, the function $y = 2x^2$ is twice the value of the function $y = x^2$. Consequently, the graph of $y = 2x^2$ is obtained by stretching the graph of $y = x^2$, two times along the y–axis.

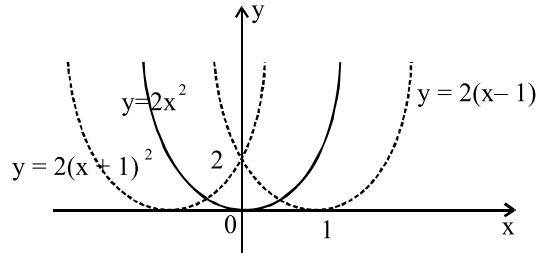


In general, the graph of $y = ax^2$ for $a > 0$ can be obtained by stretching the parabola $y = x^2$ a times along y axis (more precisely, by stretching for $a > 1$ and by compressing for $0 < a < 1$).

Similarly, the graph of $y = ax^2$ for $a < 0$ can be obtained by stretching the parabola $y = -x^2$, $|a|$ times along y-axis.



Now compare the functions $y = 2(x - 1)^2$ and $y = 2x^2$. The function $y\ 2(x - 1)^2$ takes on the same value as the function $y = 2x^2$, but with the corresponding value of the argument increased by unity. Consequently, the graph of the function $y = 2(x - 1)^2$ can be obtained by shifting the parabola $y = 2x^2$ one unit to the right. As a result, we shall get the parabola $y = 2(x - 1)^2$ whose axis of symmetry is parallel to the y-axis and whose vertex is (1, 0).



www.pdfworld.net

www.pdfworld.net

Similarly the graph of $y = 2(x + 1)^2$ is obtained by shifting $y = 2x^2$ one unit to the left.

Now let us draw $y = 2x^2 + 1$. The graph of $y = 2x^2 + 1$ is a parabola with vertex (0, 1) and y-axis as its axis of symmetry. This parabola can be obtained by shifting the parabola $y = 2x^2$ along the y-axis by 1 units upward.

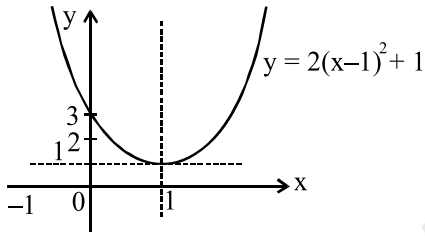
To contract the graph of $y = 2x^2 - 4x + 3$.

We rewrite the function $y = 2x^2 - 4x + 3$ by isolating a perfect square

$y = 2(x^2 - 2x + 1) + 1$

$\Rightarrow \quad y = 2(x - 1)^2 + 1.$

The graph of $y = 2(x - 1)^2 + 1$ is a parabola with vertex (1, 1), whose axis is a straight line passing through its vertex parallel to the y-axis. Also note that the parabola opens upward since $a > 0$.



In general, $y = ax^2 + bx + c$

$$\Rightarrow \quad y = a.\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right)$$

$$\Rightarrow \quad y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

From the above considered particular cases if follows that the graph of $y = ax^2 + bx + c$ is a parabola with axis $x = -\frac{b}{2a}$ and vertex $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$, which opens upward if $a > 0$ and downward if $a < 0$.

The graph of a quadratic function can be constructed with the following technique :

(i) Plot the axis $x = -\frac{b}{2a}$ and the vertex $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

(ii) Find the real roots of the quadratic function, if any, and plot the corresponding points of the parabola on the x-axis.

(iii) Mark (0, c), the point of intersection of the parabola with the y-axis.

(iv) The parabola is drawn, with month opening upwards if $a > 0$ and downward if $a < 0$.

STUDY TIP It is easy to check that the x co-ordinate of the vertex is $\frac{\alpha + \beta}{2}$ where α, β are roots of the quadratic function. So, once we mark the roots (real) on the x-axis, we draw the axis of the parabola as a vertical line mid-way between the roots. If the roots are equal, the axis is drawn there itself.

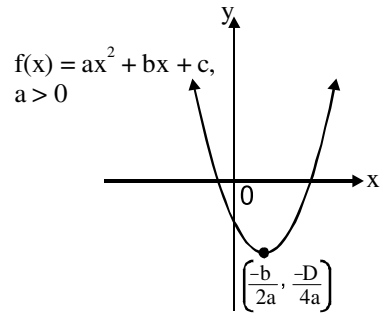
Extreme Point and Extreme Value of a Quadratic Function

For the quadratic function $P(x) = ax^2 + bx + c$,

(a) if $a > 0$, the vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is called the minimum point of the graph. The minimum value of the function is $f\left(\frac{-b}{2a}\right) = \frac{-D}{4a}$.

(b) if $a < 0$, the vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is called the maximum point of the graph. The maximum value of the function if $f\left(\frac{-b}{2a}\right) = \frac{-D}{4a}$.

The figure illustrate these points.

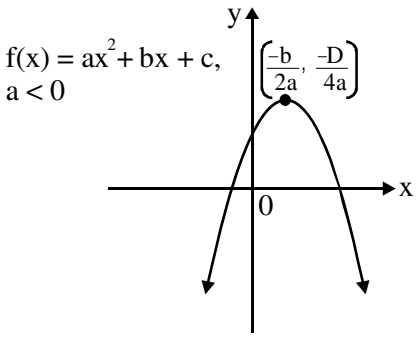


$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is the minimum point.

(a)

www.pdfworld.net

www.pdfworld.net



$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ is the maximum point.

(b)

➤ **Example 1**. Give the coordinates of the extreme point of the graph of each function, and the corresponding maximum or minimum value of the function.

(a) $f(x) = 2x^2 + 4x - 16$

(b) $f(x) = -x^2 - 6x - 8$

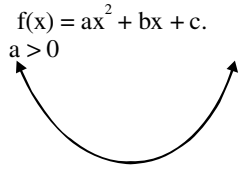
➤ ***Solution***

(a) The vertex of the graph of this function is $(-1, 18)$. If opens upward since $a > 0$, so the vertex $(-1, -18)$ is the minimum point and -18 is the minimum value of the function.

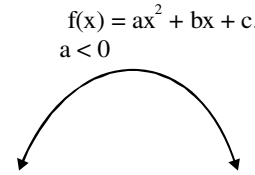
(b) The vertex $(-3, 1)$ is the maximum point and $f(-3) = 1$ is the maximum value of the function.

End Behaviour of the Graph of a Quadratic Function

We know that if the value of a is positive for the quadratic function $f(x) = ax^2 + bx + c$, the graph opens upward, and if a is negative, the graph opens downward. The sign of a determines the end behaviour of the graph. If $a > 0$, as x approaches $-\infty$ or ∞ (writing $x \to -\infty$ or $x \to \infty$), the value of f(x) approaches ∞ (written $f(x) \to \infty$). The other situations similar to this are summarized below.



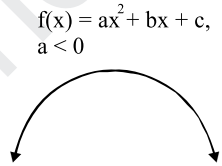
If $a > 0$,
as $x \to -\infty$, $f(x) \to \infty$;
as $x \to \infty$, $f(x) \to \infty$.



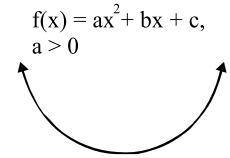
If $a < 0$,
as $x \to -\infty$, $f(x) \to -\infty$;
as $x \to \infty$, $f(x) \to -\infty$.

Concavity

Using the quadratic function graph as a illustration, we see that if $a > 0$, the graph is at all times opening upward. If water were to be poured from above, the graph would, in a sense, "hold water." We say that this graph is concave up for all values in its domain. On the other hand, if $a < 0$, the graph opens downward at all times, and it would similarly " dispel water" if it were poured from above. In this case, the graph is concave down for all values in its domain.



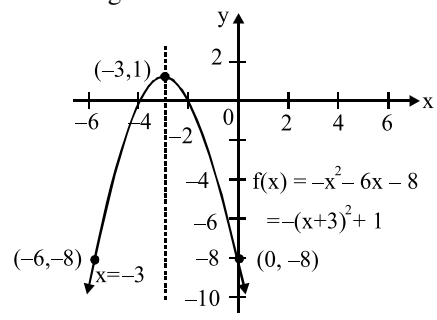
This graph is concave down.



This graph is concave up.

A formal discussion of concavity requires concepts beyond the scope of this text. It is studied more rigorously in calculus.

The graph of $f(x) = -x^2 - 6x - 8 = -(x + 3)^2 + 1$ is shown in the figure



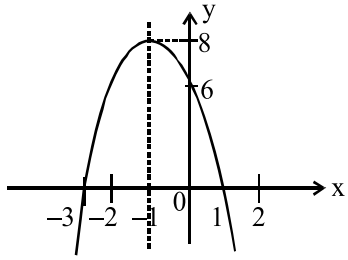
www.pdfworld.net

www.pdfworld.net

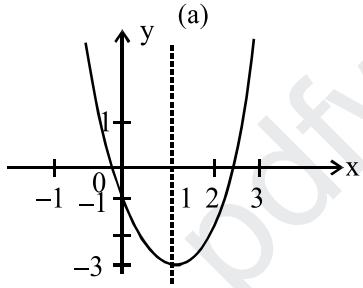
The parabola is the graph of $y = x^2$, translated 3 units to the left and 1 unit upward. It opens downward because of the negative sign before $(x + 3)^2$. The line $x = -3$ is its axis of symmetry, since if it were folded along this line, the two halves would coincide. The vertex, $(-3, 1)$, is the highest point on the graph. The domain is $(-\infty, \infty)$ and the range is $(-\infty, 1]$. The function increases on the interval $(-\infty, -3]$ and decreases on $[-3, \infty)$. Since $f(0) = -8$, the y-intercept is -8 , and since $f(-4) = f(-2) = 0$, the x-intercepts are -4 and -2.

➤ **Example 2.** Construct the graph of

(a) $y = -2x^2 - 4x + 6$ (b) $y = 2x^2 - 4x - 1$



(a)



(b)

➤ ***Solution***

(a) The vertex is $(-1, 8)$ using the formula $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$. The axis of symmetry is $x = -1$. $y = -2x^2 - 4x + 6 = -2(x - 1)(x + 3)$. Hence the roots are $x = 1, -3$. Observe that $\frac{1 + (-3)}{2} = -1$ is the abscissa of the vertex. The graph has an y intercept of 6 units. Since $a = -2 < 0$, the parabola is drawn opening downward.

(b) We can transform $y = 2x^2 - 4x - 1$ as $y = 2(x - 1)^2 - 3$. Hence the vertex is $(1, -3)$.

The roots are $1 \pm \sqrt{\frac{3}{2}}$.

STUDY TIP **Graph of $f(x) = a(x - h)^2 + k$**

The graph of $f(x) = a(x - h)^2 + k$, $a \neq 0$,

(a) is a parabola with vertex (h, k), and the vertical line $x = h$ as the axis of symmetry;

(b) opens upward if $a > 0$ and downward if $a < 0$;

(c) is wider than $y = x^2$ if $0 < |a| < 1$ and narrower than $y = x^2$ if $|a| > 1$

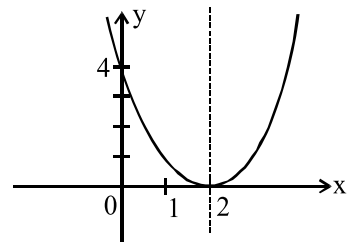
➤ **Example 3.** Construct the graphs of

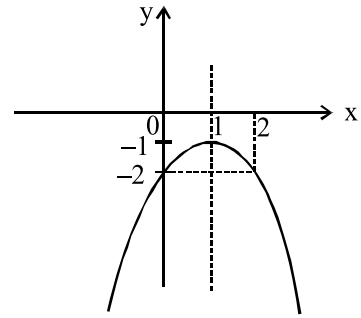
(a) $y = x^2 - 4x + 4$ (b) $y = -x^2 + 2x - 2$

➤ ***Solution***

(a) The vertex is $(2, 0)$. Notice that the graph will touch the x-axis at $(2, 0)$ where $x = 2$ appears as a repeated root. The graph cuts the y-axis at $(0, 4)$.

(b) The vertex is $(1, -1)$. Since there are no real roots, the graph does not intersect the x-axis. The y intercept is -2.





Signs of a, b, c and D

Consider the quadratic function $y = ax^2 + bx + c$.

1. The sign of 'a' is associated with the concavity of the parabola. If $a > 0$, then the parabola is

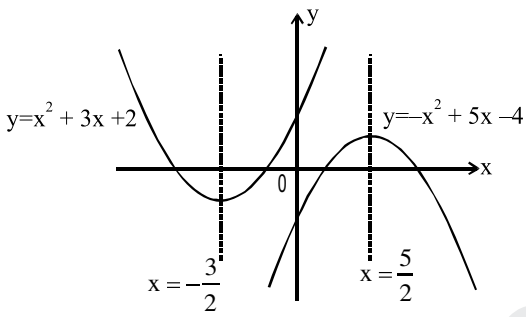
www.pdfworld.net

www.pdfworld.net

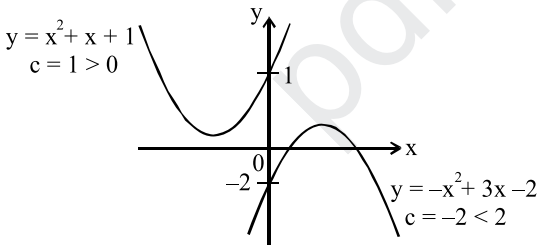
concave up and it is drawn with mouth opening upwards. If $a < 0$, then the parabola is concave down and it opens downward.

2. The sign of b is associated with the position of the axis of the parabola. Since the axis has the equation

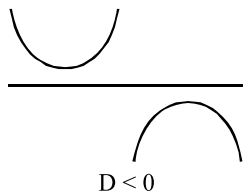
$x = -\frac{b}{2a}$, the axis is drawn to the left of y-axis if a & b are of same sign and it is drawn to the right of the y-axis if a & b are of opposite signs.



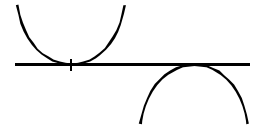
3. The sign of c is associated with the y-intercept of the parabola. If $c > 0$ then the parabola cuts the positive y-axis. If $c < 0$ then the parabola cuts the negative y-axis.



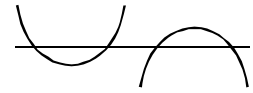
4. The sign of D is associated with the position of the parabola with respect to the x-axis.



D < 0

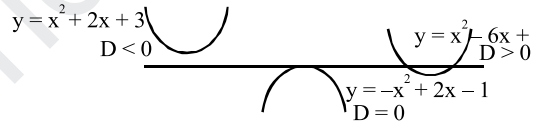


D = 0

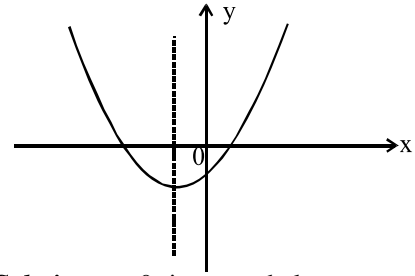


D > 0

The y-coordinate of the vertex is $-\frac{D}{4a}$. If $a > 0$, the vertex lies above, on or below the x-axis according as D is negative, zero or positive respectively. The number of times the parabola intersects the x-axis is equal to number of distinct real roots of the corresponding equation.



➤ **Example 4.** Predict the signs of a, b, c and D in the following graph:



➤ ***Solution*** $a > 0$ since parabola opens upward.

$b > 0$ since $-\frac{b}{2a} < 0$ and $a > 0$.

$c < 0$ since y intercept is negative.

$D > 0$ two distinct real roots.

Roots using sign change of function

If the graph of a continuous function f(x) crosses the x-axis between β_1 and β_2 then f(x) changes sign

www.pdfworld.net

www.pdfworld.net

between β_1 and β_2 i.e. $f(\beta_1)$ and $f(\beta_2)$ have opposite signs.

Hence, if $f(\beta_1)$ and $f(\beta_2)$ have opposite signs, then $f(x) = 0$ has atleast one real root between β_1 and β_2. In case of a quadratic equation $f(x) = 0$ has exactly one real root between β_1 and β_2 in such situation. Also if $f(\beta_1)$ and $f(\beta_2)$ have same sign then $f(x) = 0$ may or maynot have real root between β_1 and β_2.

If a quadratic expression in x can be made to change its sign by giving real values to x, then the roots of the corresponding equation must be real.

Consider, for example, the expression $a^2(x-\beta)(x-\gamma) + b^2(x-\gamma)(x-\alpha) + c^2(x-\alpha)(x-\beta)$, where the quantities are all real, and α, β, γ are supposed to be in order of magnitude. The expression is clearly positive if $x = \alpha$, and is negative if $x = \beta$. Hence the expression can be made to change its sign, and therefore the roots of the equation $a^2(x-\beta)(x-\gamma) + b^2(x-\gamma)(x-\alpha) + c^2(x-\alpha)(x-\beta) = 0$ are real for all real values of a, b, c α, β, γ.

➤ **Example 5.** The coefficients of the equation $ax^2 + bx + c = 0$ where $a \neq 0$, satisfy the inequality $(a + b + c)(4a - 2b + c) < 0$. Prove that this equation has 2 distinct real solutions.

➤ ***Solution*** $ax^2 + bx + c = 0 \quad (a \neq 0)$

given $(a + b + c)(4a - 2b + c) < 0$

$f(1) = a + b + c \qquad f(-2) = 4a - 2b + c$

$f(1) \cdot f(-2) < 0$

since $f(1)$ and $f(-2)$ have opposite signs

$\therefore$ exactly one root lies between -2 and 1

$\therefore$ this quadratic equation has 2 distinct real roots.

➤ **Example 6.** If $c < 0$ and the equation $ax^2 + bx + c = 0$, does not have any real root then prove that

(i) $a - b + c < 0 \qquad$ (ii) $9a + 3b + c < 0$.

➤ ***Solution*** Let $f(x) = ax^2 + bx + c$

$c = f(0) < 0$.

Since $f(x)$ has no real root the graph of $f(x)$ must open downward and should not cross the x-axis.

$\Rightarrow \quad f(x) = ax^2 + bx + c < 0$ for all $x \in R$

and hence $f(-1) = a - b + c < 0$

and $f(3) = 9a + 3b + c < 0$.

➤ **Example 7.** For what values of p, the number 6 lies between the roots of the equation $x^2 + 2(p-3)x + 9 = 0$

➤ ***Solution*** Since coefficient of x^2 is positive the graph of $f(x) = x^2 + 2(p-3)x + 9$ opens upward. If we have $f(6) < 0$ then one root will be less than 6 and other greater than 6.

$\Rightarrow \quad 36 + 12(p-3) + 9 < 0$

$\Rightarrow \quad 12p + 9 < 0$

$\Rightarrow \quad p < -3/4$

$p \in (-\infty, -3/4)$

➤ **Example 8.** If $a < b < c < d$, then prove that the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and distinct.

➤ ***Solution*** Consider $f(x) = (x-a)(x-c) + 2(x-b)(x-d)$

$f(a) = 2(a-b)(a-d) > 0$

$f(b) = (b-a)(b-c) < 0$

$f(c) = 2(c-b)(c-d) < 0$

$f(d) = (d-a)(d-c) > 0$

We see that $f(x)$ changes sign between a and b and again between c and d. Hence $f(x) = 0$ has one real between a and b and other between c and d.

Practice Problems [H]

1. Draw the graph of the following functions

(a) $y = x^2 + 5x + 6$

(b) $y = 4x^2 + 4x + 1$

(c) $y = x^2 + x + 1$

2. (a) $y = 3x - x^2 - 2$

(b) $y = 2x - x^2 - 1$

(c) $y = x - x^2 - 1$

3. For what values of p does the vertex of the parabola $y = x^2 + 2px + 13$ lie at a distance of 5 units from the origin?

4. Determine the intervals of the decrease of the function:

(a) $y = x^2 - 3x + 1 \qquad$ (b) $y = -x^2 - 4x + 8$

5. Determine the intervals of the increase of the function:

(a) $y = \frac{1}{3}x^2 + x - \sqrt{2} \qquad$ (b) $y = -2x^2 + 8x - 3$

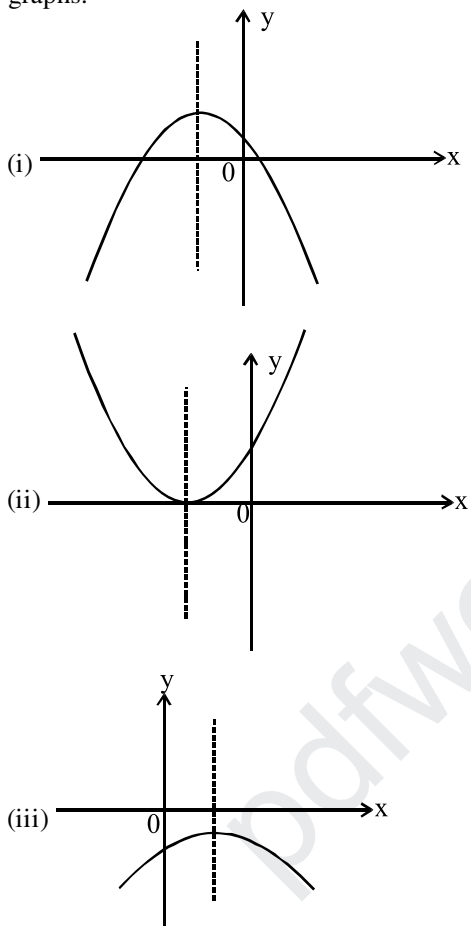
6. At what values of a does the function $f(x) = -x^2 + (a-1)x + 2$ increases on the interval $(1, 2)$?

www.pdfworld.net

www.pdfworld.net

7. Find the values of k for which the curve $y = x^2 + kx + 4$ touches the x axis.

8. Predict the signs of a, b, c and D in the following graphs:



9. If $ax^2 + bx + c = 0$ has no real roots and $a + b + c < 0$, then find the sign of the number c.

10. Prove that both the roots of the equation $(x-4)(x-5) + (x-2)(x-5) + (x-2)(x-4) = 0$ are always real.

11. If a, b, $c \in R$ and $a - b + 2c = 0$ then prove that the roots of the equation $ax^2 + bx + c = 0$ are real and distinct.

12. Given $f(x) = ax^2 + bx + c$, $f(-1) > -4$, $f(1) < 0$ and $f(3) > 5$, find the sign of 'a'.

2.12 QUADRATIC INEQUALITY

A quadratic inequality is any inequality that can be put in one of the forms

$2x^2 + bx + c < 0$ $\quad ax^2 + bx + c > 0$

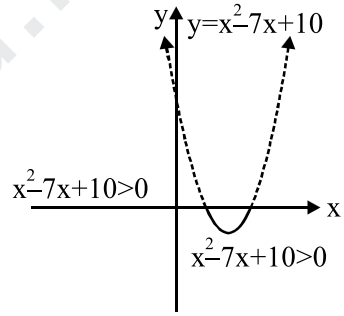
$ax^2 + bx + c \le 0$ $\quad ax^2 + bx + c \ge 0$

where a, b and c are real numbers and $a \neq 0$.

Consider, for example $x^2 - 7x + 10 > 0$.

Because we are interested in the inequality $x^2 - 7x + 10$ where $y > 0$, focus on the portion of the graph of $y = x^2 - 7x + 10$ where y is positive. This occurs where the graph lies above the x-axis, namely, when $x < 2$ or $x > 5$. Thus, the solution set to $x^2 - 7x + 10 > 0$ is $(-\infty, 2) \cup (5, \infty)$.

The solution set for $x^2 - 7x + 10 < 0$, occurs when the graph lies below the x-axis, between $x = 2$ and $x = 5$. Thus (2, 5), is the solution set for $x^2 - 7x + 10 < 0$.



The doted portion of $y = x^2 - 7x + 10$ represents $x^2 - 7x + 10 > 0$

Solving Quadratic Inequalities

Any quadratic expression $f(x) \equiv ax^2 + bx + c$ has the same sign between its real roots and changes sign only when the graph of f(x) passes through any of its real roots. If f(x) has no real roots, then f(x) has the same sign for all real values and the sign of f(x) is determined by the sign of the coefficient a.

If f is a quadratic function, the solution sets of $f(x) = 0$, $f(x) < 0$, and $f(x) > 0$ can be found graphically based on the following summary.

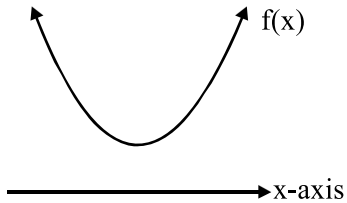
pdfworld.net

www.pdfworld.net

www.pdfworld.net

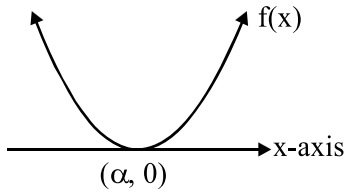
Different positions for Graphs of Quadratic Functions

I. a > 0 and D < 0



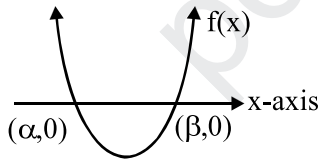
Question	Solution Set
$f(x) = 0$	ϕ
$f(x) < 0$	ϕ
$f(x) > 0$	$(-\infty, \infty)$

II. a > 0 and D = 0



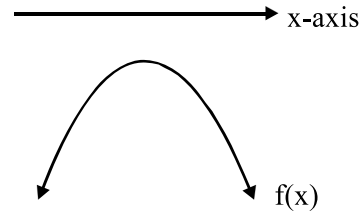
Question	Solution Set
$f(x) = 0$	$\{\alpha\}$
$f(x) < 0$	ϕ
$f(x) > 0$	$(-\infty, \alpha) \cup (\alpha, \infty)$

III. a > 0 and D > 0



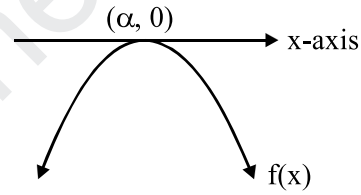
Question	Solution Set
$f(x) = 0$	$\{\alpha, \beta\}$
$f(x) < 0$	(α, β)
$f(x) > 0$	$(-\infty, \alpha) \cup (\beta, \infty)$

IV. a < 0 and D < 0



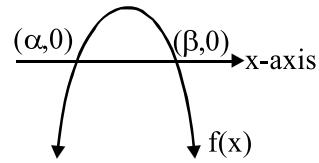
Question	Solution Set
$f(x) = 0$	ϕ
$f(x) < 0$	$(-\infty, \infty)$
$f(x) > 0$	ϕ

V. a < 0 and D = 0



Question	Solution Set
$f(x) = 0$	$\{\alpha\}$
$f(x) < 0$	$(-\infty, \alpha) \cup (\alpha, \infty)$
$f(x) > 0$	ϕ

VI. a < 0 and D > 0



Question	Solution Set
$f(x) = 0$	$\{\alpha, \beta\}$
$f(x) < 0$	$(-\infty, \alpha) \cup (\beta, \infty)$
$f(x) > 0$	(α, β)

Thus for real values of x the expression $ax^2 + bx + c$ has always the same sign as 'a' except for values of x which lie between the roots of the corresponding equation $ax^2 + bx + c = 0$.

STUDY TIP The values of 'x' satisfying the inequality, $ax^2 + bx + c > 0\ (a \neq 0)$ are :

pdfworld.net

www.pdfworld.net

www.pdfworld.net

(i) If $D > 0$, i.e. the equation $ax^2 + bx + c = 0$ has two different roots $a < b$.
Then $a > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$
$a < 0 \Rightarrow x \in (a, b)$

(ii) If $D = 0$, i.e. roots are equal, i.e. $a = b$.
Then $a > 0 \Rightarrow x \in (-\infty, a) \cup (a, \infty)$
$a < 0 \Rightarrow x \in \phi$

(iii) If $D < 0$, i.e. the equation $ax^2 + bx + c = 0$ has no real roots.
Then $a > 0 \Rightarrow x \in R$
$a < 0 \Rightarrow x \in \phi$

➤ **Example 1.** If $f(x) = x^2 + 2x + 2$, then solve the following inequalities :
(i) $f(x) \geq 0$ (ii) $f(x) \leq 0$
(iii) $f(x) > 0$ (iv) $f(x) < 0$

➤ ***Solution*** $f(x) = x^2 + 2x + 2$
Let us find $D = b^2 - 4ac = (2)^2 - 4 < 0$
$\Rightarrow$ $D < 0$
$\Rightarrow$ roots of the corresponding equation $(f(x) = 0)$ are imaginary.
Observe that the coefficient of $x^2 = a = 1 > 0$
As $a > 0$ and $D < 0$, we get :
$f(x) > 0 \ \forall \ x \in R$
(i) $f(x) \geq 0$ is true $\forall \ x \in R$
(ii) $f(x) \leq 0$ is true for no value of x i.e., $x \in \phi$
(iii) $f(x) > 0$ is true $\forall \ x \in R$
(iv) $f(x) < 0$ is true for no value of x i.e., $x \in \phi$

➤ **Example 2.** If $f(x) = x^2 + 4x + 4$, then solve the following inequalities :
(i) $f(x) \geq 0$ (ii) $f(x) \leq 0$
(iii) $f(x) > 0$ (iv) $f(x) < 0$

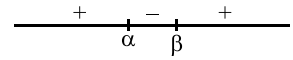
➤ ***Solution*** $f(x) = x^2 + 4x + 4 = (x + 2)^2$
Let us find D
$\Rightarrow$ $b^2 - 4ac = (4)^2 - 4(1)(4) = 0 \Rightarrow D = 0$
$\Rightarrow$ roots of the corresponding equation $(f(x) = 0)$ are real and equal.
Also observe that $a = 1 > 0$
As $D = 0$ and $a > 0$, we get :
$\Rightarrow$ $f(x) \geq 0 \ \forall \ x \in R$
(i) $f(x) \geq 0$ is true $\forall \ x \in R$
(ii) $f(x) \leq 0$ is true for $x \in \{-2\}$
(iii) $f(x) > 0$ is true $\forall \ x \in R - \{-2\}$
(iv) $f(x) < 0$ is true for no value of x i.e., $x \in \phi$

Method of Intervals

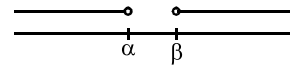
1. Express the inequality in the form $ax^2 + bx + c > 0$ or $ax^2 + bx + c < 0$.
2. Solve the equation $ax^2 + bx + c = 0$. The real solutions are the critical numbers.
3. Locate these critical numbers on a number line, thereby dividing the number line into test intervals.
4. Choose one representative number for each test interval. If substituting that value into the original inequality produces a true statement, then all real numbers in the test interval belong to the solution set. The solution set is the union of all such test intervals.

Assuming that the discriminant $D > 0$

- Make the coefficient of x^2 positive.
- Factorize the expression and represent the left hand side of inequality in the form $(x - \alpha)(x - \beta)$.

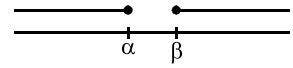


- If $(x - \alpha)(x - \beta) > 0$, then x lies outside α and β.



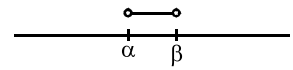
$\Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$

- If $(x - \alpha)(x - \beta) \geq 0$, then x lies on and outside α and β.



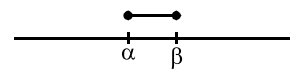
$\Rightarrow x \in (-\infty, \alpha] \cup [\beta, \infty)$

- If $(x - \alpha)(x - \beta) < 0$, then x lies inside α and β.



$\Rightarrow x \in (\alpha, \beta)$

- If $(x - \alpha)(x - \beta) \leq 0$, then x lies on and inside α and β.



$\Rightarrow x \in [\alpha, \beta]$

➤ **Example 3.** Solve $2x^2 + x < 15$
➤ ***Solution*** Write the inequality in standard form
$2x^2 + x - 15 < 0$

www.pdfworld.net

www.pdfworld.net

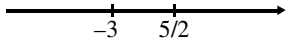
Solve the related quadratic equation

$2x^2 + x - 15 = 0 \quad \Rightarrow \quad x = 5/2, x = -3$

Factorize

$(2x - 5)(x + 3) < 0$

Use sign analysis of the product $(2x - 5)(x + 3)$



If $x < -3$, $2x - 5 < 0$ and $x - 3 < 0$

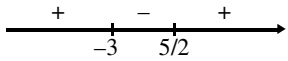
hence $(2x - 5)(x + 3) > 0$

If $-3 < x < \frac{5}{2}$, $(2x - 5) < 0$ and $x + 3 > 0$

hence $(2x - 5)(x + 3) < 0$

If $x > \frac{5}{2}$, $(2x - 5) > 0$ and $(x + 3) > 0$

hence $(2x - 5)(x + 3) > 0$



This indicates that $(2x - 5)(x + 3)$ is negative in the interval $\left(-3, \frac{5}{2}\right)$.

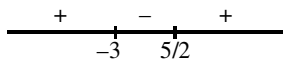
Another option is to use test points from each of the subintervals.

For the interval $(-\infty, -3)$, we substitute $x = -4$ into the inequality.

$(2.(-4)-5)(-4 + 3) = 13 > 0$

For $\left(-3, -\frac{5}{2}\right)$ we put $x = 0 \Rightarrow -15 < 0$

For $\left(\frac{5}{2}, \infty\right)$ we put $x = 3 \Rightarrow 6 > 0$



Conclusion : $\left(-3, \frac{5}{2}\right)$ is the solution of the inequality.

➤ **Example 4**. Solve the following quadratic inequality:

$x^2 - 2x - 3 < 0$

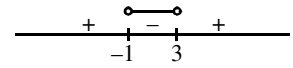
➤ ***Solution*** $x^2 - 2x - 3 < 0$

Let us find $D = b^2 - 4ac = (-2)^2 - 4(1)(-3) = 16 > 0$

$\Rightarrow \quad D > 0$

Now factorize

$x^2 - 3x + x - 3 < 0$



$\Rightarrow \quad (x - 3)(x + 1) < 0$

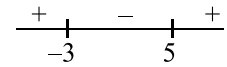
$\Rightarrow \quad x \in (-1, 3)$

➤ **Example 5.** For what values of k is the inequality $x^2 - (k - 3)x - k + 6 > 0$ valid for all real x ?

➤ ***Solution*** The expression $x^2 - (k - 3)x - k + 6$ is positive for all x if coefficient of x^2 is positive, which is true and discriminant $D < 0$

$\Rightarrow \quad (k - 3)^2 - 4(-k + 6) < 0$

$\Rightarrow \quad k^2 - 2k - 15 < 0$



$\Rightarrow \quad (k + 3)(k - 5) < 0$

$k \in (-3, 5)$,

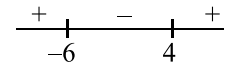
➤ **Example 6.** For what values of k, is the quadratic trinomial $(k - 2)x^2 + 8x + k + 4$ negative for all values of x?

➤ ***Solution*** The quadratic expression is negative for all x if $k - 2 < 0$ and

$D = 8^2 - 4(k - 2)(k + 4) < 0$

$16 - (k^2 - 2k - 8) < 0$

$k^2 + 2k - 24 > 0$



$(k + 6)(k - 4) > 0$

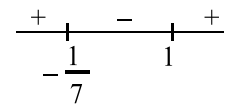
$k < -6$ or $k > 4$.

Taking intersection with $k < 2$, we get $k < -6$.

➤ **Example 7.** For what values of m does the equation $mx^2 - (m + 1)x + 2m - 1 = 0$ possess no real roots.

➤ ***Solution*** The equation has no real roots if $D < 0$

$\Rightarrow \quad (m + 1)^2 - 4m(2m - 1) < 0$



$\Rightarrow \quad -7m^2 + 6m + 1 < 0$

$\Rightarrow \quad m \in \left(-\infty, -\frac{1}{7}\right) \cup (1, \infty)$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

➤ **Example 8.** Find the values of k so that y = kx is secant to the curve $y = x^2 + k$.

➤ ***Solution*** put $y = kx$ in $y = x^2 + k$

$kx = x^2 + k = 0$

$x^2 - kx + k = 0$ for line to be secant, $D > 0$

$k^2 - 4k > 0$ $k(k-4) > 0$

hence $k > 4$ or $k < 0$

$k \in (-\infty, 0) \cup (4, \infty)$

➤ **Example 9.** Show that $(x-1)(x-3)(x-4)(x-6) + 10$ is positive for all real values of x.

➤ ***Solution*** Taking the first and last factors together, and also the other two the given expression becomes

$(x^2 - 7x + 6)(x^2 - 7x + 12) + 10$

$= (x^2 - 7x)^2 + 18(x^2 - 7x) + 82$

Assuming this as a quadratic expression in $(x^2 - 7x)$,

$D = 18^2 - 4 \times 82 = 4(81 - 82) = -4 < 0$

which is clearly always positive for real values of c.

➤ **Example 10.** Solve the inequality

$ax^2 - 2x + 4 > 0.$ (1)

➤ ***Solution*** Equating to zero the coefficient of x^2 and the discriminant of the quadratic trinomial $ax^2 - 2x + 4$, we find the first singular value of the parameter a = 0 and the second singular value a = 1/4 (and if a > 1/4, then D < 0; and if a ≤ 1/4, then D ≥ 0).

Let us solve inequality (1) in each of the following four cases :

(1) $a > 1/4$; (2) $0 < a \le 1/4$;

(3) $a = 0$; (4) $a < 0$.

(1) If $a > 1/4$, then the trinomial $ax^2 - 2x + 4$ has a negative discriminant and a positive leading coefficient. Hence, the trinomial is positive for any x, that is, the solution of inequality (1) in this cases is represented by the set of all real numbers.

(2) If $0 < a \le 1/4$, then the trinomial $ax^2 - 2x + 4$ has the following roots :

$$x_{1,2} = \frac{1 \pm \sqrt{1-4a}}{a},$$

where $$\frac{1-\sqrt{1-4a}}{a} \le \frac{1+\sqrt{1-4a}}{a}.$$

Hence, the solution of inequality (1) is represented by the following collection:

$$x < \frac{1-\sqrt{1-4a}}{a}; \; x > \frac{1+\sqrt{1-4a}}{a}.$$

(3) If a = 0, then inequality (1) takes the form: $-2x + 4 > 0$, whence we get: $x < 2$.

(4) If $a < 0$, then we have

$$\frac{1+\sqrt{1-4a}}{a} < \frac{1-\sqrt{1-4a}}{a}.$$

Hence, in this case the solution of inequality (1) is represented by the following system:

$$\frac{1+\sqrt{1-4a}}{a} < x < \frac{1-\sqrt{1-4a}}{a}.$$

Answer :

(1) if $a > 1/4$, then $-\infty < x < +\infty$;

(2) if $0 < a \le 1/4$, then $x < \frac{1-\sqrt{1-4a}}{a}$; $x > \frac{1+\sqrt{1-4a}}{a}$;

(3) if $a = 0$, then $x < 2$;

(4) if $a < 0$, then $\frac{1+\sqrt{1-4a}}{a} < x < \frac{1-\sqrt{1-4a}}{a}$.

➤ **Example 11.** Find the values of a so that the function $f(x) = (a+1)x^2 - 3ax + 4a$ is negative for atleast one real x.

➤ ***Solution***

Case A :

If $a + 1 < 0$ then the graph of f(x) is a parabola which opens downward. Hence f(x) is definitely negative for some real x.

Case B :

If $a + 1 > 0$ then the graph of f(x) is a parabola which opens upward. Here f(x) is negative for atleast one real x if its graph crosses the x-axis, for which $D > 0$ is required.

$9a^2 - 4(a+1).4a > 0$

$\Rightarrow \quad 7a^2 + 16a < 0$

$\Rightarrow \quad -16/7 < a < 0$

Taking intersection we get $-1 < a < 0$

www.pdfworld.net

www.pdfworld.net

Case C :
If $a + 1 = 0$ then the graph of $f(x) = 3x - 4$, is a straight line which goes below the x-axis for $x < 4/3$. Hence $f(x)$ is definitely negative for some real x.
So the final answer is $a < 0$

➤ **Example 12.** Prove the following inequality

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \le (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2).$$

(This is known as Cauchy-Schwartz Inequality)

➤ ***Solution*** Let $E = (a_1x + b_1)^2 + (a_2x + b_2)^2 + ... + (a_nx + b_n)^2$
We can see that the above sum is positive for every $x \in R$. Expanding, we have

$$E = (a_1^2 + a_2^2 + ... + a_n^2)\,x^2 + 2(a_1b_2 + a_2b_2 + ... + a_nb_n)\,x + (b_1^2 + b_2^2 + ... + b_n^2)$$

which will be positive for every $x \in R$ only if

coeff. of $x^2 + a_1^2 + a_2^2 + ... + a_n^2 > 0$, which is true

and $D \le 0$

$\Rightarrow$ $(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \le (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2).$

which is the desired result.

Practice Problems [I]

1. Solve the inequalities :
(i) $x^2 - 3x - 4 \le 0$ (ii) $4x^2 + 4x + 1 \le 0$

2. Determine in each of the following problems the set of values of x for which the given inequality is true :
(A) $x^2 + 4x + 12 > 0$ (B) $x^2 - 4x + 3 < 0$
(C) $21 - 7(2x - 9) > 7x^2$
(D) $3x^2 - 5x - 8 \le 0$

3. Solve the inequalities :

(i) $\begin{cases} x^2 - 2x - 3 > 0 \\ x + 4 \ge 0 \end{cases}$

(ii) $\begin{cases} x^2 - 4 \ge 0 \\ x^2 - 2x - 8 \ge 0 \\ -x^2 + 5x - 4 \ge 0 \end{cases}$

(iii) $\dfrac{x^2 - 13x + 40}{\sqrt{19x - x^2 - 78}} \le 0$

4. Solve $x^2 + ax + a > 0$ for different real values of a.

5. Find the integral values of k for which the equations $(k - 12)\,x^2 + 2(k - 12)\,x + 2 = 0$ possess no real roots.

6. Determine the values of k for which the equation $\dfrac{x^2 + x + 2}{3x + 1} = k$ has both roots real.

7. For what values of a is the inequality $ax^2 + 2ax + 0.5 > 0$ valid throughout the entire number axis ?

8. For what integral k is the inequality $x^2 - 2(4k - 1)x + 15k^2 - 2k - 7 > 0$ valid for any real x?

9. Find all values of a for which the inequality $(a - 1)\,x^2 - (a + 1)\,x + a + 1 > 0$ is satisfied for all real x.

10. Find all values of 'a' for which the inequality $(a + 4)x^2 - 2ax + 2a - 6 < 0$ is satisfied for all $x \in R$.

11. For what values of 'p' do the graphs of $y = 2px + 1$ and $y = (p - 6)x^2 - 2$ not intersect ?

12. Find the least integral value of k for which the equation $x^2 - 2(k + 2)x + 12 + k^2 = 0$ has two different real roots.

13. Find the values of the parameter 'a' such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality, $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} < 2$.

14. If $a + b + c = 0$ then prove that $4ax^2 + 3bx + 2c = 0$ must have real roots.

www.pdfworld.net

www.pdfworld.net

15. Given the inequality $ax + k^2 > 0$. For which values of a

(a) is the inequality valid for all values of x and k ?

(b) are there values of x and k for which the inequality is valid ?

(c) is there a value of x such that the inequality is valid for any value of k ?

(d) is there a value of k for which the inequality is valid for every value of x ?

(e) is there a value of k such that the inequality is valid for all values of x ?

(f) if there a value of x for which the inequality is valid for every value of k ?

2.13 RANGE OF QUADRATIC FUNCTION

Consider the quadratic function $f(x) = a x^2 + b x + c$

(i) Range when $x \in R$

If $a > 0 \quad \Rightarrow \quad f(x) \in \left[-\frac{D}{4a}, \infty\right)$

$a < 0 \quad \Rightarrow \quad f(x) \in \left(-\infty, \frac{D}{4a}\right]$

Maximum or Minimum Value of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

➤ **Example 1.** Find maximum or minimum values of the following polynomials over $x \in R$.

(i) $f(x) = 4x^2 - 12x + 15$

(ii) $f(x) = -3x^2 + 5x - 4$

➤ ***Solution***

(i) $f(x) = 4x^2 - 12x + 15$

As $a = 4 > 0$

f(x) has minimum value at vertex

$$f_{min} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}$$

$D = (-12)^2 - 4 \times 4 \times 15 = 144 - 240 = -96$

$$\Rightarrow \quad f_{min} = \frac{-(-96)}{4 \times 4} = \frac{96}{16} = 6 \text{ at } x = -\frac{-12}{2 \times 4} = \frac{3}{2}$$

$$\therefore \quad f_{min} = 6 \text{ at } x = \frac{3}{2}$$

$f_{max} = \infty$

Alternative : Method of perfect square

Let us isolate a perfect square

$$f(x) = 4\left(x^2 - 3x + \frac{9}{4}\right) + 15 - 9 = 4\left(x - \frac{3}{2}\right)^2 - 6$$

$$f_{min} = -6 \text{ at } x = \frac{3}{2}$$

(ii) $f(x) = -3x^2 + 5x - 4$

As $a = -3 < 0$

f(x) has maximum value at vertex

$$f_{max} = \frac{-D}{4a} \text{ at } x = -\frac{b}{2a}$$

$D = (5)^2 - 4(-3)(-4) = 25 - 48 = -23$

$$f_{max} = -\frac{(-23)}{4(-3)} = -\frac{23}{12} \text{ at } x = \frac{-(5)}{2(-3)} = \frac{5}{6}$$

$$\therefore \quad f_{max} = \frac{-23}{12} \text{ at } x = \frac{5}{6}$$

$f_{min} = -\infty$

Alternative : Method of perfect square

$$f(x) = -3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) - 4 + \frac{25}{12}$$

$$= -3\left(x - \frac{5}{6}\right)^2 - \frac{23}{12}$$

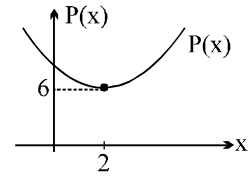
$$f_{max} = \frac{-23}{12} \text{ at } x = \frac{5}{6}$$

➤ **Example 2.** Let $P(x) = ax^2 + bx + 8$ is a quadratic polynomial. If the minimum value of P(x) is 6 when x = 2, find the values of *a* and *b*.

➤ ***Solution*** $P(x) = ax^2 + bx + 8$ ....(1)

$P(2) = 4a + 2b + 8 = 6$ ....(2)

$$\therefore \quad -\frac{b}{2a} = 2; \qquad \therefore \quad 4a = -b$$



www.pdfworld.net

www.pdfworld.net

from (2), we get

$-b+2b=-2 \quad \Rightarrow \quad b=-2$

$\therefore \quad 4a=-(-2) \Rightarrow \quad a=1/2$

➤ **Example 3.** If $\min (x^2+(a-b)x+(1-a-b)) > \max(-x^2+(a+b)x-(1+a+b))$, prove that $a^2+b^2<4$.

➤ ***Solution*** Let $f(x)=x^2+(a-b)x+(1-a-b)$

$f(x)_{min}=-\frac{D_1}{4}$, where D_1 is the discriminant of $f(x)$.

Let $g(x)=-x^2+(a+b)x-(1+a+b)$.

$g(x)_{max}=\frac{-D_2}{-4}$ where D_2 is the discriminant of $g(x)$.

Thus $-\frac{(a-b)^2-4(1-a-b)}{4} > -\frac{(a+b)^2-4(1+a+b)}{-4}$

$\Rightarrow \quad 4(1-a-b)-(a-b)^2>(a+b)^2>(a+b)^2-4(1+a+b)$

or $\quad 8>2(a^2+b^2) \Rightarrow a^2+b^2<4$.

➤ **Example 4**. Consider the quadratic polynomial $f(x)=x^2-4ax+5a^2-6a$.

(a) Find the smallest positive integral value of 'a' for which $f(x)$ is positive for every real x.

(b) Find the largest distance between the roots of the equation $f(x)=0$.

➤ ***Solution***

(a) $D<0, 16a^2-4(5a^2-6a)<0, 4a^2-5a^2+6a<0$,

$a^2-6a>0$,

$a(a-6)>0 \Rightarrow a>6 \quad$ or $\quad a<0$

$\therefore$ smallest +ve integer = 7

(b) $d=|\alpha-\beta|, (\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$=16a^2-4(5a^2-6a)=-4a^2+24a=-4(a^2-6a)$

$=-4[(a-3)^2-9]=36-4(a-3)^2$

$\therefore \quad |\alpha-\beta|^2_{max.}=36$ when $a=3 \quad d_{max}=6$

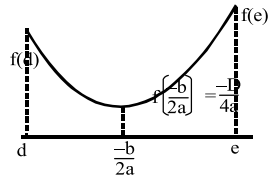
(ii) Range in restricted domain

Consider $f(x)=ax^2+bx+c, \quad x \in [d, e]$.

Evaluate $f(x)$ at the following points and choose the least and greatest values of f :

(i) f(d) (ii) f(e)

(iii) If $x=\frac{-b}{2a} \in (d, e)$ then find $\left(-\frac{b}{2a}\right)$ or $-\frac{D}{4a}$



otherwise leave the vertex point since it does not lie in the restricted domain.

If the least value is m and the greatest value is M then the range is [m, M].

STUDY TIP If x belongs to a union of several intervals then the range is equal to the union of ranges found on each of these intervals.

➤ **Example 5.** Find the range of $f(x)=2x^2-3x+2$ in [0, 2].

➤ ***Solution*** $f(0)=2$

$f(2)=4$

$f\left(\frac{3}{4}\right)=\frac{7}{8}$ since $x=\frac{-b}{2a}=\frac{3}{4} \in [0, 2]$

$f\left(\frac{3}{4}\right)$ is the least value and f(2) is the greatest value. Then the range is $\left[\frac{7}{8}, 4\right]$.

Alternative : Method of perfect square

$$f(x)=2\left(x-\frac{3}{4}\right)^2+\frac{7}{8}, x \in [0, 2].$$

To obtain the least value $x=\frac{3}{4}$ is required to make the square zero. The greatest value is obtained using $x=2$ since $\left|2-\frac{3}{4}\right| > \left|0-\frac{3}{4}\right|$.

Hence the range is [0, 2].

www.pdfworld.net

www.pdfworld.net

➤ **Example 6.** Find the range of
$y = 3 - 2\sin^2\theta - 6\sin\theta$.

➤ ***Solution*** Let $x = \sin\theta$, $x \in [-1, 1]$.
$f(x) = 3 - 2x^2 - 6x$, $x \in [-1, 1]$
$f(-1) = 3 - 2 + 6 = 7$
$f(1) = 3 - 2 - 6 = -5$

$$x = \frac{-b}{2a} = \frac{-6}{2 \times 2} = \frac{-3}{2} \notin [-1, 1].$$

Hence, we donot include the vertex. Range is
$y \in [-5, 7]$.

Alternative :

$$f(x) = 3 - 2\left(x^2 + \frac{2.3}{2}x + \left(\frac{3}{2}\right)^2\right) + \frac{9}{2}$$

$$= \frac{15}{2} - 2\left(x + \frac{3}{2}\right)^2, \; x \in [-1, 1]$$

The least value is obtained when the square is made larger using $x = 1$.
The greatest value is obtained when the square is made smaller using $x = -1$.

Here we cannot make the square zero since $x = \frac{-3}{2}$ is not available.

Practice Problems [J]

1. Find the maximum and minimum values of the functions :
   $f(x) = 2x^2 - 6x + 3$ where (i) $x \in R$
   (ii) $x \in [2, \infty)$ (iii) $x \in [1, 5]$
2. Find the least and the greatest value of the functions in the indicated intervals :
   (a) $y = 3x^2 - x + 5$ on the interval $[1, 2]$
   (b) $y = -4x^2 + 5x - 8$ on the interval $[2, 3]$
   (c) $y = x^2 - 2x + 5$ on the interval $[-1, 2]$
   (d) $y = -x^2 + 6x - 1$ on the interval $[0, 4]$
3. Find the difference between the least and greatest values of $y = -2x^2 + 3x - 2$ for $x \in [0, 2]$.
4. If x be real find the maximum value of $3 - 6x - 8x^2$ and the corresponding value of x.
5. If equation $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then find the least value of $3a + b$.
6. Find the range of $y = 3^{x+1} - 2 \cdot 3^{2x} - 2$
7. Find the values of k so that the least value of the expression $x^2 + 2kx + k^2 + 3k$ for $x \in [0, 2]$ is 4.

2.14 RATIONAL FUNCTION

A rational function of x is defined as the ratio of two polynomials of x, say P(x) and Q(x) where $Q(x) \neq 0$ i.e. f(x) is a rational function of x if

$$f(x) = \frac{P(x)}{Q(x)}; \; Q(x) \neq 0,$$

Following are some examples of rational functions of x.

$$f(x) = \frac{2x + 1}{x^2 + x + 1} \; ;$$

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 5x + 6} \; ; \; x \neq 2, \; x \neq 3$$

$$f(x) = \frac{x^4 + 2x^3 + 3x + 1}{(x - 1)^2} \; ; \; x \neq 1$$

➤ **Example 1.** If for any real x, we have

$$-1 \leq \frac{x^2 + nx - 2}{x^2 - 3x + 4} \leq 2, \text{ then find the values of n.}$$

➤ ***Solution*** The right hand inequality can be written as

$$\frac{x^2 + nx - 2}{x^2 - 3x + 4} - 2 \leq 0$$

$$\Rightarrow \quad \frac{x^2 - (n + 6)x + 10}{x^2 - 3x + 4} \geq 0$$

$$\Rightarrow \quad x^2 - (n + 6)x + 10 \geq 0$$

[As $x^2 - 3x + 4 > 0 \; \forall \; x \in R$, since $D < 0$ and $a > 0$]

The above inequality will be true $\forall \; x \in R$ if its $D \leq 0$

This gives $-\sqrt{40} - 6 \leq n \leq \sqrt{40} - 6$ ...(1)

The left hand inequality can be written as

$$\frac{x^2 + nx - 2}{x^2 - 3x + 4} + 1 \geq 0$$

www.pdfworld.net

www.pdfworld.net

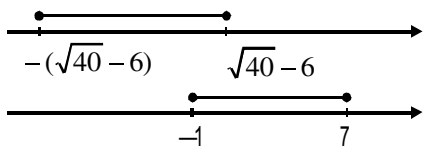
$$\Rightarrow \quad \frac{2x^2+(n-3)x+2}{x^2-3x+4} \geq 0$$

$\Rightarrow \quad 2x^2+(n-3)x+2 \geq 0$

The above inequality will be true $\forall\ x \in R$ if $D \leq 0$ which gives $-1 \leq n \leq 7$.

Drawing the number lines for inequalities (1), (2) and taking their intersection gives

$x \in [-1,\ \sqrt{40}\ -6]$



➤ **Example 2.** For what values of the parameter k in the inequality $\left|\frac{x^2+kx+1}{x^2+x+1}\right| < 3$ satisfied for all real values of x ?

➤ ***Solution*** $\left|\frac{x^2+kx+1}{x^2+x+1}\right| < 3$

$$\Rightarrow \quad -3 < \frac{x^2+kx+1}{x^2+x+1} < 3$$

$\Rightarrow$ since $x^2+x+1 > 0$,

$\therefore \quad -3(x^2+x+1) < x^2+kx+1 < 3(x^2+x+1)$

$\therefore \quad 4x^2+(k+3)x+4 > 0$ ...(1)

and $2x^2-(k-3)x+2 > 0$ ...(2)

Since $4 > 0$ and $2 > 0$

the inequality (1) will be valid

if $(k+3)^2 - 4.4.4 < 0 \Rightarrow -11 < k < 5$ ...(3)

and the inequality (2) will be valid

if $(k-3)^2 - 4.2.2. < 0 \Rightarrow -1 < k < 7$ ...(4)

The conditions (3) and (4) will hold simultaneously if $-1 < k < 5$.

Maximum and minimum values of a rational function

Consider $y = \frac{ax^2+bx+c}{px^2+qx+r}$ ...(i)

To find maximum and minimum values which y can take

cross multiply (i) to get :

$y(px^2+qx+r) = ax^2+bx+c$

$\Rightarrow \quad (a-py)x^2+(b-qy)x+(c-ry) = 0$

Since x is real, $D \geq 0$

$\Rightarrow \quad (b-qy)^2-4(a-py)(c-ry) \geq 0$

On solving this inequality we will get the values which y can take.

Case-I :

$y \in [A, B]$

If y can take values between A and B, then,

Maximum value of $y = y_{max} = B$,

Minimum value of $y = y_{min} = A$.

Case-II :

$y \in (-\infty, A] \cup [B, \infty)$

If y can take values outside A and B, then,

Maximum value of $y = y_{max} = \infty$ i.e. not defined.

Minimum value of $y = y_{min} = -\infty$ i.e. not defined.

Case-III :

$y \in (-\infty, \infty)$ i.e. $y \in R$

If y can take all values, then

Maximum value of $y = y_{max} = \infty$ i.e. not defined.

Minimum value of $y = y_{min} = -\infty$ i.e. not defined

➤ **Example 3.** If x be a real, show that the expression $\frac{x^2+2x-11}{x-3}$ can take all values which do not lie in the open interval (4, 12).

➤ ***Solution*** Let $y = \frac{x^2+2x-11}{x-3}$

Writing this as a quadratic equation in x, we have

$x^2+x(2-y)+(3y-11) = 0$ ....(1)

The values of x and y are related by this equation and for each value of y, there is a value of x, which is a root of this quadratic equation. In order that this x (or root) is real, discriminant ≥ 0.

$(2-y)^2-4(3y-11) \geq 0$

$y^2-16y+48 \geq 48$

$(y-4)(y-12) \geq 0$

$\therefore \quad y \leq 4$ or $y \geq 12$

Hence y (or the given expression) does not take any value between 4 and 12.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

If any value between 4 and 12, say 5, is given for y this equation (1) becomes $x^2 - 3x + 4 = 0$ whose roots are imaginary.

➤ **Example 4.** Show that, by giving real values to x, $\dfrac{4x^2+36x+9}{12x^2+8x+1}$ can be made to assume any real value.

➤ ***Solution*** Put $\lambda = \dfrac{4x^2+36x+9}{12x^2+8x+1}$

then $x^2(4-12\lambda) + (36-8\lambda)x + 9 - \lambda = 0$.

Now in order that x may be real it is necessary and sufficient that

$(36-8\lambda)^2 - 4(4-12\lambda)(9-\lambda) > 0$,

or that $\lambda^2 - 8\lambda + 72 > 0$,

or $(l-4)^2 + 56 > 0$,

which is clearly true for all real values of λ. Thus we can find real values of x corresponding to any real value of λ.

➤ **Example 5.** Find the range of the expression

$$y = \frac{\tan^2\theta - 2\tan\theta - 8}{\tan^2\theta - 4\tan\theta - 5},$$

for all permissible values of θ.

➤ ***Solution*** Let $x = \tan\theta$, where $x \in R$ for all permissible values of θ.

$$\Rightarrow \quad y = \frac{x^2-2x-8}{x^2-4x-5}$$

$\Rightarrow \quad x^2y - 4xy - 5y = x^2 - 2x - 8$

$\Rightarrow \quad (y-1)x^2 + 2x(1-2y) + 8 - 5y = 0$

$\therefore \quad x \in R$ hence $D \geq 0$

$\Rightarrow \quad 4(1-2y)^2 - 4(y-1)(8-5y) \geq 0$

$\Rightarrow \quad (4y^2 - 4y + 1) - (13y - 8 - 5y^2) \geq 0$

$\Rightarrow \quad 9y^2 - 17y + 9 \geq 0$ ....(1)

since coefficient of $y^2 > 0$

and $D = 289 - 324 < 0$

Hence (1) is always true.

Therefore range of y is $(-\infty, \infty)$

➤ **Example 6.** Find the values of m for which the expression :

$$\frac{2x^2-5x+3}{4x-m}$$ can take all real values for

$$x \in R - \left\{\frac{m}{4}\right\}$$

➤ ***Solution*** Let $y = \dfrac{2x^2-5x+3}{4x-m}$

$\Rightarrow \quad 2x^2 - (4y+5)x + 3 + my = 0$

$\Rightarrow \quad$ Ax x is real , discriminant ≥ 0

$\Rightarrow \quad (4y+5)^2 - 8(3+my) \geq 0$

$\Rightarrow \quad 16y^2 + (40-8m)y + 1 \geq 0$

A quadratic in y is non-negative for all values of y if coefficient of y^2 is positive and discriminant ≤ 0.

$\Rightarrow \quad (40-8m)^2 - 4(16)(1) \leq 0$

$\Rightarrow \quad (5-m)^2 - 1 \leq 0$

$\Rightarrow \quad (m-5-1)(m-5+1) \leq 0$

$\Rightarrow \quad (m-6)(m-4) \leq 0$

$\Rightarrow \quad m \in [4, 6]$

Consider $m = 4$

$$y = \frac{(2x-3)(x-1)}{4(x-1)}$$

$$y = \frac{2x-3}{4}, \qquad x \neq 1$$

Since $x \neq 1, y \neq -\dfrac{1}{4}$

Hence y cannot take all real values for $m = 4$. So $m = 4$ is not acceptable.

Similarly $m = 6$ is to be rejected.

So for the given expression to take all real values, m should take values : $m \in (4, 6)$.

In such problems the end-points of the result are rejected, in general due to cancellation of common factor from numerator and denominator and the resulting expression being incapable of assuming all real values.

➤ **Example 7.** Find the range of the expression

$$y = \frac{(\cot^2\theta + 5)(\cot^2\theta + 10)}{\cot^2\theta + 1},$$

for all permissible values of θ.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Let $x = \cot^2\theta + 1$, where $x \in [1, \infty)$ for all permissible values of θ.

Then

$$y = \frac{(x+4)(x+9)}{x} = x + 13 + \frac{36}{x}$$

$$= \left(\sqrt{x} - \frac{6}{\sqrt{x}}\right)^2 + 25$$

Hence the range of y is $[25, \infty)$.

Practice Problems [K]

1. Given $y = \frac{x^2+2}{x^2-1}$, determine the values of y for which x is real .
2. Prove that $\left|\frac{12x}{4x^2+9}\right| \le 1$ for all real values of x, the equality being satisfied only if $|x| = 3/2$.
3. If x is real, show that $\frac{2x^2-3x+2}{2x^2+3x+2}$ lies between 1/7 and 7.
4. If x be real and $0 < b < c$ show that $\frac{x^2-bc}{2x-b-c}$ cannot lie between b and c.
5. For real values of x, prove that $\frac{11x^2+12x+6}{x^2+4x+2}$ cannot lie between –5 and 3.
6. Show that, if x be real $\frac{x^2-6x+5}{x^2+2x+1}$ can never be less than $-\frac{1}{3}$.
7. If x is real, show that the expression $\frac{4x^2+36x+9}{12x^2+8x+1}$ can have any real value.
8. Show that the expression $\frac{mx^2+3x-4}{m+3x-4x^2}$ will be capable of all values when x is real, provided that m has any value between 1 and 7.
9. Find all values of 'a' for which $\frac{ax^2-7x+5}{5x^2-7x+a}$ takes all real values.
10. If $\frac{mx^2+3x+4}{x^2+3x+4} < 5$ for all $x \in R$, find possible values of m.

2.15 RESOLUTION OF A SECOND DEGREE EXPRESSION IN X AND Y

Let us find the condition that a quadratic function of x and y may be resolved into two linear factors. This is of great importance in analytical geometry.

Denote the function by f(x, y) where

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.

If we write this in descending powers of x, and equate it to zero , we get

$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0$.

Solving this quadratic in x we have

$$x = \frac{-(hy+g) \pm \sqrt{(hy+g)^2 - a(by^2+2fy+c)}}{a},$$

or $ax + hy + g$

$$= \pm\sqrt{y^2(h^2-ab) + 2y(hg-af) + (g^2-ac)}\,.$$

Now in order that f(x, y) may be the product of two linear factors of the form px + qy + r, the quantity under the radical sign must be a perfect square ; hence

$(hg - af)^2 = (h^2 - ab)(g^2 - ac)$.

Transposing and dividing by a, we obtain

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

CAUTION We note that f(x, y) can be written as a product of two linear factors with real coefficients if apart from the above condition we have either (i) $h^2 - ab > 0$, or (ii) $h^2 - ab = 0$, $hg - af = 0$, $g^2 - ac \ge 0$

➤ **Example 1.** Find the values of k for which the expression

www.pdfworld.net

www.pdfworld.net

$12x^2 - 10xy + 2y^2 + 11x - 5y + k$ is the product of two linear factors.

➤ ***Solution*** Rearrange as a quadratic equation in x.

$12x^2 + x(11 - 19y) + (2y^2 - 5y + k) = 0$

Solving

$$x = \frac{(10y - 11) \pm \sqrt{(10y - 11)^2 - 48(2y^2 - 5y + k)}}{24}$$

The factors will be linear only if

$(10y - 11)^2 - 48(2y^2 - 5y + k)$ is a perfect square and $h^2 - ab > 0$

$\Rightarrow$ if $4y^2 + 20y + (121 - 48k)$ is a perfect square and

$(-5)^2 - 12.2 = 1 > 0$

$\Rightarrow$ discriminant $400 - 16(121 - 48k) = 0$

(i.e.) if $k = 2$.

➤ **Example 2.** Find whether the expression $x^2 + y^2 + 4x + 4$ can be resolved into the product of two linear factors.

➤ ***Solution*** Rearrange as a quadratic equation in x.

$x^2 + 4x + y^2 + 4 = 0$

Solving

$$x = \frac{-4 \pm \sqrt{16 - 4(y^2 + 4)}}{2} = -2 \pm \sqrt{-y^2}$$

$$= -2 \pm iy$$

We can see that the condition

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

is satisfied but the given expression cannot be resolved into product of two linear factors with real coefficients. This happened because $h^2 - ab < 0$.

➤ **Example 3.** Show that the equation $x^2 - xy + y^2 = 4(x + y - 4)$ will not be satisfied for any real value of x and y except $x = 4$ and $y = 4$.

➤ ***Solution*** Rewriting the given equation as a quadratic in y

$y^2 - y(x + 4) + (x^2 - 4x + 16) = 0$

Since y is real, discriminant ≥ 0

$(x + 4)^2 - 4(x^2 - 4x + 16) \geq 0$

$\Rightarrow$ $-3x^2 + 24x - 48 \geq 0$

$\Rightarrow$ $x^2 - 8x + 16 \leq 0$, on division by (-3)

$(x - 4)^2 \leq 0$

But $(x - 4)^2$ cannot be less than zero

$\therefore$ $(x - 4)^2$ must be equal to zero. $\therefore$ $x = 4$

Similarly we can prove that $y = 4$.

Note that an interchange of x and y does not alter the equation.

➤ **Example 4.** If x, y be real and $9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0$, show that $x \in [3, 6]$, $y \in [1, 10]$.

➤ ***Solution*** If we write the given equation as a quadratic in x we get

$9x^2 + x(2y - 92) + y^2 - 20y + 244 = 0$

Since x is real we must have

$(2y - 92)^2 - 36(y^2 - 20y + 244) \geq 0$ or $(y - 46)^2 - 9(y^2 - 20y + 244) \geq 0$

$y^2 - 11y + 10 \leq 0 \Rightarrow (y - 1)(y - 10) \leq 0 \Rightarrow y \in [1, 10]$

Again if we write the given equation as a quadratic in y we get

$y^2 + y(2x - 20) + 9x^2 - 92x + 244 = 0$

As before $(2x - 20)^2 - 4(9x^2 - 92 + 244) \geq 0$

$\Rightarrow$ $(x - 10)^2 (9x^2 - 92x + 244) \geq 0$ or $x^2 - 9x + 18 \leq 0$

$\Rightarrow$ $x \in [3, 6]$.

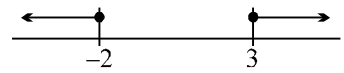
➤ **Example 5.** If x is real and $4y^2 + 4xy + x + 6 = 0$, then find the complete set of values of x for which y is real.

➤ ***Solution*** $4y^2 + 4xy + x + 6 = 0$ $\quad y \in R$

$\therefore$ $D \geq 0$

$16x^2 - 16(x + 6) \geq 0$

$\Rightarrow$ $x^2 - x - 6 \geq 0$ $\quad \Rightarrow$ $\quad (x - 3)(x + 2) \geq 0$



$\therefore$ $x \in (-\infty, -2] \cup [3, \infty)$

➤ **Example 6.** Find the greatest and the least real values of x & y satisfying the relation,

$x^2 + y^2 = 6x - 8y$.

➤ ***Solution*** writing as a quadratic in y,

$y^2 + 8y + x^2 - 6x = 0$

$y \in R$

$\Rightarrow$ $D \geq 0$

$\Rightarrow$ $x^2 - 6x - 16 \leq 0$

$\Rightarrow$ $-2 \leq x \leq 8$

Similarly range of y can also be obtained :

$-2 \leq x \leq 8$ & $-9 \leq y \leq 1$

www.pdfworld.net

www.pdfworld.net

➤ **Example 7.** If x, y and z are three real numbers such that $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$, then show that each of x, y and z lie in the closed interval $\left[\frac{2}{3}, 2\right]$.

➤ ***Solution*** Here, $x^2 + y^2 + (4 - x - y)^2 = 6$

$x^2 + (y - 4)x + (y^2 + 5 - 4y) = 0$

Since x is real $\Rightarrow D \geq 0$

$\Rightarrow (y - 4)^2 - 4(y^2 - 4y + 5) \geq 0$

$\Rightarrow -3y^2 + 8y - 4 \geq 0$

$\Rightarrow -(3y - 2)(y - 2) \geq 0$

$\Rightarrow \frac{2}{3} \leq y \leq 2$

Similarly, we can show that x and $z \in \left[\frac{2}{3}, 2\right]$.

Practice Problems [L]

1. Resolve the expressions into factors:
   (i) $3x^2 + xy - 4y^2 + 8x + 13y - 3$
   (ii) $3x^2 + 2xy - 8y^2 - 7x + 16y - 6$
2. Find the value(s) of λ for which the given expression can be resolved into factors, and for each value of λ resolve the expression into factors:
   (i) $x^2 - xy + 3y - 2 + \lambda(x^2 - y^2)$
   (ii) $x^2 + y^2 - 6y + 4 + \lambda(x^2 - 3y + 2)$.
3. For what values of m will the expression $5x^2 + 5y^2 + 4mxy - 2x + 2y + my + m$ be capable of resolution into two linear factors with real coefficients ?
4. If $(x^2 - 3x - a)y + 4x^2 - 11x + 2a = 0$, find a so that x may be expressed as a rational function of y.
5. If the expression $3x^2 + 2pxy + 2y^2 + 2ax - 4y + 1$ can be resolved into linear factors, prove that p must be one of the roots of $p^2 + 4ap + 2a^2 + 6 = 0$.
6. Show that in the equation $x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$, for every real value of x there is a real value of y, and for every real value of y there is a real value of x.
7. Prove that if the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied for real values of x and y, x must lie between 1 and 3 and y must lie between –1/3 and 1/3.
8. If $5x^2 + 4xy + y^2 - 24x - 10y + 24 = 0$, where x and y are real numbers, show that x must lie between $2 - \sqrt{5}$ and $2 + \sqrt{5}$ inclusive, and y between –4 and 6 inclusive.
9. If $x^2 + xy - y^2 + 2x - y + 1 = 0$, where x and y are real, show that y cannot lie between 0 and $-\frac{8}{5}$, but x can have any real value.

2.16 LOCATION OF ROOTS

In the previous sections, we dealt about the nature of roots of a quadratic equation where we found information about the roots when the coefficients were given.

In the present topic we need to find values of the coefficients when some information about the roots is given. There is one problem to determine the location of roots relative to the point 0 which means finding the signs of roots. Our interest also lies in the location of the roots in a given interval. With respect to this interval we can ask the following questions: Under what conditions : Do both the roots occur in the interval ? Is there exactly one root ? Is there atleast one root ? We can formulate a variety of questions based on these ideas. The student must concentrate on the graphical approach in developing the algebraic conditions associated with location of roots.

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and α, β be roots of the equation $f(x) = 0$ i.e. $ax^2 + bx + c = 0$.

In this topic we consider only the real roots of f(x). For simplicity we can assume that $\alpha \leq \beta$.

where $\alpha = \frac{-b - \sqrt{D}}{2a}$ and $\beta = \frac{-b + \sqrt{D}}{2a}$,

$D = b^2 - 4ac$.

1. Condition for both roots to be positive

   Sum of roots $> 0 \Rightarrow -\frac{b}{a} > 0$

   Product of roots $> 0 \Rightarrow \frac{c}{a} > 0$

   For real roots $\Rightarrow D \geq 0$

www.pdfworld.net

www.pdfworld.net

2. Condition for both roots to be negative

Sum of roots $< 0 \quad \Rightarrow \quad -\frac{b}{a} < 0$

Product of roots $> 0 \quad \Rightarrow \quad \frac{c}{a} > 0$

For real roots $\quad \Rightarrow \quad D \geq 0$

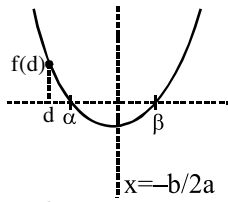
3. Condition for one root negative and other positive i.e. roots opposite in signs.

Product of roots $< 0 \quad \Rightarrow \quad \frac{c}{a} < 0.$

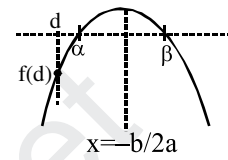
Note that $D > 0$ is not required since

$b^2 - 4ac > 0$ is satisfied by $\frac{c}{a} < 0.$

4. Condition for both roots to be greater than a given number d.



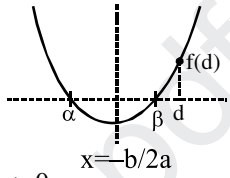
(i) $a > 0$
$f(d) > 0$
$-\frac{b}{2a} > d$
$D \geq 0$



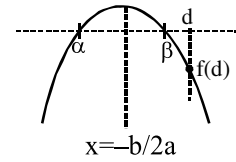
(ii) $a < 0$
$f(d) < 0$
$-\frac{b}{2a} > d$
$D \geq 0$

Combining (i) and (ii) we get $af(d) > 0, \; -\frac{b}{2a} > d \; , D \geq 0$

5. Condition for both roots to be less than a given number d.



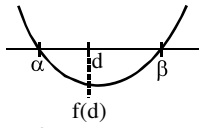
(i) $a > 0$
$f(d) > 0$
$-\frac{b}{2a} < d$
$D \geq 0$



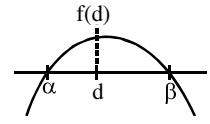
(ii) $a < 0$
$f(d) < 0$
$-\frac{b}{2a} < d$
$D \geq 0$

Combining (i) and (ii) we get $af(d) > 0, \; -\frac{b}{2a} < d \; , D \geq 0.$

6. Condition for one root to be less then d and other greater than d. i.e. a given number d lies between the roots.



(i) $a > 0$
$f(d) < 0$



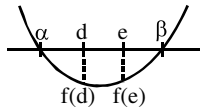
(ii) $a < 0$
$f(d) > 0$

www.pdfworld.net

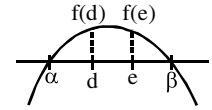
www.pdfworld.net

There is no need to take $D > 0$ since the above is sufficient.
Combining (i) and (ii) we get $af(d) < 0$.

7. Condition for one root to be less than d and other greater then e.



(i) $a > 0$
$f(d) < 0$
$f(e) < 0$

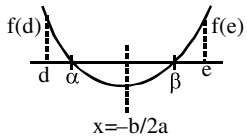


(ii) $a < 0$
$f(d) > 0$
$f(e) > 0$

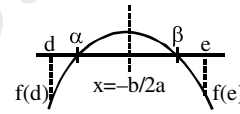
Combining (i) and (ii) we get $af(d) < 0$, $af(e) < 0$.

CAUTION One should not use $f(d) \cdot f(e) > 0$ since it allows other possibilities.

8. Condition for both roots to lie in a given interval (d, e).



(i) $a > 0$
$f(d) > 0$
$f(e) > 0$
$d < \frac{-b}{2a} < e$
$D \geq 0$



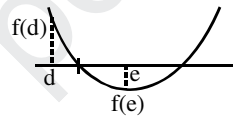
(ii) $a < 0$
$f(d) < 0$
$f(e) < 0$
$d < \frac{-b}{2a} < e$
$D \geq 0$

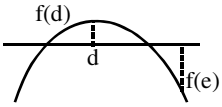
Combining (i) and (ii) we get $af(d) > 0$, $af(e) > 0$, $d < \frac{-b}{2a} < e$, $D \geq 0$

9. Condition for exactly one root to lie in a given interval (d, e).

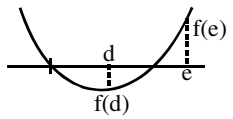
Case-A :

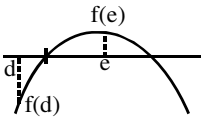
(i) $f(d) > 0$
$f(e) < 0$





(ii) $f(d) < 0$
$f(e) > 0$



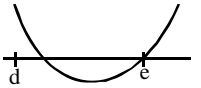


Combining (i) and (ii) we get $f(d) \cdot f(e) < 0$

Case-B :
One root is d and other lies in (d, e)

Solve $f(d) = 0$
Find the other root and check whether it lies in (d, e).

Case–C :
Solve $f(e) = 0$



Find the other root and check whether it lies in (d, e).
Final answer is union of the cases A, B and C.

www.pdfworld.net

www.pdfworld.net

➤ **Example 1.** If the roots of the quadratic equation,
$(a^2 - 7a + 13)x^2 - (a^3 - 8a - 1)x + \log_{1/2}(a^2 - 6a + 9) = 0$
lie on either side of origin, then find the range of value of 'a'.

➤ ***Solution*** $f(0) < 0 \Rightarrow \log_{1/2}(a-3)^2 < 0$
$(a-3)^2 > 1 \Rightarrow (a-4)(a-2) > 0$
$\Rightarrow (-\infty, 2) \cup (4, \infty)$

➤ **Example 2.** Find the range of values of a for which the equation $x^2 - (a-5)x + \left(a - \frac{15}{4}\right) = 0$ has at least one positive root.

➤ ***Solution*** $x^2 - (a-5)x + \left(a - \frac{15}{4}\right) = 0 \quad D \geq 0$

$\Rightarrow (a-5)^2 - 4\left(a - \frac{15}{4}\right) \geq 0$

$\Rightarrow a^2 - 10a + 25 - 4a + 15 \geq 0$
$\Rightarrow a^2 - 14a + 40 \geq 0$
$\Rightarrow (a-4)(a-10) \geq 0$
$\Rightarrow a \in (-\infty, 4] \cup [10, \infty)$

Case I : When both roots are positive

$D \geq 0, a - 5 > 0, a - \frac{15}{4} > 0$

$\Rightarrow D \geq 0, a > 5, a > \frac{15}{4}$

$\Rightarrow a \in [10, \infty)$

Case II : when exactly one root is positive

$\Rightarrow a - \frac{15}{4} \leq 0, \qquad a \leq 0\ \frac{15}{4}$

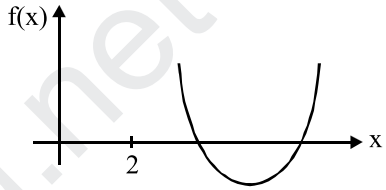
then finally $a \in (-\infty, 15/4] \cup [10, \infty)$

➤ **Example 3.** Find the set of values of 'p' for which the quadratic equation,
$(p-5)x^2 - 2px - 4p = 0$ has atleast one positive root.

➤ ***Solution*** For real roots
$D \geq 0 \Rightarrow p(p-4) \geq 0; \quad p \neq 5$
$\Rightarrow (-\infty, 0] \cup [4, 5) \cup (5, \infty)$
For both non positive roots sum ≤ 0;
product ≥ 0 & $D \geq 0 \Rightarrow [4, 5)$
Hence for atleast one positive root
$p \in (-\infty, 0] \cup (5, \infty)$

➤ **Example 4.** Consider the quadratic equation
$x^2 - (m-3)x + m = 0$ and answer the questions that follow.

(a) Find values of m so that both the roots are greater than 2.



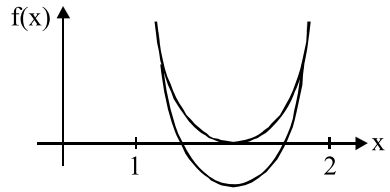
$D \geq 0$
$\Rightarrow (m-3)^2 = 4m \geq 0$
$\Rightarrow m^2 - 10m + 9 \geq 0$
$\Rightarrow (m-1)(m-9) \geq 0$
$\Rightarrow m \in (-\infty, 1] \cup [9, \infty)$ .... (i)
$f(2) > 0$
$\Rightarrow 4 - (m-3)2 + m > 0$
$\Rightarrow m < 10$ .... (ii)

$-\frac{b}{2a} > 2 \Rightarrow \frac{m-3}{2} > 2$

$\Rightarrow m > 7$ .... (iii)

Intersection of (i), (ii) and (iii) gives $m \in [9, 10)$

(b) Find the values of m so that both roots lie in the interval (1, 2)



$D \geq 0 \Rightarrow \quad m \in (-\infty. 1] \cup [9, \infty)$
$f(1) > 0 \quad \Rightarrow \quad 1 - (m-3) + m > 0$
$\Rightarrow 4 > 0 \quad \Rightarrow \quad m \in R$

www.pdfworld.net

www.pdfworld.net

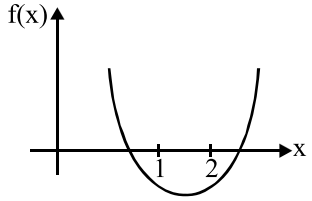
$f(2) > 0 \quad \Rightarrow \quad m < 10$

$1 < -\frac{b}{2a} < 0 \quad \Rightarrow \quad 1 < \frac{m-3}{2} < 2$

$\Rightarrow \quad 5 < m < 7$

Intersection gives $m \in \phi$.

(c) One root is greater than 2 and other smaller than 1

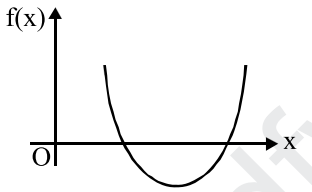


$f(1) < 0 \quad \Rightarrow \quad 4 < 0 \quad \Rightarrow \quad m \in \phi$

$f(2) < 0 \quad \Rightarrow \quad m < 10$

Intersection gives $m \in \phi$.

(d) Find the value of m for which both roots are positive.



$D \geq 0 \Rightarrow \quad m \in (-\infty, 1] \cup [9, \infty)$

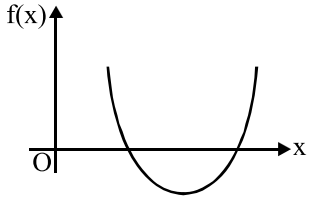
$f(0) > 0 \quad \Rightarrow \quad m > 0$

$-\frac{b}{2a} > 0 \quad \Rightarrow \quad \frac{m-3}{2} > 0$

$\Rightarrow \quad m > 3$

Intersection gives $m \in [9, \infty)$

(e) Find the values of m for which one root is positive and other is negative.



$f(0) < 0 \quad \Rightarrow \quad m < 0$

(f) Roots are equal in magnitude and opposite in sign.

Sum of roots $= 0 \Rightarrow \quad m = 3$

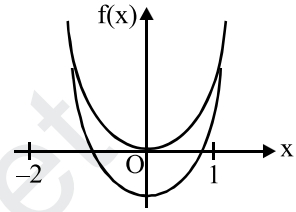
and $f(0) < 0 \quad \Rightarrow \quad m < 0$

$\therefore \quad m \in \phi$

➤ **Example 5.** Find all the values of 'a' for which both the roots of the equation.

$(a-2)x^2 + 2ax + (a+3) = 0$ lies in the interval $(-2, 1)$.

➤ ***Solution***



$(a-2)\, f(-2) > 0$

$\Rightarrow \quad (a-2)[(a-2)\,4 - 4a + a + 3] > 0$

$\Rightarrow \quad (a-2)(a-5) > 0$

$\Rightarrow \quad a < 2, a > 5$

$(a-2) f(1) > 0$

$\Rightarrow \quad (a-2)(4a+1) > 0$

$\Rightarrow \quad a < -\frac{1}{4}, a > 2$

$D \geq 0$

$\Rightarrow \quad 4a^2 - 4(a+3)(a-2) \geq 0$

$\Rightarrow \quad a \leq 6$

$-\frac{b}{2a} < 1 \quad \Rightarrow \quad \frac{2(a-1)}{a-2} > 0$

$\Rightarrow \quad a \in (-\infty, 1) \cup (2, \infty)$

$-2 < -\frac{b}{2a} \quad \Rightarrow \quad \frac{-2a}{2(a-2)} > -2 \quad \Rightarrow \frac{a-4}{a-2} > 0$

$\Rightarrow \quad a \in (-\infty, 2) \cup (4, \infty)$

Complete solution is $a \in \left(-\infty, -\frac{1}{4}\right) \cup (5, 6]$.

➤ **Example 6.** Find the values of a for which one root of equation $(a-5)x^2 - 2ax + a - 4 = 0$ is smaller than 1 and the other greater than 2.

➤ ***Solution*** Let $f(x) = (a-5)x^2 - 2ax + a - 4$ $(a \neq 5)$ as 1 and 2 lie between the roots of $f(x) = 0$

www.pdfworld.net

www.pdfworld.net

Consider $(a-5)\,f(1)<0 : (a-5)(a-5-2a+a-4)<0$

$\Rightarrow \quad (a-5)(-9)<0$

$\Rightarrow \quad a-5>0$

$\Rightarrow \quad a \in (5, \infty)$ ...(1)

Consider $(a-5)\,f(2)<0 : (a-5)(4(a-5)-4a+a-4)<0$

$\Rightarrow \quad (a-5)(a-24)<0$

$\Rightarrow \quad a \in (5, 24)$ ...(2)

Hence, the values of a satisfying (1) and (2) at the same time are $a \in (5, 24)$

➤ **Example 7.** Find the values of a so that the equation

$x^2+(3-2a)x+a=0$ has exactly one root in $(-1, 2)$.

➤ ***Solution***

(A) $f(-1)\,.\,f(2)<0 \quad \Rightarrow \quad (3a-2)(10-3a)<0$

$$\Rightarrow \quad a \in \left(-\infty, \frac{2}{3}\right) \cup \left(\frac{10}{3}, \infty\right).$$

(B) Let $f(-1)=0 \Rightarrow 3a-2=0 \Rightarrow a=\frac{2}{3}$

$$\alpha\beta = a \Rightarrow \quad -1\,.\,\beta = \frac{2}{3} \Rightarrow \beta = -\frac{2}{3} \in (-1, 2)$$

Hence $a=\frac{2}{3}$ is acceptable.

(C) Let $f(2)=0 \Rightarrow 10-3a=0 \qquad \Rightarrow a=\frac{10}{3}$

$$\alpha\beta = a \Rightarrow 2\beta = \frac{10}{3} \Rightarrow \beta = \frac{5}{3} \in (-1, 2)$$

Hence $a=\frac{10}{3}$ is also acceptable.

Finally, $\quad a \in \left(-\infty, \frac{2}{3}\right] \cup \left[\frac{10}{3}, \infty\right)$

➤ **Example 8.** For what values of 'a' exactly one root of the equation $2^a x^2 - 4^a x + 2^a - 1 = 0$, lies between 1 and 2.

➤ ***Solution*** Since exactly one root of the given equation lies between 1 and 2 we have $f(1)\,f(2)<0$

Here $f(x) = 2^a x^2 - 4^a x + 2^a - 1$

$\therefore \quad (2^a - 4^a + 2^a - 1)(4.2^a - 2.4^a + 2^a - 1) < 0$

$\Rightarrow \quad (4^a - 2.2^a + 1)(4.4^a - 5.2^a + 1) < 0$

$\Rightarrow \quad (2^a - 1)^2 (2.2^{2a} - 5.2^a + 1) < 0$

$$\Rightarrow \therefore 2.2^{2a} - 5.2^a + 1 < 0 \quad \frac{1}{2} < 2^a < 1$$

$\Rightarrow \quad \log_2 (1/2) < a < \log_2 1$

$\Rightarrow \quad -1 < a < 0 \qquad \therefore \qquad a \in (-1, 0)$

➤ **Example 9.** Find all values of a for which the equation $2x^2 - 2(2a+1)x + a(a+1) = 0$ has two roots, one of which is greater than a and the other is smaller than a.

➤ ***Solution*** Here coefficient of x^2 is positive

Let $f(x) = 2x^2 - 2(2a+1)x + a(a+1)$

$\therefore \quad f(a) = 2a^2 - 2(2a+1)a + a(a+1)$

$= -a^2 - a < 0$

$\Rightarrow \quad -a(a+1) < 0 \qquad \Rightarrow \qquad a(a+1) > 0$

$a \in (-\infty, -1) \cup (0, \infty)$

➤ **Example 10.** Find the values of the parameter k for which the equation $x^4 - (k-3)x^2 + k = 0$ has

(i) four real roots (ii) exactly two real roots (iii) no real root

➤ ***Solution*** The equation $x^4 - (k-3)x^2 + k = 0$ ...(1) can be transformed to $t^2 - (k-3)t + k = 0$ ...(2) where $x^2 = t$

(i) Now the equation (1) will have four real roots if the equation (2) has both roots as positive and distinct. These values of k are included in the interval $(9, \infty)$.

Again (1) will have exactly two real roots if either

(a) (2) has equal roots and this equal root is positive (since $t = x^2$)

(b) (2) has one positive and one negative root.

Now (2) has equal roots if $k = 1$ and $k = 9$

(ii) $k = 1, t = -1 \Rightarrow$ No real roots and $k = 9, t = 3$

$\Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

www.pdfworld.net

www.pdfworld.net

Thus at k = 9 the original equation has exactly two roots.

Again (2) has one positive and one negative root if $k \in (-\infty, 0)$. Thus the required values of k are 9 and the ones lying in the interval $(-\infty, 0)$.

(iii) Equation (1) will not have and real roots if equation (2) has either no real roots or has both roots negative. Now (2) has no real root if $m \in (1, 9)$ and has both roots negative if $k \in (0, 1)$. Thus required values of k are contained in the interval (0, 9).

➤ **Example 11.** Solve the equation $9^{-|x-2|} - 4.3^{-|x-2|} - a = 0$ for every real number a.

➤ ***Solution*** Let $y = 3^{-|x-1|}$ and noting that $0 < 3^{-|x-2|} \le 1$ for every x, we get the equation

$$y^2 - 4y - 1 = 0$$

for which we have to find the roots lying in the interval $0 < y \le 1$. The abscissa of the vertex of $f(y) = y^2 - 4y - a$ is equal to 2 so that if the quadratic has roots, then the greater root exceeds 2 and does not interest us. It therefore remains to write down the condition under which there is exactly one root in the interval $0 < y \le 1$.

First of all, y = 1 is a root when a = –3. Further more, there is exactly one root in the interval $0 < y < 1$ if $f(0).f(1) < 0 \Rightarrow -a(-a-3) < 0 \Rightarrow -3 < a < 0$.

Thus, the equation has exactly one root in the interval

$$0 < y \le 1 \text{ for } -3 \le a < 0 .$$

This is the smaller root $y = 2 - 2\sqrt{4+a}$. Now solving the equation $3^{-|x-2|} = 2 - \sqrt{4+a}$, which for the values of a thus found has a solution , we get

$$|x-2| = -\log_3 (2 - \sqrt{4+a}\,,$$

$$x_{1,2} = 2 \pm \log_2 (2 - \sqrt{4+a}\,) \text{ for } -3 \le a < 0$$

Practice Problems [M]

1. Find all values of p for which the roots of the equation $(p-3)x^2 - 2px + 5p = 0$ are positive.

2. For what values of p does the equation $2x^2 - (p^3 + 8p - 1)x + p^2 - 4p = 0$ possess roots of opposite signs?

3. For what value of m will the equation $\frac{x^2 - bx}{ax + c} = \frac{m-1}{m+1}$ have roots equal in magnitude but opposite in sign?

4. If $0 < m < 3$, then show that the roots of the equation $(m-2)x^2 - (8-2m)x - (8-3m) = 0$ are real. Find the range of values of m for which one root is positive and the other is negative.

5. Find all values of m for which the equation $m \in R, m \ne -1, (1+m)x^2 - 2(1+3m)x + (1+8m) = 0$ gives roots according to the following conditions:

(i) Both roots are
(a) Imaginary (b) Equal
(c) Real and distinct (d) Positive
(e) Negative

(ii) Roots are opposite in sign

(iii) Roots are equal in magnitude

(iv) Roots are equal is magnitude but opposite in sign

(v) Atleast one root is positive.

(vi) Atleast one root is negative.

(vii) Roots are reciprocal to each other

(viii) Negative root is greater than positive root numerically

(ix) Positive root is greater than negative root numerically.

6. For what values of a, does the equation $ax^2 - (a+1)x + 3 = 0$ have roots lying between 1 and 2?

7. For which values of a is one of the roots of the polynomial $(a^2 + a + 1)x^2 + (a-1)x + a^2$ greater than 3 and the other less than 3?

8. Find all the values of a for which the roots of the equation $x^2 + x + a = 0$ exceed a.

9. For what values of 'a' does the equation $2\log_3^2 x - |\log_3 x| + a = 0$ possess four solutions?

10. Find all the values of the parameter a for which both roots of the quadratic equation $x^2 - ax + 2 = 0$ lie between 0 and 3 .

www.pdfworld.net

www.pdfworld.net

11. Find all values of m for which the equation $m \in R, m \neq -1, (1+m)x^2 - 2(1+3m)x + (1+8m) = 0$ gives roots according to the following conditions:
(i) Exactly one root in the interval (2, 3)
(ii) One root smaller than 1 and other root greater than 1
(iii) Both roots smaller than 2
(iv) Atleast one root in the interval (2, 3)
(v) Atleast one root greater than 2
(vi) Roots such that both 1 and 2 lie between them
(vii) One root in (1, 2) and other root in (2, 3)

12. If the quadratic equation $ax^2 + bx + c = 0$ has roots of opposite sign lying in the interval (–2, 2), then prove that $1 + \frac{c}{4a} - \left|\frac{b}{2a}\right| > 0$.

13. For what values of a do the real roots, x_1 & x_2, of the equation $x^2 - 4ax + 1 = 0$ satisfy $x_1 \geq a$ and $x_2 \geq 0$?

2.17 SOLVING INEQUALITIES BASED ON LOCATION OF ROOT

➤ **Example 1.** Find all values of 'k' for which the inequality $k \cdot 4^x + (k-1)2^{x+2} + k > 1$ is satisfied for all $x \in R$.

➤ ***Solution*** Let $2^x = t$ where $t \in (0, \infty)$.

The inequality is $kt^2 + 4(k-1)t + k - 1 > 0$ for $t > 0$

Let $f(t) = kt^2 + 4(k-1)t + k - 1$

Case A :

f(t) is positive for all $t \in R$.

$k > 0$

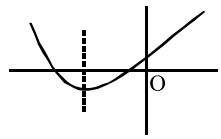
$D < 0 \Rightarrow \quad 4(k-1)^2 - (k-1)k < 0$

$\Rightarrow \quad (k-1)(3k-4) < 0$

$\Rightarrow \quad 1 < k < \frac{4}{3}$.

Case B :

$f(t) > 0$ for $t > 0$



and f(t) may be non-positive for $t \leq 0$.

$a > 0 \quad \Rightarrow \quad k > 0$

$f(0) \geq 0 \quad \Rightarrow \quad k - 1 \geq 0$

$\frac{-b}{2a} \leq 0 \quad \Rightarrow \quad \frac{-4(k-1)}{2k} \leq 0$

$\Rightarrow \quad \frac{k-1}{k} \geq 0 \quad \Rightarrow \quad k < 0$ or $k \geq 1$

There is no need of discriminant.

Taking intersection, we get $k \geq 1$.

Case C :

Let the leading coefficient be zero i.e. $k = 0$

Then $f(t) = -4t - 1$ which is always negative since $t > 0$. The inequality is not satisfied.

Finally, we have $k \in [1, \infty)$.

➤ **Example 2.** Find all possible parameters 'a' for which, $f(x) = (a^2 + a - 2)x^2 - (a+5)x - 2$ is non positive for every $x \in [0, 1]$.

➤ ***Solution*** $f(x) = (a+2)(a-1)x^2 - (a+5)x - 2 \leq 0$

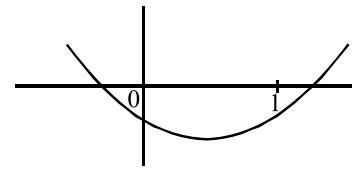
for $a = -2$; $f(x) = -3x - 2$

which is negative in [0, 1]

for $a = 1$; $f(x) = -6x - 2$

which is negative in [0, 1]

Now **Case I** :



for $a^2 + a - 2 > 0$ i.e. $a > 1$ or $a < -2$ — (1)

(i) $f(0) \leq 0$ always true

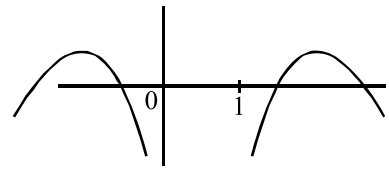
(ii) $f(1) \leq 0$

$\Rightarrow \quad a \in [-3, 3]$ — (2)

from (1) and (2) $a \in [-3, -2) \cup (1, 3]$

Case II $a^2 + a - 2 < 0$

i.e. $-2 < a < 1$



In this case :

(i) $D \geq 0 \quad \Rightarrow \quad (a+2)^2 \geq 0 \quad \Rightarrow \quad a \in R$

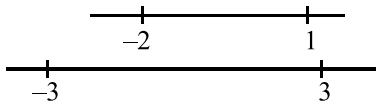
(ii) $f(1) < 0$ & $f(0) < 0$ (always true)

www.pdfworld.net

www.pdfworld.net

$\Rightarrow \quad a \in (-3, 3)$

common solution $(-2, 1)$



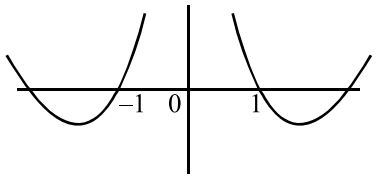
Case III $a < 0$ & $D < 0$

since $D \geq 0 \quad \Rightarrow$ no solution in this case

combining all $a \in [-3, 3]$

➤ **Example 3.** Find all values of k for which the inequality, $2x^2 - 4k^2x - k^2 + 1 > 0$ is valid for all real x which do not exceed unity in the absolute value.

➤ ***Solution*** **Case I** when $D < 0$ which gives the result



Case II $f(1) \geq 0 \; ; \; D \geq 0 \; ; \; -\frac{b}{2a} > 1$

or $f(-1) \geq 0 \; ; \; D \geq 0 \; ; \; -\frac{b}{2a} < -1 \; ;$

No solution in case II

$$-\frac{1}{\sqrt{2}} < k < \frac{1}{\sqrt{2}}$$

➤ **Example 4.** Find the values of a for which the inequality $\tan^2 x + (a+1)\tan x - (a-3) < 0$ holds for atleast one $x \in \left(0, \frac{\pi}{2}\right)$

➤ ***Solution*** We are required to find the values of a for which

$$\tan^2 x\,(a+1)\tan x - (a-3) < 0$$

for some $x \in \left(0, \frac{\pi}{2}\right)$

i.e. $y^2 + (a+1)y - (a-3) < 0$ for some $y > 0$

[assume $\tan x = y$]

i.e. $f(y) = y^2 + (a+1)y + (3-a)$.

Since the coeff. of y^2 is positive the plot of f represents a parabola whose mouth opens upward.

According to the required condition, $f(y) < 0$ for some positive y

$\Rightarrow$ a portion of the parabola must lie in the fourth quadrant

$\Rightarrow$ atleast one root of $f(y) = 0$ is positive.

Case I (roots opposite in sign)

product of roots < 0 i.e. $a > 3$.

Case II (both roots positive)

$$D \geq 0 \Rightarrow (a+1)^2 + 4(a-3) \geq 0 \Rightarrow a^2 + 6a - 11 \geq 0$$

$\Rightarrow \quad a \leq -3 - 2\sqrt{5}$ or $a \geq -3 + 2\sqrt{5}$ ...(1)

and product of roots > 0

$\Rightarrow \quad a < 3$ ...(2)

and sum of roots > 0

$\Rightarrow \quad a < -1$ ...(3)

Intersection of inequalities (1), (2) and (3), gives

$$a \leq -3 - 2\sqrt{5}.$$

Now, union of the inequalities obtained in both the cases gives

$$a \in (-\infty, -3, -2\sqrt{5}\,] \cup (3, \infty).$$

Practice Problems [N]

1. Find all real values of m for which the inequality $mx^2 - 4x + 3m + 1 > 0$ is satisfied for all positive x.
2. Find all values of m for which all $x \in [1, 2]$ is a solution of the inequality $x^2 - mx + 1 < 0$.
3. Find all values of a for which $x^2 - x + a - 3 < 0$ for atleast one negative x.
4. Prove that for any value of a the inequality $(a^2+3)x^2 + (a+2)x - 4 < 2$ is true for atleast one negative x.
5. For what values of $m \in R_1$ $m \neq -1$, the function $f(x) = (1+m)x^2 - 2(1+3m)x + (1+8m)$ is

www.pdfworld.net

www.pdfworld.net

(a) positive for all $x \geq 2$
(b) negative for all $x \geq 2$
(c) less than 1 for all $x \in (-\infty, 0)$
(d) greater than 5 for all $x \in (-1, 1)$

6. Find all values of 'a' for which $9^x - a.\,3^x - a + 5/4 \leq 0$ is satisfied for atleast one $x \in R$.

THEORY OF EQUATIONS

2.18 POLYNOMIALS

An expression of the form

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$

in which a_0, a_1, a_2,....., a_n are called coefficients (constants free from x) and n is a non-negative integer, is called a polynomial in x.

If $a_0 \neq 0$, the polynomial is of degree n, a_0x^n is the leading term, and a_0 is the leading coefficient. Also a_n is the constant term.

A polynomial in one variable is written in descending powers of the variable when the exponents on the terms decrease from left to right. Writting with descending powers of x, the polynomial is said to be in standard form.

The n + 1 coefficients can be determined, if the values of the polynomial are known for n + 1 values of x.

Polynomial of two or more variables

If a polynomial contains more than one variable, the degree of a term is the sum of the exponents on all the variables. The degree of the polynomial is the greatest of the degrees of any of its terms.

A polynomial of two variables x and y is the algebraic sum of several terms of the form ax^ry^s where r and s are positive integers. r + s is the degree of the term. For example, the degree of $3x^2y^7$ is 2 + 7 or 9. Also $x^5 + 3x^4y + 2xy^3 + 4y^2 - 2y + 1$ is a polynomial of x and y and its degree is 5.

Similarly an expression whose terms are of the form $ax^ry^sz^t$ where r, s and t are positive integer is a polynomial of three variables x, y and z.

Homogeneous polynomial

A polynomial of more than one variable is said to be homogeneous, if the degree of each term is the same. Thus, $2x^7 + 5x^5y^2 - 3x^4y^3 + 4x^2y^5$ is a homogeneous polynomial of degree 7 in x and y.

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$ is a homogeneous polynomial of degree 2 in x, y and z.

➤ **Example 1.** If the value of the polynomial $a_0x^2 + a_1x + a_2$ are 10, 19 and 7 for x = 1, 2 and –2 respectively, determine the co-efficients.

➤ ***Solution*** Here $a_0 . 1^2 + a_1 . 1 + a_2 = 10$,

or $a_0 + a_1 + a_2 = 10$ ...(1)

$a_0 . 2^2 + a_1 . 2 + a_2 = 19$,

or $4a_0 + 2a_1 + a_2 = 19$ ...(2)

$a_0(-2)^2 + a_1(-2) + a_2 = 7$,

or $4a_0 - 2a_1 + a_2 = 7$ ...(3)

From (1), (2) and (3) we get $a_0 = 2$, $a_1 = 3$, $a_2 = 5$.

Identical polynomials

Two polynomials in x are identical if the coefficients of their like powers of x are equal.

Thus, $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$
$= b_0x^n + b_1x^{n-1} + b_2x^{n-2} + ...+ b_{n-1}x + b_n$

implies that $a_0 = b_0$, $a_1 = b_1$,....., $a_n = b_n$.

➤ **Example 2.** Obtain the condition that $x^3 + 3px + q$ may have a factor of the form $(x - a)^2$.

➤ ***Solution*** If $(x - a)^2$ is a factor we may assume that

$x^3 + 3px + q = (x - a)^2(x + b) = (x^2 - 2ax + a^2)(x + b)$
$= x^3 + (b - 2a)x^2 + (a^2 - 2ab)x + a^2b.$

Since the two polynomials are identical we have

$b - 2a = 0$ ...(1)

$a^2 - 2ab = 3p$ ...(2)

$a^2b = q$ ...(3)

From (1) $b = 2a$. Putting $b = 2a$ in (2) $3p = -3a^2$, or, $p = -a^2$.

From (3) $2a^3 = q \Rightarrow 4a^6 = q^2 \Rightarrow 4(-p)^3 = q^2 \Rightarrow 4p^3 + q^2 = 0$.

➤ **Example 3.** Using the method of undetermined coefficients, show that the expression

$(x + 1)(x + 2)(x + 3)(x + 4) + 1$ is the square of a trinomial.

➤ ***Solution*** If the given expression is the square of a trinomial, then the following equality is true:

www.pdfworld.net

www.pdfworld.net

$(x+1)(x+2)(x+3)(x+4)+1=(x^2+ax+b)^2$, where a and b are the desired coefficients.
On performing the operations indicated in this equality, we compare the coefficients of x^3 and x^2 in both sides. We obtain the system

$$\begin{cases} 2a=10, \\ a^2+2b=35, \end{cases}$$

Whence we find a = 5 and b = 5. Make sure that for these values of a and b the coefficients of x and x^0 (i.e. constant terms) also coincide. Thus, the given expression is equal to $(x^2+5x+5)^2$.

➤ **Example 4.** Find the factorization of the polynomial $P(x)=(x^2+x+1)(x^2+x+2)-12$.

➤ ***Solution*** $P(x)=(x^2+x+1)((x^2+x+1)+1)-12$
$=(x^2+x+1)^2+(x^2+x+1)-12$.
Let $x^2+x+1=y$. We then have $y^2+y-12=(y+4)(y-3)$, since the roots of the trinomial y^2+y-12 are equal to (–4) and 3. Passing from y to x, we get
$P(x)=(x^2+x+5)(x^2+x-2)$.
Since the trinomial $x^2+x-2=(x-1)(x+2)$,
$P(x)=(x-1)(x+2)(x^2+x+5)$.

Polynomial Equation

If a polynomial in x becomes equal to another polynomial for certain values of x, then the algebraical expression of such a relation is called an equation. Any value of x which satisfies the equation is called a root of the equation. Determination of all possible roots is known as the complete solution of the equation.

By bringing all the terms to one side, the equation can be arranged in descending powers of x in the following form:

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n=0$$

Therefore, a polynomial equation is obtained by equating a polynomial to zero. The highest power of x in the equation is called the degree of the equation. An equation remains unchanged if all the terms are divided by the same non-zero number. It is said to be complete if it contains all the powers of x from n to 0; otherwise it is incomplete.
If α be a complex number such that $f(\alpha)=0$, then α is called a **root or zero** of the polynomial f.
$3x^3+4x^2-7$ is a polynomial of degree 3 with real coefficients; 1 is a zero of this polynomial.
x^3+2x-i is a polynomial of degree 3 with complex coefficients; i is a zero of this polynomial.

➤ **Example 5.** If x = 1 and x = 2 are roots of the equation $x^3+ax^2+bx+c=0$ and a + b = 1, then find the value of b.

➤ ***Solution*** $a+b+c=-1$
$\Rightarrow$ $c=-2$
and $8+4a+2b+c=0$
$\Rightarrow$ $4a+2b=-6$
$\Rightarrow$ $2a+b=-3$
$\Rightarrow$ $a=-4,\ b=5$
hence $a=-4;\ b=5;\ c=-2$

2.19 REMAINDER THEOREM

Remainder Theorem

Let P(x) be any polynomial of degree greater than or equal to one and 'a' be any real number. If P(x) is divided (x – a), then the remainder is equal to P(a).

Proof :

Since x – a is of first degree, the remainder will be a constant. If we represent the quotient by Q and the remainder by R, then $f(x)=(x-a)\times Q+R$. By substituting a for x, we get the result.

Factor Theorem

Let P(x) be polynomial of degree greater than of equal to 1 and 'a' be a real number such that P(a) = 0, then (x – a) is a factor of P(x). Conversely, if (x – a) is a factor of P(x), then P(a) = 0.

If f(x) has integer coefficients and a is an integer root of f(x) and m is any integer different from a, then a – m divides f(m).

www.pdfworld.net

www.pdfworld.net

Proof :

On dividing f(x) by x – m we get

$f(x) = (x - m)\, q(x) + f(m)$,

where q(x) has integer coefficients. So for x = a, we get

$0 = f(a) = (a - m)\, q(a) + f(m)$ or $f(m) = (a - m)\, q(a)$.

Hence (a – m) divides f(m).

➤ **Example 1.** Let f(x) be a polynomial, having integer coefficients and let f(0) = 1989 and f(1) = 9891. Prove that f(x) has no integer roots.

➤ ***Solution*** If a is an integer root, then $a \neq 0$ as $f(0) \neq 0$. Also a must be odd since it must divide $f(0) = a_n = 1989$. But $a \neq 1$ as $f(1) \neq 0$. So taking m = 1 in the above Note, we see that the even number (a – 1) divides the odd number f(1) = 9891, a contradiction.

STUDY TIP When one root of an equation can be found by inspection using remainder theorem, the degree of the equation can be lowered by means of division.

In the equation $(a - x)^4 + (x - b)^4 = (a - b)^4$, since x = a and x = b both satisfy the equation, (x – a) (x – b) will divide $(a - x)^4 + (x - b)^4 - (a - b)^4$, and as the quotient will be of the second degree, the equation formed by equating it to zero can be solved.

➤ **Example 2.** Find p and q so that (x + 2) and (x – 1) may be factors of the polynomial $f(x) = x^3 + 10x^2 + px + q$.

➤ ***Solution*** Since (x + 2) is a factor f(–2) must be zero $\therefore -8 + 40 - 2p + q = 0$ ...(1)

Since (x – 1) is a factor, f(1) must be zero

$\therefore \quad 1 + 10 + p + q = 0$ ...(2)

From (1) and (2), by solving we get p = 7 and q = –18

➤ **Example 3.** Show that (2x + 1) is a factor of the expression $f(x) = 32x^5 - 16x^4 + 8x^3 + 4x + 5$.

➤ ***Solution*** Since (2x + 1) is to be a factor of f(x), $f\left(-\frac{1}{2}\right)$ should be zero.

$$f\left(-\frac{1}{2}\right) = 32\left(-\frac{1}{2}\right)^5 - 16\left(-\frac{1}{2}\right)^4 + 8\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right) + 5$$

Hence (2x + 1) is a factor of f(x).

➤ **Example 4.** Without actual division prove that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

➤ ***Solution*** Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2$ be the given polynomials. Then

$g(x) = x^2 - 3x + 2 = (x - 1)(x - 2)$

In order to prove that f(x) is exactly divisible by & (x), it is sufficient to prove that x – 1 and x – 2 are factors of f(x). For this it is sufficient to prove that f(1) = 0 and f(2) = 0.

Now, $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$\Rightarrow \quad f(1) = 2 \times 1^4 - 6 \times 1^3 + 3 \times 1^2 + 3 \times 1 - 2$

and, $f(2) = 2 \times 2^2 - 6 \times 2^3 \times 2^2 + 3 \times 2 - 2$

$\Rightarrow \quad f(1) = 2 - 6 + 3 + 3 - 2$ and $f(2) = 32 - 48 + 12 + 6 - 2$

$\Rightarrow \quad f(1) = 8 - 8$ and $f(2) = 50 - 50$

$\Rightarrow \quad f(1) = 0$ and $f(2) = 0$

$\Rightarrow \quad (x - 1)$ and $(x - 2)$ are factors of f(x).

$\Rightarrow \quad g(x) = (x - 1)(x - 2)$ is a factors of f(x).

Hence, f(x) is exactly divisible by g(x).

➤ **Example 5.** Show that $(x - 1)^2$ is a factor of $x^n - nx + n - 1$.

➤ ***Solution*** Let $f(x) = x^n - nx + n - 1$. Since f(x) = 0, therefore by the factor theorem f(x) is divisible by x – 1.

We can write $f(x) = (x^n - 1) - n(x - 1)$,

$= (x - 1)\{x^{n-1} + x^{n-2} + ... + 1 - n\}$

$= (x - 1)\, g(x)$,

where $g(x) = x^{n-1} + x^{n-2} + ... + 1 - n$.

since g(1) = 0, therefore by the factor theorem x – 1 is a factor of g(x), and consequently f(x) is divisible by $(x - 1)^2$.

➤ **Example 6.** A polynomial in x of degree greater than 3 leaves the remainder 2, 1 and –1 when divided by (x – 1); (x + 2) & (x + 1) respectively. Find the remainder, if the polynomial is divided by, $(x^2 - 1)(x + 2)$.

➤ ***Solution*** $f(x) = Q_1(x - 1) + 2 = Q_2(x + 2) + 1 = Q_3(x + 1) - 1$

$\Rightarrow \quad f(1) = 2 \; ; \quad f(-2) = 1 \; ; \quad f(-1) = -1$

again $f(x) = Q_r(x^2 - 1)(x + 2) + ax^2 + bx + c$

Hence $a + b + c = 2 \; ; \; 4a - 2b + c = 1$

and $a - b + c = -1$

www.pdfworld.net

www.pdfworld.net

➤ **Example 7.** Find all cubic polynomials p(x) such that $(x-1)^2$ is a factor of p(x) + 2 and $(x+1)^2$ is a factor of p(x) – 2.

➤ ***Solution*** If $(x-\alpha)$ divides a polynomial q(x) then $q(\alpha) = 0$. Let $p(x) = ax^3 + bx^2 + cx + d$. Since (x – 1) divides p(x) + 2, we get a + b + c + d + 2 = 0.

Hence d = –a – b – c– 2 and

$$p(x) + 2 = a(x^3 - 1) + b(x^2 - 1) + c(x - 1)$$
$$= (x - 1)\{a(x^2 + x + 1) + b(x + 1) + c\}.$$

Since $(x-1)^2$ divides p(x) + 2, we conclude that (x – 1) divides $a(x^2 + x + 1) + b(x + 1)$ c. This implies that 3a + 2b + c = 0. Similarly, using the information that $(x+1)^2$ divides p(x) – 2, we get two more relations: –a + b – c + d – 2 = 0; 3a – 2b + c = 0. Solving these for a, b, c, d, we obtain b = d = 0, and a = 1, c = –3. Thus there is only one polynomial satisfying the given condition : $p(x) = x^3 - 3x$.

➤ **Example 8.** Show that the polynomial $x(x^{n-1} - na^{n-1}) + a^n(n-1)$ is divisible by $(x-a)^2$.

➤ ***Solution*** Let $f(x) = x(x^{n-1} - na^{n-1}) + a^n(n-1)$ then $f(a) = a(a^{n-1} - na^{n-1}) + a^n(n-1) = 0$.

Also $f'(x) = nx^{n-1} - na^{n-1} \Rightarrow f'(a) = 0$

Since both f(a) and f′(a) are zero. The polynomial f(x) must be divisible by $(x-a)^2$.

➤ **Example 9.** Find the remainder when

(i) $x^{100} + x + 2$ is divided by x – 1

(ii) $x^{100} + x + 2$ is divided by $x^2 - 1$

➤ ***Solution***

(i) Let $f(x) = x^{100} + x + 2$ then answer is clearly f(1) = 4.

(ii) Note that the remainder when f(x) is divided by x + 1 is f(–1) = 2

Using f(1) = 4, f(–1) = 2 (remainders) we write

$$\frac{f(x)}{x-1} = g(x) + \frac{4}{x-1}; \frac{f(x)}{x+1} = h(x) + \frac{2}{x+1}$$

where g(x) and h(x) are two polynomials.

On subtracting we get

$$\frac{2f(x)}{x^2-1} = g(x) - h(x) + \frac{2x+6}{x^2-1}$$

$$\text{or } \frac{f(x)}{x^2-1} = \frac{1}{2}g(x) - \frac{1}{2}h(x) + \frac{x+3}{x^2-1}$$

$\Rightarrow$ The remainder when f(x) is divided by $x^2 - 1$ is x + 3.

Practice Problems [O]

1. Show by substitution that x – 1, x – 5, x + 2 and x + 4 are factors of $x^4 - 23x^2 - 18x + 40$.
2. Show that $x^4 + 3ax^3 - 7x -$ is divisible by 2x + 3 when $a = \frac{121}{162}$.
3. Prove that $4ax^3 - 2x^2 + 3bx + 5$ is divisible by 2x – 1 and x + 2 if $b = -3\frac{1}{6}$.
4. If f(x) is a rational integral function of x and a, b are unequal, show that the remainder in the division of f(x) by (x – a) (x – b) is $[(x-a)f(b) - (x-b)f(a)]/(b-c)$.
5. For what values of parameters m and n is the polynomial $5x^6 - px^5 + 2qx^2 - 2x + 3$ divisible by $x^2 - 1$.
6. Find a quadratic function of x which shall vanish when $x = -\frac{3}{7}$ and have the values 24 and 62 when x = 3 and x = 4 respectively.
7. If a, b, c, d are in G.P. then prove that $ax^2 + bx^2 + cx + d$ is divisible by $ax^2 + c$.
8. Show that $x^3 + \alpha x^2 + \beta x + \gamma$ will be a perfect cube when $\alpha^2 = 27\gamma$ and $3\alpha\gamma = \beta^2$.
9. Prove that $ax^3 + bx + c$ is divisible by $x^2 + px + 1$ if $a^2 - c^2 = ab$.

2.20 ROOTS OF A POLYNOMIAL EQUATION

Fundamental Theorem of Algebra

Every polynomial equation of degree ≥ 1 with given coefficients (real or imaginary) has atleast one root (real or imaginary). This theorem was discovered by the famous German mathematician Karl Friedrich Gauss.

www.pdfworld.net

www.pdfworld.net

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ be a polynomial of degree n with the leading coefficient a_0. As a result of the fundamental theorem, the equation $f(x) = 0$ has a real or imaginary root α_1 so that $f(\alpha_1) = 0$. By the remainder theorem $f(x)$ is divisible by $(x - \alpha_1)$. Therefore, $f(x) = (x - \alpha_1) \times f_1(x)$, where $f_1(x)$ is polynomial of degree $n - 1$ with the leading coefficient a_0. By the fundamental theorem again, the equation $f_1(x) = 0$ has a real or imaginary root α_2. Therefore, $f_1(\alpha_2) = 0$.

Consequently $f_1(x) = (x - \alpha_2) = (x - \alpha_2) \times f_2(x)$, where $f_2(x)$ is a polynomial of degree $n - 2$ with the leading coefficient a_0. The same procedure is repeated until we come to a polynomial of the first degree, $f_{n-1}(x)$ with the root α_n and the leading coefficient a_0. So,

$$f_{n-1}(x) = a_0(x - \alpha_n).$$

Combining all these relations, we get

$$f(x) = a_0 (x - \alpha_1)(x - \alpha_2) ..... (x - \alpha_n).$$

From the above relation we conclude that every polynomial in x of degree n can be factorized into n linear factors of the form $(x - \alpha)$. All of them may not be distinct, some may be equal. If x is replaced by any one of $\alpha_1, \alpha_2, ..., \alpha_n$ $f(x)$ will reduce to zero and no other value of x, different from $\alpha_1, \alpha_2, ..., \alpha_n$ will make $f(x)$ zero.

Theorem : A polynomial $f(x)$ of the n-th degree cannot vanish for more than n values of x unless all its coefficients are zero.

For otherwise the product of more than n expressions of the form $x - \alpha$ would be a factor of the polynomial. That is to say, a polynomial of the nth degree would have a factor of degree higher than n, which is impossible.

Corollary 1 : If a polynomial in x of degree n vanishes for more than n distinct values of x, it vanishes identically.

For, if $f(x) = a_0(x - \alpha_1) (x - \alpha_2) .... (x - \alpha_n)$ ...(1)

vanishes for $x = \alpha$, different from $\alpha_1, \alpha_2, ..., \alpha_n$, we have $0 = a_0(\alpha - \alpha_1) (\alpha - \alpha_2) ..... (\alpha - \alpha_n)$. Therefore, $a_0 = 0$. Hence from (1) all the coefficients in $f(x)$ are zeros and $f(x)$ vanishes identically.

Corollary 2 : Two polynomials in x of degree n will be identical, if their equality holds for more than n values of x.

Equating the difference of the two polynomials to zero, we get an equation of degree n, which is satisfied by more than n values of x, and so all the coefficients will be zeros. Hence the original polynomials must be identically equal.

➤ **Example 1.** Form a fourth degree polynomial equation whose roots are $0, \pm a, \frac{c}{b}$.

➤ ***Solution*** The equation has to be satisfied by

$$x = 0, x = a, x = -a, x = \frac{c}{b};$$

therefore it is $x(x + a)(x - a)\left(x - \frac{c}{b}\right) = 0$;

that is, $x(x^2 - a^2)(bx - c) = 0$,

or $bx^4 - cx^3 - a^2bx^2 + a^2cx = 0$.

Identity

$f(x) = 0$ is said to be an identity in x if it is satisfied by all values of x in the domain of $f(x)$

Thus an identity in x is satisfied by all values of x whereas an equation in x is satisfied by some particular values of x.

(i) $(x + 2)^2 = x^2 + 4x + 4$ is an identity in x.

Here highest power of x in the given relation is 2 and this relation is satisfied by three different values $x = 0$, $x = 1$ and $x = -1$ and hence it is an identity because a polynomial equation of n th degree cannot have more than n distinct roots.

(ii) $x^9 - 2x^7 + 5 = 0$ is an equation in x because it is not satisfied by $x = 0$.

Equivalent equations : Two equations are said to be equivalent if they have the same roots.

The symbol ≡ is used to distinguish an identity from an equation.

In an identity in x coefficients of similar powers of x on the two sides are equal.

Thus if $ax^3 + bx^2 + cx + d = 2x^3 - 5x^2 + 3x - 6$ be an identity in x, then $a = 2$, $b = -5$, $c = 3$, $d = -6$.

www.pdfworld.net

www.pdfworld.net

➤ **Example 2.** Solve the equation

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^3$$

➤ ***Solution*** the equation is satisfied by x = a, by x = b, and by x = c; and the equation is not an identity, since the coefficient of x^3 is not zero.
Hence the roots of the cubic are a, b, c.

➤ **Example 3.** Show that, if

$(a-\alpha)^2x+(a-\beta)^2y+(a-\gamma)^2z=(a-\delta)^2$,
$(b-\alpha)^2x+(b-\beta)^2y+(b-\gamma)^2z=(b-\delta)^2$,
$(c-\alpha)^2x+(c-\beta)^2y+(c-\gamma)^2z=(c-\delta)^2$,

then will

$(d-\alpha)^2x+(d-\beta)^2y+(c-\gamma)^2z=(d-\delta)^2$, where d has any value whatever.

➤ ***Solution*** The equation

$(X-\alpha)^2x+(X-\beta)^2y+(X-\gamma)^2z=(X-\delta)^2$

is a quadratic equation is X, and it has the three roots a, b, c. It is therefore satisfied when any other quantity d is put for X.

Equal Roots

A polynomial of degree n has n linear factors and a number of them may be repeated. The factor $(x-\alpha)$ may occur twice, thrice, or more times but not more than n times. In this case, the equation f(x) = 0 said to have n roots, two or more being equal to one another. If $f(x)=a_0(x-\alpha)^r(x-\beta).....(x-\delta)$, then α is called a multiple root of f(x) = 0 and r is called the multiplicity of α.

➤ **Example 4.** Find the zeros of the polynomial $(x-1)(x+2)^2(x-3)^2$ and give the multiplicity of each.

➤ ***Solution*** The zeros are 1, –2, –2, 3, 3, i.e., 1 is a single root, and – 2 and 3 are roots with multiplicity two each.

Imaginary Roots

If a complex number of the form $\alpha+i\beta$, $\beta\neq0$, be a root of f(x) = 0 whose coefficients are all real, then $\alpha-i\beta$ will be another root of f(x) = 0, ($i^2=-1$)

Therefore f(x) is divisible by $(x-\alpha-i\beta)(x-\alpha+i\beta)$, that is, by $(x-\alpha)^2+\beta^2$.
Thus a polynomial in x with real coefficients can be resolved into factors which are linear or quadratic functions of x with real coefficients.

For a cubic equation $ax^3+bx^2+cx+d=0$, whose roots are α, β, γ either all the three roots are real or one root is real (the other two roots being imaginary).

In an equation with complex coefficients, complex roots may not occur in conjugate pairs. For example, the roots of the quadratic equation $x^2-7xi-12=0$ are 3i and 4i. Here 3i is a root but its complex conjugate, –3i, is not a root.

➤ **Example 5.** Solve the equation
$2x^3-15x^2+46x-42=0$, having given that one root is $3+\sqrt{-5}$.

➤ ***Solution*** Since $3\pm\sqrt{-5}$ are roots of the equation

$(x-3-\sqrt{-5})(x-3+\sqrt{-5})$

must be a factor of the left-hand side of the equation, which may be written

$\{(x-3)^2+5\}(2x-3)=0$.

The roots required are $3\pm\sqrt{-5}$, $\frac{3}{2}$.

➤ **Example 6.** Form the equation of lowest degree with real coefficients which has 2 + 3i, 3 – 2i as two of its roots.

➤ ***Solution*** Since in an equation with real coefficients, complex roots occur in conjugate pairs, therefore the required equation must have at least four roots, namely, $2\pm3i$, $3\pm2i$, i.e. it must have

$\{x-(2+3i)\}\{x-(2-3i\}\{x-(3+2i)\}\{x-(3-2i)\}$

as a factor,

i.e., it must have $\{(x-2)^2+9\}\{(x-3)^2+4\}$ as a factor.

Since the coefficients are real, therefore

$\{(x-2)^2+9)\}\{(x-3)^2+4\}=0$

www.pdfworld.net

www.pdfworld.net

is the equation of the lowest degree with real coefficients having $2 + 3i$, $3 - 2i$ as two of the roots.

Roots as Quadratic Surds

In an algebraic equation with rational coefficients, the roots in the form of a quadratic surd occur in conjugate pairs. Let the quadratic surd $p + \sqrt{q}$ be a root of the algebraic equation $f(x) = 0$ whose coefficients are all rational. Then $p - \sqrt{q}$ is also a root of the equation.

If $\sqrt{p}$ and $\sqrt{q}$ are two dissimilar quadratic surds and $\sqrt{p}$ and $\sqrt{q}$ is a root of $f(x) = 0$ with rational coefficients then $\pm\sqrt{p} \pm \sqrt{q}$ are the roots of $f(x) = 0$.

1. In an equation with rational coefficients, if $1+\sqrt[3]{2}$ is a root we cannot easily claim that $1-\sqrt[3]{2}$ is another root of the equation, since it involves a cubic surd.
2. In an equation with irrational coefficients, irrational roots may not occur in pairs. For example, the roots of the equation $x^2 - (2 + \sqrt{3})x + 2\sqrt{3} = 0$ are 2 and $\sqrt{3}$.

➤ **Example 7.** Solve the equation $x^4 - 2x^2 - 22x^2 + 62x - 15 = 0$, having given that one root is $2 + \sqrt{3}$.

➤ ***Solution*** Since both $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are roots of the equation,

$$(x - 2 - \sqrt{3})\,(x - 2 + \sqrt{3}),$$

that is $x^2 - 4x + 1$, must be a factor of the left-hand side of the equation. Thus we have

$$(x^2 - 4x + 1)\,(x^2 + 2x - 15) = 0.$$

The roots are $2 \pm \sqrt{3}$, -5, 3.

➤ **Example 8.** Obtain a polynomial of lowest degree with integral coefficient, whose one of the zeroes is $\sqrt{5} + \sqrt{2}$.

➤ ***Solution*** Let $P(x) = x - (\sqrt{5} + \sqrt{2})$

$$= [(x - \sqrt{5}) - \sqrt{2}]$$

Now following the method used in the pervious example, using the conjugate, we get

$$P_1(x) = [(x - \sqrt{5}) - \sqrt{2}]\ [(x - \sqrt{5}) + \sqrt{2}]$$

$$= (x^2 - 2\sqrt{5}\,x + 5) - 2 = (x^2 + 3 - 2\sqrt{5}\,x)$$

$$P_2(x) = [(x^2 + 3) - 2\sqrt{5}\,x]\,[(x^2 + 3) + 2\sqrt{5}\,x]$$

$$= (x^2 + 3)^2 - 20\,x^2 = x^4 + 6x^2 + 9 - 20x^2$$

$$= x^4 - 14x^2 + 9$$

$P(x) = ax^4 = 14ax^2 + 9a$, where $a \in Q$, $a \neq 0$.

The other zeroes of this polynomial are

$$\sqrt{5} - \sqrt{2}, -\sqrt{5} + \sqrt{2}, -\sqrt{5} - \sqrt{2}.$$

Rational root theory

If p and q are two numbers prime to each other and $\frac{p}{q}$ is a root of $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$, where $a_0, a_1, a_2, ..., a_n$ are all integers, then p is a factor of a_n and q is a factor of a_0.

Proof :

Since $\frac{p}{q}$ is a root,

$$a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + ... + a_{n-1}\frac{p}{q} + a_n = 0,$$

or, $a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + ... + a_{n-1}pq^{n-1} + a_nq^n = 0,$

or, $a_0p^n = -q(a_1p^{n-1} + a_2qp^{n-2} + ... + a_n\,q^{n-1})$.

Both the factors on the right-hand side are integers. Therefore, q is a factor of a_0p^n. But q is prime to p, so it is a factor of a_0.

www.pdfworld.net

www.pdfworld.net

Similarly, since

$a_n q^n = -p(a_0 p^{n-1} + a_1 p^{n-2} q + ... + a_{n-1} q^{n-1})$,

We can prove that p is a factor of a_n. Thus the rational roots of an equation can be found by trials.

Corollary : Every rational root of the equation $x^n + a_1 x^{n-1} + ... + a_n = 0$, where each $a_i (i = 1, 2, ..., n)$ is an integer, must be an integer. Moreover, every such root must be a divisor of the constant term a_n.

Proof :

Put $a_0 = 1$ in the preceding theorem. Since q must be a divisor of a_0, therefore we must have $q = 1$.

Hence every rational root must be an integer. Moreover, if p be such a root, then p must be a divisor of a_n.

➤ **Example 9.** Find the rational roots of

$6x^4 - x^3 + x^2 - 5x + 2 = 0$.

➤ ***Solution*** The factors of 6 are 1, 2, 3, 6 and those of 2 are 1 and 2. So the possible rational roots are ±1, $\pm 2, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}$ of which $\frac{1}{2}$ and $\frac{2}{3}$ only satisfy the equation. Thus the roots are $\frac{1}{2}$ and $\frac{2}{3}$.

➤ **Example 10.** Find the integral roots of the equation

$x^4 - x^3 - 19x^2 + 49x - 30 = 0$

➤ ***Solution*** If α be any integral root, then α must be a divisor of the constant term –30. Therefore the only integers which an possibly be the roots of the given equation are ±1, ±2, ±3, ±5, ±6, ±10, ±15 and 30.

By factor theorem we find that 1 and 2 are roots of the equation.

Thus $x^4 - x^3 - 19x^2 + 49x - 30 \equiv (x - 1)(x - 2)(x^2 + 2x - 15) \equiv (x - 1)(x - 2)(x - 3)(x + 5)$.

The given equation may, therefore, be written as

$(x - 1)(x - 2)(x - 3)(x + 5) = 0$

Hence the root are 1, 2, 3 and – 5.

➤ **Example 11.** Show that the equation

$$\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + ... + \frac{L^2}{x-l} = x - m$$

where a, b, c,.....,l are real numbers, all different, cannot have any imaginary root.

➤ ***Solution*** If possible let $\alpha + i\beta$, $\beta \neq 0$, be an imaginary root of the equation, then $\alpha - i\beta$ is also a root.

$$\therefore \quad \frac{A^2}{\alpha+i\beta-a} + \frac{B^2}{\alpha+i\beta-b} + \frac{C^2}{\alpha+i\beta-c} + ... + \frac{L^2}{\alpha+i\beta-l} = \alpha + i\beta - m ..... \quad (1)$$

$$\text{and} \quad \frac{A^2}{\alpha-i\beta-a} + \frac{B^2}{\alpha-i\beta-b} + \frac{C^2}{\alpha-i\beta-c} + ... + \frac{L^2}{\alpha-i\beta-l} = \alpha - i\beta - m .... \quad (2)$$

Subtracting (1) from (2),

$$\left[\frac{A^2}{(\alpha-a)^2+\beta^2} + \frac{B^2}{(\alpha-b)^2+\beta^2} + ... + \frac{L^2}{(\alpha-l)^2+\beta^2} + 1\right] \times 2i\beta = 0.$$

This is possible only when $\beta = 0$. Hence the equation cannot have any imaginary root.

Practice Problems [P]

1. One of the roots of $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ is $\sqrt{5} - 3$. Find the other roots.
2. Find the equation of the lowest degree with rational coefficients having $2 + \sqrt{3}$ and $\sqrt{5} - 2$ as two of its roots.
3. Two roots of the equation $x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$ are $\alpha + i\beta$ and $\beta + i\alpha$. Find all the roots of the equation.

www.pdfworld.net

www.pdfworld.net

4. Solve the equation $x^4 + 2x^2 - 16x^2 - 22x + 7 = 0$, having given that $2 + \sqrt{3}$ is one root.

5. Find the biquadratic equation with rational coefficients one root of which is $\sqrt{3} - \sqrt{5}$.

6. Prove that the equation $\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} = 1$ has no imaginary roots.

7. Show that the roots of the equation $\frac{1}{x-1} + \frac{2}{x-2} + \frac{3}{x-3} + \frac{4}{x-4} + \frac{5}{x-5} = 6$ are all real.

8. Find all the quadratic factors of $x^8 + x^4 + 1$ with real coefficients .

9. Find the rational roots of the equation $2x^3 - 3x^2 - 11x + 6 = 0$.

10. Find the values of $p \in N$, so that the equation $x^3 - px + 1 = 0$ has atleast one rational root.

2.21 POLYNOMIAL EQUATION REDUCIBLE TO QUADRATIC EQUATION

I. An equation of the form
$(x-a)(x-b)(x-c)(x-d) = k$,
where $a < b < c < d$, $b - a = d - c$, can be solved by a change of variable.

i.e. $y = \frac{(x-a) + (x-b) + (x-c) + (x-d)}{4}$

$y = x - \frac{(a+b+c+d)}{4}$

➤ **Example 1.** Two solve

$$(x+a)(x+2a)(x+3a)(x+4a) = \frac{9}{16}a^4.$$

➤ ***Solution*** Taking together the first and last of the factors on the left, and also the second and third, the equation becomes of the form we are now considering. We have

$$(x^2 + 5ax + 4a^2)(x^2 + 5ax + 6a^2) = \frac{9}{16}a^4.$$

Hence

$$(x^2 + 5ax)^2 + 10a^2(x^2 + 5ax) + 24a^2 = \frac{9}{16}a^4,$$

$$\therefore \quad x^2 + 5ax = -\frac{25}{4}a^2, \text{ or else } x^2 + 5ax = -\frac{15}{4}a^2.$$

Hence $x + \frac{5}{2}a = 0$, or $x + \frac{5}{2}a = \pm\frac{a}{2}\sqrt{10}$.

Thus the roots are $-\frac{5}{2}a, -\frac{5}{2}a \pm \frac{a}{2}\sqrt{10}$.

Alternative :

Put $y = x + \frac{a+2a+3a+4a}{4} \Rightarrow y = x + \frac{5a}{2}$

We get $\left(y - \frac{3a}{2}\right)\left(y - \frac{a}{2}\right)\left(y + \frac{a}{2}\right)\left(y + \frac{3a}{2}\right) = \frac{9}{16}a^4$

$$\Rightarrow \quad \left(y^2 - \frac{9a^2}{2}\right)\left(y^2 - \frac{a^2}{2}\right) = \frac{9}{16}a^4$$

We can solve this as a quadratic equation in y^2.

II. An equation of the form
$(x-a)(x-b)(x-c)(x-d) = kx^2$
where $ab = cd$, can be reduced to a collection of two quadratic equations by the substitution $y = x + \frac{ab}{x}$.

➤ **Example 2.** Solve the equation
$(x+2)(x+3)(x+8)(x+12) = 4x^2$

➤ ***Solution*** Since $(-2)(-12) = (-3)(-8)$ we can write the equation as

$(x+2)(x+12)(x+3)(x+8) = 4x^2$ ...(1)

$\Rightarrow \quad (x^2 + 14x + 24)(x^2 + 11x + 24) = 4x^2$ ...(2)

Check that $x = 0$ is not a root of (1).

Dividing by x^2 on both sides of (2) we get

$$\Rightarrow \quad \left(x + \frac{24}{x} + 14\right)\left(x + \frac{24}{x} + 11\right) = 4 \qquad ...(3)$$

Put $x + \frac{24}{x} = y$ then equation (3) can be reduced to
$(y + 14)(y + 11) = 4$

www.pdfworld.net

www.pdfworld.net

or $y^2 + 25y + 150 = 0$

$\therefore \quad y_1 = -15$ and $y_2 = -10$

Thus the original equation is equivalent to the collection of equations :

$$\begin{cases} x + \dfrac{24}{x} = -15 \\ x + \dfrac{24}{x} = -10 \end{cases}$$

i.e., $\begin{cases} x^2 + 15x + 24 = 0 \\ x^2 + 10x + 24 = 0 \end{cases}$

Solving this collection, we get

$$x_1 = \frac{-15 - \sqrt{129}}{2}, \; x_2 = \frac{-15 + \sqrt{129}}{2}, \; x_3 = -6,$$

$x_4 = -4$.

III. An equation of the form $(x-a)^4 + (x-b)^4 = k$ can also be solved by a change of variable, i.e., making a substitution

$$y = \frac{(x-a) + (x-b)}{2}$$

➤ **Example 3.** Solve the equation

$(6-x)^4 + (8-x)^4 = 16$

➤ ***Solution*** After a change of variable

$$y = \frac{(6-x) + (8-x)}{2}$$

$\therefore \quad y = 7 - x$

or $x = 7 - y$

Now put $x = 7 - y$ in the given equation we get

$(y-1)^4 + (y+1)^4 = 16$

$\therefore \quad y^4 + 6y^2 - 7 = 0$

$y^2 = -7$ gives imaginary roots

then $y_1 = -1$ and $y_2 = 1$

Thus $x_1 = 8$ and $x_2 = 6$ are the real roots.

IV. Reciprocal Equation

A reciprocal equation is one in which the coefficients are the same whether read in order backwards or forwards; or in which all the coefficients when read in order backwards differ in sign from the coefficients read in order forwards. Thus

$ax^3 + bx^2 + bx + a = 0$

$ax^4 + bx^3 + cx^2 + bx + a = 0$

$ax^5 + bx^4 + cx^3 - cx^2 - bx - a = 0$ are reciprocal equations.

To Solve $ax^4 + bx^3 + cx^2 + bx + a = 0$.

Divide by x^2. then we have

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0.$$

Now put $\quad x + \dfrac{1}{x} = y$

then $x^2 + \dfrac{1}{x^2} = y^2 - 2$.

Hence $\quad a(y^2 - 2) + by + c = 0$.

Let the two roots of the quadratic in y be α and β; then the roots of the original equation will be the four roots of the two equations

$$x + \frac{1}{x} = \alpha \text{ and } x + \frac{1}{x} = \beta.$$

To Solve $ax^5 + bx^4 + cx^3 - cx^2 - bx - a = 0$.

We have $\quad a(x^5 - 1) + bx(x^3 - 1) + cx^2(x - 1) = 0$,

that is $(x-1)\{a(x^4 + x^3 + x^2 + x + 1)$

$\quad + bx(x^2 + x + 1) + cx^2\} = 0$.

Hence $x = 1$, or else

$ax^4 + (b+a)x^2 + (a+b+c)x^2 + (b+a)x + a = 0$.

The last equation is a reciprocal equation of the fourth degree and is solved as before.

➤ **Example 4.** Solve the equation

$2x^4 + x^3 - 11x^2 + x + 2 = 0$ ...(1)

➤ ***Solution*** Since $x = 0$ is not a solution of the given equation. Dividing by x^2 in both sides of (1) we get

$$2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0 \quad \text{...(2)}$$

Putting $x + \dfrac{1}{x} = y$ in (2) we get $2(y^2 - 2) + y - 11 = 0$

$\Rightarrow \quad 2y^2 + y - 15 = 0$

$\Rightarrow \quad \therefore \quad y_1 = -3$ and $y_2 = \dfrac{5}{2}$

Consequently, the original equation is equivalent to

$$\begin{cases} x + \dfrac{1}{x} = -3 \\ x + \dfrac{1}{x} = \dfrac{5}{2} \end{cases}$$

www.pdfworld.net

www.pdfworld.net

We find that

$$x_1 = \frac{-3-\sqrt{5}}{2}, x_2 = \frac{-3+\sqrt{5}}{2}, x_3 = \frac{1}{2}, x_4 = 2.$$

➤ **Example 5.** Solve

$$6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0.$$

➤ ***Solution*** One root is –1. Dividing by $x + 1$, we have $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$. Dividing by x^2 and grouping the terms,

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0.$$

If $z = x + \frac{1}{x}$, this becomes

$6(z^2 - 2) + 5z - 38 = 0$, that is $6z^2 + 5z - 50 = 0$;

$\therefore \quad z = \frac{5}{2}$ or $-\frac{10}{3}$, and x is given by

$x + \frac{1}{x} = \frac{5}{2}$ or $-\frac{10}{3}$.

We find $x = 2, \frac{1}{2}, -3$ or $-\frac{1}{3}$. Thus the roots are $-1, 2, \frac{1}{2}, -3, -\frac{1}{3}$.

V. Trinomial Equation

Equations of the form $ax^{2n} + bx^n + c = 0, a \neq 0, n \in N$

Substitute $x^n = t$

➤ **Example 6.** Solve the biquadratic equation $x^4 - 5x^2 + 4 = 0$

➤ ***Solution*** Put $x^2 = t$

We get $t^2 - 5t + 4 = 0$

$\Rightarrow \quad t = 1, 4$

$\Rightarrow \quad x = \pm 1, \pm 2.$

➤ **Example 7.** Solve the biquadratic equation

$2x^4 + 2x^2 + 3 = 0$

➤ ***Solution*** We first reduce the equation to

$$x^4 + x^2 + \frac{3}{2} = 0$$

$$x^4 + x^2 + \frac{3}{2}$$

$$= \left(x^4 + 2\sqrt{\frac{3}{2}}x^2 + \frac{3}{2}\right) - 2\left(\sqrt{\frac{3}{2}} - 1\right)x^2$$

$$= \left(x^4 + \sqrt{\frac{3}{2}}\right)^2 - (\sqrt{6} - 1)x^2$$

$$= \left(x^2 + x\sqrt{\sqrt{6}} + \sqrt{\frac{3}{2}}\right)\left(x^2 - x\sqrt{\sqrt{6}} + \sqrt{\frac{3}{2}}\right)$$

The first equation $x^2 + x\sqrt{\sqrt{6}} + \sqrt{\frac{3}{2}} = 0$

has a negative discriminant :

$$D = (\sqrt{\sqrt{6} - 1})^2 - 4\sqrt{\frac{3}{2}} = -1 - \sqrt{6}$$

and, consequently, its roots

$$x_{1,2} = -\frac{\sqrt{\sqrt{6} - 1}}{2} \pm i\frac{\sqrt{\sqrt{6} + 1}}{2}$$

Similarly we find the roots of the second equation :

$$x^2 - x\sqrt{\sqrt{6} - 1} + \sqrt{\frac{3}{2}} = 0$$

They are

$$x_{3,4} = \frac{\sqrt{\sqrt{6} - 1}}{2} \pm i\frac{\sqrt{\sqrt{6} + 1}}{2}.$$

Practice Problems [Q]

Solve the following equations :

1. $(x^2 + 2)^2 + 8x^2 = 6x(x^2 + 2)$
2. $2x^4 - x^3 - 11x^2 - x + 2 = 0$
3. $\frac{2x}{3x^2 - x + 2} - \frac{7x}{3x^2 + 5x + 2} = 1$
4. $(x - 1)^4 + (x - 5)^4 = 82$
5. $(12x - 1)(6x - 1)(4x - 1)(3x - 1) = 5$
6. $x^2 + \frac{x^2}{(x + 1)^2} = 3$

www.pdfworld.net

www.pdfworld.net

2.22 RELATION BETWEEN ROOTS AND COEFFICIENTS

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$

and $f(x) = 0$ have n roots $\alpha_1, \alpha_2, ..., \alpha_n$.

Then $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$
$= a_0(x-\alpha_1)(x-\alpha_2) ..... (x-\alpha_n)$,

or, $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0}$

$= (x-\alpha_1)(x-\alpha_2) ... (x-\alpha_n)$,

or, $x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0}$

$= x^n - \sum\alpha_1x^{n-1} + \sum\alpha_1\alpha_2x^{n-2} - \sum\alpha_1\alpha_2\alpha_3x^{n-3}$
$+ .... + (-1)^n\alpha_1\alpha_2...\alpha_n$.

Equating the coefficients of like powers of x on each side of the identity gives

$$\sum\alpha_1 = -\frac{a_1}{a_0},$$

$$\sum\alpha_1\alpha_2 = \frac{a_2}{a_0},$$

$$\sum\alpha_1\alpha_2\alpha_3 = \frac{a_3}{a_0},$$

.....

$$\sum\alpha_1\alpha_2.....\alpha_{n-1} = (-1)^{n-1}\frac{a_{n-1}}{a_0}.$$

$$\alpha_1\alpha_2.....\alpha_n = (-1)^n\frac{a_n}{a_0}$$

Therefore, the sum of the roots is equal to $-\frac{a_1}{a_0}$,

the sum of their products in pair is equal to $\frac{a_2}{a_0}$ and so on.

For a **cubic equation** $ax^3 + bx^2 + cx + d = 0$, whose roots are α, β, γ

S_1 = Sum of the roots = $\Sigma\alpha = \alpha + \beta + \gamma = -\frac{b}{a}$

S_2 = Sum of the product of the roots taken two roots at a time = $\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

S_3 = Product of the roots = $\alpha\beta\gamma = -\frac{d}{a}$

This can be verified by writing

$ax^3 + bx^2 + cx + d \equiv a(x-\alpha)(x-\beta)(x-\gamma)$

and comparing coefficients of like powers.

Similarly for a biquadratic equation

$ax^4 + bx^3 + cx^2 + dx + e = 0$ whose roots are $\alpha, \beta, \gamma, \delta$,

$$S_1 = \Sigma\alpha = -\frac{b}{a} \qquad S_2 = \Sigma\alpha\beta = \frac{c}{a}$$

$$S_3 = \Sigma\alpha\beta\gamma = -\frac{d}{a} \qquad S_4 = \alpha\beta\gamma\delta = \frac{e}{a}$$

STUDY TIP It should be remarked that the sum of the roots of any polynomial equation will be zero provided the term one degree lower than the highest is absent.

➤ **Example 1**. Solve the equation

$$a^4\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^4\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^4\frac{(x-a)(x-b)}{(c-a)(c-b)} = x^4.$$

➤ ***Solution*** The equation is clearly satisfied by x = a. by x = b, and by x = c. Also, since the coefficient of x^3 is zero, the sum of the roots is zero. Hence the remaining root must be $-a-b-c$.

➤ **Example 2.** Let α and β be the roots of the equation $x^3 + ax^2 + bx + c = 0$ satisfying the relation $\alpha\beta + 1 = 0$. Prove that $c^2 + ac + b + 1 = 0$.

➤ ***Solution*** If α, β and γ be the roots of the given equation, then we have

$\alpha + \beta + \gamma = -a$ ...(1)

$\alpha\beta + \alpha\gamma + \beta\gamma = b$ ...(2)

and $\alpha\beta\gamma = -c$ ...(3)

Also, we have $\alpha\beta + 1 = 0$ ...(4)

www.pdfworld.net

www.pdfworld.net

Putting $\alpha\beta = -1$ in equation (3), we have $\gamma = c$
Putting the value of γ in equation (1),
we have $\alpha + \beta = -a - c$
Now, putting the above values in equation (2), we have

$\alpha\beta + \gamma(\alpha + \beta) = b$
i.e. $-1 - c(a + c) = b$
i.e. $c^2 + ac + b + 1 = 0$

➤ **Example 3.** Let u, v be two real numbers such that u, v and uv are roots of a cubic polynomial with rational coefficients. Prove or disprove uv is rational.

➤ ***Solution*** Let $x^3 + ax^2 + bx + c = 0$ be the cubic polynomial of which u, v and uv are the roots and a, b, c are all rationals.

$u + v + uv = -a$
$\Rightarrow \quad u + v = -a - uv, \qquad uv + uv^2 + u^2 v = b$
and $u^2v^2 = -c$
$b = uv + uv^2 + u^2v = uv(1 + v + u)$
$= uv(1 - a - uv) \qquad = (1 - a)uv - u^2v^2$
$= (1 - a)uv + c$

i.e., $uv = \dfrac{(b - c)}{1 - a}$ and since a, b, c are rational, uv is rational.

STUDY TIP When it is known that two or more roots of an equation are connected by any given relation, then it becomes easier to find these roots with the help of relations between roots and coefficients.

➤ **Example 4.** Solve completely the equation
$x^4 - 5x^3 + 11x^2 - 13x + 6 = 0$
using the fact that two of its roots α and β are connected by the relation $3\alpha + 2\beta = 7$.

➤ ***Solution*** From the given relation $\beta = \dfrac{7 - 3\alpha}{2}$.

Replacing x by $\dfrac{7 - 3x}{2}$ in
$f(x) = x^4 - 5x^3 + 11x^2 - 13x + 6$, we get

$$\left(\frac{7-3x}{2}\right)^4 - 5\left(\frac{7-3x}{2}\right)^3 + 11\left(\frac{7-3x}{2}\right)^2 - 13\frac{7-3x}{2} + 6$$

Let it reduce to
$\phi(x) = 81x^4 - 486x^3 + 1152x^2 - 1242x + 495.$
The H.C.D. of f(x) and $\phi(x)$ is $x - 1$.

$\therefore \quad \alpha = 1$ and $\beta = \dfrac{7 - 3}{2} = 2.$

Let the other roots of $f(x) = 0$ by γ and δ.
By the relation between roots and coefficients,
$1 + 2 + \gamma + \delta = 5, \quad$ or, $\quad \gamma + \delta = 2$
and $1 \cdot 2 \cdot \gamma \cdot \delta = 6, \quad$ or, $\quad \gamma\delta = 3.$

Solving a quadratic equation we get $\gamma = 1 + i\sqrt{2}$ and $\delta = 1 - i\sqrt{2}$.

Hence the roots are $1, 2, 1 \pm i\sqrt{2}$.

➤ **Example 5.** Solve the cubic equation $9x^3 - 27x^2 + 26x - 8 = 0$, given that one of the roots of this equation is double to other.

➤ ***Solution*** Let the roots be α, 2α and β.

Now, $3\alpha + \beta = -\dfrac{-27}{9} = 3$
$\Rightarrow \quad \beta = 3(1 - \alpha)$ ...(1)

$2\alpha^2 + 3\alpha\beta = \dfrac{26}{9}$ ...(2)

$2\alpha^2\beta = +\dfrac{8}{9}$ ...(3)

From Eq. (1) and Eq. (2), we get

$2\alpha^2 + 3\alpha \times 3(1 - \alpha) = \dfrac{26}{9}$
$\Rightarrow \quad 63\alpha^2 - 81\alpha + 26 = 0$
$\Rightarrow \quad (21\alpha - 13)(3\alpha - 2) = 0$

So $\alpha = \dfrac{13}{21}$ or $\dfrac{2}{3}$

If $a = \dfrac{13}{21} \qquad \therefore \qquad \beta = 3\left(1 - \dfrac{13}{21}\right) = \dfrac{24}{21} = \dfrac{8}{7}$

This leads to $2\alpha^2\beta = 2 \times \dfrac{169}{441} \times \dfrac{8}{7} \neq \dfrac{8}{9}$.

(a contradiction)

www.pdfworld.net

www.pdfworld.net

So taking $a = \frac{2}{3}$,

$\therefore \quad \beta = 3\left(1 - \frac{2}{3}\right) = 3 \times \frac{1}{3} = 1$

Hence $\alpha + 2\alpha + \beta = \frac{2}{3} + \frac{4}{3} + 1 = 3,$

$$2\alpha^2 + 3\alpha\beta = 2 \times \frac{4}{9} + \frac{3 \times 2}{3} \times 1 = \frac{26}{9},$$

and $2\alpha^2\beta = 2 \times \frac{4}{9} \times 1 = \frac{8}{9}$

Thus, the roots are $\frac{2}{3}, \frac{4}{3}$ and 1.

➤ **Example 6.** Show that all the roots of the equation $ax^3 + x^2 + x + 1 = 0$ cannot be real, where $a \in R$.

➤ ***Solution*** If $a = 0$ then all roots are obviously non-real therefore we have $a \neq 0$.

Let the roots be x_1, x_2, x_3 then we have

$$x_1 + x_2 + x_3 = -\frac{1}{a}, \; x_1x_2 + x_2x_3 + x_2x_1 = \frac{1}{a},$$

$$x_1x_2x_3 = -\frac{1}{a}$$

Now, if possible, let all the roots be real. Consider the expression

$$x_1^2\, x_2^2 + x_2^2\, x_3^2 + x_3^2\, x_1^2$$

$$= (x_1\, x_2 + x_2\, x_3 + x_3\, x_1)^2 - 2x_1\, x_2\, x_3\, (x_1 + x_2 + x_3)$$

$$= \frac{1}{a^2} - 2\left(-\frac{1}{a}\right)\left(-\frac{1}{a}\right) = -\frac{1}{a^2} < 0$$

which is contradictory

since $x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2$ must be essentially non-negative if all the roots are real.

Practice Problems [R]

1. Solve the equation $2x^3 + x^2 - 7x - 6 = 0$, given that the difference between two of the roots is 3.
2. Solve the equation $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, the roots being in A.P.
3. Solve $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$, two roots being 1 and 7.
4. Solve $2x^3 - x^2 - 22x - 24 = 0$, two of the roots being in the ratio of 3 : 4.
5. Solve $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$, two of the roots being equal but opposite in sign.
6. Solve $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$, the product of two of the roots being 2.
7. If one root of the equation $x^3 + 2ax^2 - b = 0$, is equal to the sum of the other two, then show that $a^3 = b$.
8. Solve $3x^3 - 26x^2 + 52x - 24 = 0$, the roots being in G.P.

2.23 TRANSFORMATION OF POLYNOMIAL EQUATION

Let $\alpha, \beta, \gamma, \ldots$ be the roots of $f(x) = 0$, and suppose that we require the equation whose roots are $\phi(\alpha), \phi(\beta), \phi(\gamma), \ldots$ where $\phi(x)$ is a given function of x.

Let $y = \phi(x)$ and suppose that from this equation we can find x as a single-valued function of y, which we denote by $\phi^{-1}(y)$.

Transforming the equation $f(x) = 0$ by the substitution $x = \phi^{-1}(y)$, we obtain $f\{\phi^{-1}(y)\} = 0$, which is the equation required.

A case in which x is not a single-valued function of y is also shown below.

STUDY TIP The following transformations are often required.

Let $\alpha, \beta, \gamma, \ldots$ be the roots of $f(x) = 0$, then

(1) the equation whose roots are $-\alpha, -\beta, -\gamma, \ldots$ if $f(-x) = 0$;

(2) the equation whose roots are $1/\alpha, 1/\beta, 1/\gamma$, is $f(1/x) = 0$

(3) the equation whose roots are $k\alpha, k\beta, k\gamma, \ldots$ is $f(x/k) = 0$.

www.pdfworld.net

www.pdfworld.net

(4) The equation whose roots are $\alpha - h, \beta - h, \gamma - h$ is $f(x + h) = 0$.

➤ **Example 1.** Solve $6x^3 - 11x^2 + 6x - 1 = 0$ if roots of the equation are in H.P.

➤ ***Solution*** Putting $x = \frac{1}{y}$ in the given equation we get

$$\frac{6}{y^3} - \frac{11}{y^2} + \frac{6}{y} - 1 = 0 \Rightarrow 6 - 11 + 6y^2 - y^3 = 0$$

$\Rightarrow \quad y^3 - 6y^2 + 11y - 6 = 0 \qquad ...(1)$

Now roots of (1) are in A.P.

Let the roots be $\alpha - \beta, \alpha, \alpha + \beta$

Then sum of roots $\alpha - \beta + \alpha + \alpha + \beta = 6$

$\Rightarrow \quad 3\alpha = 6 \qquad \therefore \quad \alpha = 2$

Product of roots $(\alpha - \beta)\,\alpha\,(\alpha + \beta) = 6$

$\therefore \quad 2(4 - \beta^2) = 6 \therefore \qquad \beta = \pm 1$

$\therefore \quad$ Roots of (1) are 1, 2, 3

Hence roots of the given equation are $1, \frac{1}{2}, \frac{1}{3}$

➤ **Example 2.** If a, b, c be the roots of equation $x^3 + px^2 + qx + r = 0$, then find a cubic equation whose roots are $a(b + c), b(c + a), c(a + b)$.

➤ ***Solution*** Given $a + b + c = -p$,

$ab + bc + ca = q, \qquad abc = -r$

$$\text{Let } x = a(b+c) = ab + ac = ab + bc + ca - bc = q - bc = q + \frac{r}{a}$$

$$\Rightarrow \quad \frac{r}{a} = x - q \quad \Rightarrow \qquad a = \frac{r}{x - q} \qquad ...(1)$$

Since a is a root of given equation so $a^3 + pa^2 + qa + r = 0$

Put a from (1)

$$\Rightarrow \quad \frac{r^3}{(x-q)^3} + p\frac{r^2}{(x-q)^2} + q\frac{r}{(x-q)} + r = 0$$

$$\Rightarrow \quad \frac{r^2}{(x-q)^3}$$

$$r^2 + pr(x - q) + x(x - q)^2 = 0$$

➤ **Example 3.** If α, β, γ be the roots of the equation $x^3 - px^2 + r = 0$, find a cubic equation whose roots are $\frac{\beta + \gamma}{\alpha}, \frac{\gamma + \alpha}{\beta}, \frac{\alpha + \beta}{\gamma}$.

➤ ***Solution*** Let

$$y = \frac{\beta + \gamma + \alpha - \alpha}{\alpha} = \frac{p - \alpha}{\alpha} = \frac{p}{\alpha} - 1 \quad \left(\because \sum \alpha = p\right),$$

$$\text{or,} \quad \alpha = \frac{p}{y+1} \quad \therefore \qquad x = \frac{p}{y+1}.$$

Replacing x by $\frac{p}{y+1}$ in the equation, we have

$$\frac{p^3}{(y+1)^3} - \frac{p^3}{(y+1)^2} + r = 0,$$

or, $\quad r(y + 1)^3 - p^3(y + 1) + p^3 = 0,$

or, $\quad ry^3 + 3ry^2 + (3r - p^3)\,y + r = 0.$

It is the required equation in y.

➤ **Example 4.** If α, β, γ are the roots of the cubic $x^3 + x + 2 = 0$, find the equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$.

➤ ***Solution*** Since α, β, γ are the roots of the cubic

$x^3 + x + 2 = 0 \qquad ...(1)$

then $\Sigma\alpha = 0, \qquad \Sigma\alpha\beta = 1,$

$\alpha\beta\gamma = -2 \qquad ...(2)$

If y is a root of the required equation, then

$$y = (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = (\alpha + \beta + \gamma - \gamma)^2 - \frac{4\alpha\beta\gamma}{\gamma}$$

$$= (0 - \gamma)^2 + \frac{8}{\gamma} \qquad \{\text{from (2)}\}$$

$$\Rightarrow \quad y = \gamma^2 + \frac{8}{\gamma}$$

[replacing γ by x which is root of (1)]

$$\therefore \quad y = x^2 + \frac{8}{\gamma} \text{ or } x^3 - yx + 8 = 0 \qquad ...(3)$$

The required equation is obtained by eliminating x between (1) and (3).

www.pdfworld.net

www.pdfworld.net

Now subtracting (1) from (3), we get

$$(1+y)\,x - 6 = 0 \qquad \text{or} \qquad x = \frac{6}{1+y}$$

substituting in (1), we get

$$\left(\frac{6}{1+y}\right)^3 + \left(\frac{6}{1+y}\right) + 2 = 0$$

or $y^3 + 6y^2 + 9y + 112 = 0$
which is the required equation.

➤ **Example 5.** If α is a root of the equation
$x^4 + px^3 - 6x^2 - px + 1 = 0$,
then show that $\dfrac{1+\alpha}{1-\alpha}$ is also a root. Hence show that the other two roots are $-\left(\dfrac{1}{\alpha}\right), \dfrac{(\alpha-1)}{(\alpha+1)}$.

➤ ***Solution*** Putting $x = \dfrac{1+\alpha}{1-\alpha}$ in the given equation, we get

$$\left(\frac{1+\alpha}{1-\alpha}\right)^4 + \left(\frac{1+\alpha}{1-\alpha}\right)^3 - 6\left(\frac{1+\alpha}{1-\alpha}\right)^2 - p\left(\frac{1+\alpha}{1-\alpha}\right) + 1 = 0$$

$\Rightarrow \quad [(1+\alpha)^4 + (1-\alpha)^4] + p[(1+\alpha)^3(1-\alpha) + (1+\alpha)^3] - 6(1+\alpha)^2(1-\alpha)^2 = 0$

$\Rightarrow \quad 2(1+6\alpha^2+\alpha^4) + 4\alpha(1-\alpha)^2 p - 6(1-2\alpha^2+\alpha^4) = 0$

$\Rightarrow \quad \alpha^4 + p\alpha^3 - 6\alpha^2 - p\alpha + 1 = 0$

which shows that $\dfrac{1+\alpha}{1-\alpha}$ is also a root.

Now replacing x by $-\dfrac{1}{x}$, in the given equation we note that the equation does not change. Hence if α is a root then $-\dfrac{1}{\alpha}$ is also a root.

Now since $-\dfrac{1}{\alpha}$ is a root $\dfrac{1+\left(-\frac{1}{\alpha}\right)}{1-\left(-\frac{1}{\alpha}\right)}$ is a root from above.

Thus the roots are $\alpha, -\dfrac{1}{\alpha}, \dfrac{1+\alpha}{1-\alpha}$ and $\dfrac{(\alpha-1)}{(\alpha+1)}$.

Practice Problems [S]

1. Find the equation each of the whose roots is greater by unity than a root of the equation $x^3 - 5x^2 + 6x - 3 = 0$.
2. Solve the equation $3x^3 - 22x^2 + 48x - 32 = 0$, the roots of which are in H.P.
3. If α, β, γ are the roots of $x^3 + x^2 - 4x + 7 = 0$, find the reduced cubic equation whose roots are $\alpha+\beta, \beta+\gamma, \gamma+\alpha$.
4. Find the equation whose roots are those of the equation $6x^3 - 5x^2 - \dfrac{1}{4} = 0$, each multiplied by c, and find the least value of c in order that the resulting equation may have integral coefficients with unity for the coefficient of the highest power.
5. If a, b, c are the roots of the equation $x^3 - px^2 + qx - r = 0$ find the value of
   (a) $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$ (b) $\dfrac{1}{b^2c^2} + \dfrac{1}{c^2a^2} + \dfrac{1}{a^2b^2}$
6. If a, b, c are the roots of $x^3 + qx + r = 0$, find the value of
   (a) $(b-c)^2 + (c-a)^2 + (a-b)^2$
   (b) $\dfrac{1}{b+c} + \dfrac{1}{c+a} + \dfrac{1}{a+b}$

2.24 SYMMETRIC FUNCTIONS OF ROOTS

Symmetrical expressions

An expression which is unaltered by interchanging any pair of the letters which it contains is said to be a symmetrical expression. Thus $a+b+c$, $bc+ca+ab$, $a^3+b^3+c^3-3abc$ are symmetrical expressions.
Expressions which are unaltered by a cyclical change of the letters involved in them are called cyclically

www.pdfworld.net

www.pdfworld.net

symmetrical expressions. For example, the expression $(b - c)(c - a)(a - b)$ is a cyclically symmetrical expression since it is unaltered by changing a into b, b into c, and c into a.

It is clear that the product, or the quotient, of two symmetrical expressions is symmetrical, for if neither of two expressions is altered by an interchange of two letters, their product, or their quotient, cannot be altered by such interchange.

It is also clear that the product, or the quotient, of two cyclically symmetrical expressions is cyclically symmetrical.

Every rational symmetric function of the roots of a polynomial equation is expressible in terms of the elementary symmetric functions of the same and therefore in terms of the coefficients of the equation. Without knowing the separate values of the roots in terms of the coefficients, we can calculate the values of symmetric functions of roots in terms of the coefficients. For example, if α, β, γ be the roots of $x^3 + px^2 + qx + r = 0$, we can find out the value of

(i) $\alpha\beta + \beta\gamma + \gamma\alpha$,

(ii) $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$,

(iii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

in terms of p, q and r and each of these relations are symmetric functions of the roots, since there will be no change in them if two of α, β, γ are interchanged. To find out the value of the above symmetric functions, we take the help of the following relations,

$$\alpha + \beta + \gamma = -p, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = q,$$
$$\alpha\beta\gamma = -r.$$

Now $\sum \alpha^2\beta = \sum \alpha \sum \alpha\beta - 3\alpha\beta\gamma = -p \times q - 3.(-r)$
$= 3r - pq$ and

$$\sum \alpha^2\beta\gamma = \alpha\beta\gamma \sum \alpha = -r \times (-p) = pr.$$

Important Formulae

(1) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
$= a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

(2) $a^2 + b^2 + c^2 - ab - bc - ca$
$= \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(3) $a^3 + b^3 + c^3 - 3abc$
$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Newton's Theorem

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n$ be the roots of
$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n = 0$

and if $S_r = \sum \alpha_1^r$, then

(i) $S_r + p_1 S_{r-1} + p_2 S_{r-2} + \ldots\ldots + S_1 p_{r-1} + r p_r = 0$ when $r < n$
and

(ii) $S_r + p_1 S_{r-1} + p_2 S_{r-2} + \ldots\ldots + p_{n-1} S_{r-n+1} + p_n S_{r-n}$
$= 0$ when $r \geq n$.

➤ **Example 1.** Let a, b, c be the three roots of the equation $x^3 + x^2 - 333x - 1002 = 0$ then find the value of $a^3 + b^3 + c^3$.

➤ ***Solution*** Let t be the root of the given cubic where t can take values a, b, c

hence $t^3 + t^2 - 333t - 1002 = 0$

or $t^3 = 1002 + 333t - t^2$

$$\therefore \quad \sum t^3 = \sum 1002 + 333 \sum t - \sum t^2$$
$$= 3006 + 333 \sum t - \left[\left(\sum t\right)^2 - 2 \sum t_1 t_2\right]$$

but $\sum t = -1$; $\sum t_1 t_2 = -333$

$\therefore \quad a^3 + b^3 + c^3 = 3006 - 333 - [1 + 666]$
$= 3006 - 333 - 667 = 3006 - 1000 = 2006$

➤ **Example 2.** If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + x = 0$,
show that $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \delta^2)$
$= (1 - q + s)^2 + (p - r)^2$.

➤ ***Solution*** Since $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$x^4 + px^3 + qr^2 + rx + x = 0,$$

therefore $x^4 + px^3 + qx^2 + rx + x$
$\equiv (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$,

Substituting $x = i, -i$ successively, we have

$$(1 - q + s) - i(p - r) = (i - \alpha)(i - \beta)(i - \gamma)(i - \delta) \quad \ldots(1)$$

$$(1 - q + s) + i(p - r) = (-i - \alpha)(-i - \beta)(-i - \gamma)(-i - \delta) \quad \ldots(2)$$

Multiplying corresponding sides of (1) and (2), we have

www.pdfworld.net

www.pdfworld.net

$(1-q+s)^2+(p-r)^2=(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)$.

➤ **Example 3.** If α, β, γ be the roots of the equation $x^3 + qx + r = 0$, then prove that

$$\frac{\alpha^5+\beta^5+\gamma^5}{5}=\frac{\alpha^3+\beta^3+\gamma^3}{3}\times\frac{\alpha^2+\beta^2+\gamma^2}{2}$$

➤ ***Solution*** Since α, β, γ are the roots of the equation

$$x^3 + qx + r = 0, \qquad ...(1)$$

therefore
$$\left.\begin{aligned}\alpha^3+q\alpha+r=0,\\ \beta^3+q\beta+r=0,\\ \gamma^3+q\gamma+r=0\end{aligned}\right\} \qquad ...(2)$$

Adding the corresponding sides of (2), we have

$$\Sigma\alpha^3 + q\Sigma\alpha + 3r = 0$$

Since $\Sigma a = 0$, this yields,

$$\Sigma\alpha^3 = -3r \qquad ...(3)$$

Also, $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = -2q$ ...(4)

Multiplying (1) throughout by x^2, we find that α, β, γ are three of the roots of the equation

$$x^5 + qx^3 + rx^2 = 0 \qquad ...(5)$$

Substituting $x = \alpha, \beta, \gamma$ successively in (v) and adding, we get

$$\Sigma\alpha^5 + q\Sigma\alpha^3 + r\Sigma\alpha^2 = 0 \qquad ...(6)$$

Substituting the values of $\Sigma\alpha^3$ and $\Sigma\alpha^2$ from (3) and (4), we get

$$\Sigma\alpha^5 = 5qr \qquad ...(7)$$

From (3), (4) and (7), we get

$$\frac{1}{5}\Sigma\alpha^5=\left(\frac{1}{3}\Sigma\alpha^3\right)\left(\frac{1}{2}\Sigma\alpha^3\right)$$

2.25 TRIGONOMETRICAL METHOD OF SOLVING CUBIC EQUATION

We consider the trigonometrical identity

$$4\cos^3\theta - 3\cos\theta = \cos 3\theta,$$

or, $$\cos^3\theta-\frac{3}{4}\cos\theta-\frac{1}{4}\cos 3\theta=0, \qquad ...(1)$$

Writting $z = r\cos\theta, r > 0$, in $z^3 + 3Hz + G = 0$, we have

$$\cos^3\theta+\frac{3H}{r^2}\cos\theta+\frac{G}{r^3}=0. \qquad ...(2)$$

If we assume that the equations (1) and (2) are identical then comparing them,

$$\frac{3H}{r^2}=-\frac{3}{4} \text{ and } \frac{G}{r^3}=-\frac{1}{4}\cos 3\theta, \text{ or}$$

$$r = 2\sqrt{-H}, \cos 3\theta = \frac{-4G}{r^3}=\frac{-G}{2\sqrt{-H^3}} \qquad ...(3)$$

From (3), θ is known and then $z = 2\sqrt{-H}\cos\theta$ is obtained.

If α is any value of θ satisfying $\cos 3\theta = \frac{-G}{2\sqrt{-H^3}}$, then $3\theta = 2n\pi \pm 3\alpha$. The three different values of $\cos\theta$ are $\cos\alpha$, $\cos\left(\frac{2\pi}{3}+\alpha\right)$ and $\cos\left(\frac{2\pi}{3}-\alpha\right)$ and hence the solutions are

$$2\sqrt{-H}\cos\alpha, 2\sqrt{-H}\cos\left(\frac{2\pi}{3}+\alpha\right)$$

and $$2\sqrt{-H}\cos\left(\frac{2\pi}{3}-\alpha\right).$$

If θ is real, $|\cos 3\theta| \le 1$.

$$\therefore \quad \left|\frac{G}{2\sqrt{-H^3}}\right| \le 1. \qquad \text{i.e.,} \qquad G^2 + 4H^3 \le 0.$$

In this case all the roots are real. Further in order that r is real, $H < 0$.

Thus the trigonometrical method to solve a cubic equation is generally applicable when the roots are real.

➤ **Example 1.** Solve $x^3 - 27x + 27 = 0$.

➤ ***Solution*** Here $H = -9$, $G = 27$, $G^2 + 4H^3 = 27^2 - 4.9^3 = 9^3 - 4.9^3 = -3.9^3 < 0$.

Therefore all the roots are real.

From above, $\cos 3\theta = \frac{-G}{2\sqrt{-H^3}} = -\frac{1}{2} = \cos\frac{2\pi}{3}$.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

$\therefore \quad 3\theta = 2n\pi + \frac{2\pi}{3}$.

Again $x = 2\sqrt{-H}\cos\theta = 6\cos\theta$.

The three different values of $\cos\theta$ are $\cos\frac{2\pi}{9}$, $\cos\frac{8\pi}{9}$ and $\cos\frac{4\pi}{9}$.

Hence the solutions are, $6\cos\frac{2\pi}{9}$, $6\cos\frac{8\pi}{9}$ and $6\cos\frac{4\pi}{9}$.

➤ **Example 2.** Show that the roots of $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$ and hence find the equation whose roots are $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$ and $\sec^2\frac{5\pi}{7}$.

➤ ***Solution*** Let $7\theta = (2n+1)\pi$ where n is any integer.

$\therefore \quad \theta = \frac{2n+1}{7}\pi$ and $\cos\theta = \cos\frac{2n+1}{7}\pi$

where $n = 0, 1, \ldots, 6$.

Explicitly $\cos\theta = \cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}, \cos\pi, \cos\frac{9\pi}{7}\ \cos\frac{11\pi}{7}, \cos\frac{13\pi}{7}$.

Again $\cos\frac{13\pi}{7} = \cos\frac{\pi}{7}$, $\cos\frac{11\pi}{7} = \cos\frac{3\pi}{7}$, $\cos\frac{9\pi}{7} = \cos\frac{5\pi}{7}$.

Therefore the distinct values of $\cos\theta$ are

$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$ and -1 (since $\cos\pi = -1$)

Now $7\theta = (2n+1)\pi$, or, $4\theta = (2n+1)\pi - 3\theta$.

$\therefore \quad \cos 4\theta = -\cos 3\theta$,

or, $2\cos^2 2\theta - 1 = -4\cos^3\theta + 3\cos\theta$,

or, $2(2\cos^2\theta - 1)^2 - 1 = -4\cos^3\theta + 3\cos\theta$,

or, $8\cos^4\theta - 8\cos^2\theta + 1 = -4\cos^3\theta + 3\cos\theta$,

or, $8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0$.

Putting $\cos\theta = x$, we get $8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$.

The roots of this equation in x are -1, $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$.

For the root -1, $x + 1$ is a factor of $8x^4 + 4x^3 - 8x^2 - 3x + 1$. Cancelling this factor we get

$8x^3 - 4x^2 - 4x + 1 = 0$.

This is the required equation.

Transforming the equation by $y = \frac{1}{x^2}$, we get the equation

whose roots are $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$ and $\sec^2\frac{5\pi}{7}$.

We have $8x^3 - 4x = 4x^2 - 1$,

or, $4x(2x^2 - 1) = 4x^2 - 1$,

or, $4x\left(\frac{2}{y} - 1\right) = \frac{4}{y} - 1$,

or, $4x(2 - y) = 4 - y$.

Squaring, $16x^2(2 - y)^2 = (4 - y)^2$,

or, $\frac{16}{y}(4 - 4y + y^2) = 16 - 8y + y^2$,

or, $y^3 - 24y^2 + 80y - 64 = 0$.

It is the equation in y whose roots are $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$ and $\sec^2\frac{5\pi}{7}$.

www.pdfworld.net

www.pdfworld.net

Practice Problems [T]

1. If α, β, γ be the roots of the $ax^3 + bx^2 + cx + d = 0$ then find the values of,

(i) $\sum\alpha^2$ (ii) $\sum\frac{1}{\alpha}$ (iii) $\sum\alpha^2\beta$

2. If a, b, c be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the values of

(i) $(b + c - 3a)(c + a - 3b)(a + b - 3c)$.

(ii) $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)$.

(iii) $\left(\frac{1}{a^2}-\frac{1}{bc}\right)\left(\frac{1}{b^2}-\frac{1}{ca}\right)\left(\frac{1}{c^2}-\frac{1}{ab}\right)$.

3. Find the roots of the equation $x^3 - 3x + 1 = 0$.

4. Solve the equation $8x^3 - 36x + 27 = 0$

2.26 COMMON ROOTS IN POLYNOMIAL EQUATIONS

If $\psi(x)$ is the highest common factor (H.C.F.) of $f(x)$ and $g(x)$, then the roots of $\psi(x) = 0$ are the common roots of $f(x) = 0$ and $g(x) = 0$.

➤ **Example 1.** Find the common roots of $x^4 + 5x^3 - 2x^2 - 50x + 132 = 0$ and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ and hence solve the equations.

➤ ***Solution*** We find that $4(x^2 - 5x + 6)$ is H.C.F. of the two equations

and hence, the common roots are the roots of

$x^2 - 5x + 6 = 0$ i.e., $x = 3$ or $x = 2$.

Now, $x^4 + 5x^3 - 22x^2 - 50x + 132 = 0$ ....(1)

and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ ....(2)

have 2 and 3 as their common roots.

➤ **Example 2.** Suppose a cubic polynomial $f(x) = x^3 + px^2 + qx + 72$ is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ (where a, b, p, q are constants and $a \neq b$). Find the sum of the squares of the roots of the cubic polynomial.

➤ ***Solution*** Since cubic is divisible by both

$x^2 + ax + b$ and $x^2 + bx + a$

$\therefore$ $x^2 + ax + b$

and $x^2 + bx + a$ must have a common roots.

$x^2 + ax + b = 0$ $\quad - x^2 + bx + a = 0$

subtract

$\Rightarrow$ $x(a - b) = (a - b)$

$x = 1$ $\quad \therefore$ $\quad$ common root is 1

$x^2 + ax + b = 0$ < 1, α

$1 \cdot \alpha = b \quad \Rightarrow \quad \alpha = b$

$x^2 + bx + a = 0$ < 1, β

$1 \cdot \beta = a \quad \Rightarrow \quad \beta = a$

$\Rightarrow$ roots of cubic be 1, a, b

product of the roots be

$1 \cdot a \cdot b = -72$ ....(1)

and $a + b + 1 = 0$ ....(2)

(from $x^2 + ax + b = 0$ put $x = 1$)

$\therefore \quad a - \frac{72}{b} = -1$

$\Rightarrow \quad a^2 + a - 72 = 0 \quad (a + 9)(a - 8) = 0$

$a = -9, 8 \quad \therefore \quad$ roots are $1, -9, 8$

$\Rightarrow$ sum of their squares $= 1 + 81 + 64 = 146$

Method of Subtraction

If α is a common root of $f(x) = 0$ and $g(x) = 0$ then α is also a common root of $f(x) - g(x) = 0$; but not all roots $f(x) - g(x) = 0$ are common roots of $f(x)$ and $g(x)$.

Proof :

Let α be a common root

Then $f(\alpha) = 0$ and $g(\alpha) = 0$ ...(1)

The equation $f(x) - g(x) = 0$ is satisfied by $x = \alpha$ because of (1)

However, consider a number β such that

$f(\beta) = g(\beta) = k \neq 0$ ...(2)

Then β satisfies the equation $f(x) - g(x) = 0$ because of (2) but we know that β is not a common root of $f(x)$ and $g(x)$.

Hence all roots of $f(x) - g(x) = 0$ are not necessarily common roots. Here we solve $f(x) - g(x) = 0$. Only

www.pdfworld.net

www.pdfworld.net

those roots of $f(x) - g(x) = 0$ which satisfy $f(x) = 0$ (or $g(x) = 0$) are the common roots.

➤ **Example 3.** Find the common roots of the equations

$x^3 - 2x^2 - x + 2 = 0$ and $x^3 + 6x^2 + 11x + 6 = 0$.

➤ ***Solution*** Substract the two equations

We get $8x^2 + 12x + 4 = 0$

$$2x^2 + 3x + 1 = 0 \quad \Rightarrow \quad x = -1, -\frac{1}{2}$$

Verify $x = -1, -\frac{1}{2}$ in one of the equations, say

$x^3 - 2x^2 - x + 2 = 0$.

Only $x = -1$ satisfies this equation.

It means that $x = -1$ is the only common root.

➤ **Example 4.** Find the values of a so that the following equations have common root(s) :

$x^3 - 2x^2 - x + a = 0$, $x^2 - 5x + 2a = 0$

➤ ***Solution*** $x^3 - 2x^2 - x + a = 0$ ...(1)

$x^2 - 5x + 2a = 0$ ...(2)

$x = 0$ is a common root when $a = 0$.

Now multiply (2) by x and substract from (1).

So as to cancel the x^3 term.

$-3x^2 + (1 + 2a)x - a = 0$ ...(3)

Multiply (2) by 3 and add to (3)

$(2a - 14)x + 5a = 0$

$$\Rightarrow \quad x = \frac{5a}{14 - 2a}$$

For this to be the common root we put this root into (2).

$$\left(\frac{5a}{14-2a}\right)^2 - 5\left(\frac{5a}{14-2a}\right) + 2a = 0$$

$\Rightarrow$ $a[25a - 25(14 - 2a) + 2(14 - 2a)^2] = 0$

$\Rightarrow$ $a(8a^2 - 37a + 42) = 0$

$\Rightarrow$ $a(a - 2)(8a - 21) = 0$

$\Rightarrow$ $a = 0, 2, \frac{21}{8}$.

The value of a for which we get common roots are

$a = 0, 2, \frac{21}{8}$.

2.27 MULTIPLE ROOTS

If $f(x)$ contains a factor $(x - \alpha)^r$, then $f'(x)$ contains a factor $(x - \alpha)^{r-1}$. Therefore if $f(x)$ and $f'(x)$ have no common factor, no factor in $f(x)$ will be repeated ; hence the equation $f(x) = 0$ has or has not equal roots, according as $f(x)$ and $f'(x)$ have or have not a common factor involving x.

Proof :

If α be an r-multiple root of $f(x) = 0$, whose degree is n, then

$f(x) = (x - \alpha)^r \phi(x)$

where $\phi(x)$ is a polynomial in x of degree $n - r$ and it is not divisible by $x - \alpha$.

Now $f'(x) = r(x - \alpha)^{r-1}\phi(x) + (x - \alpha)^r\phi'(x)$

$= (x - \alpha)^{r-1}\{r\phi(x) + (x - \alpha)\phi'(x)\}$

$= (x - \alpha)^{r-1}\psi(x)$, where $\psi(x)$

$= r\phi(x) + (x - \alpha)\phi'(x)$.

$\psi(x)$ is not divisible by $(x - \alpha)$. Therefore, α is an $(r - 1)$multiple root of $f'(x) = 0$. Consequently $(x - \alpha)^{r-1}$ is the highest common divisor of $f(x)$ and $f'(x)$.

If $f(x)$ has no other multiple root, $(x - \alpha)^{r-1}$ is the highest common divisors of $f(x)$ and $f'(x)$. So, to determine the multiple roots of $f(x) = 0$, we find out the H.C.F. of $f(x)$ and $f'(x)$ and if it is of the form $(x - \alpha)^{p-1}(x - \beta)^{q-1}$...., then α, β,.... are the multiple roots of $f(x) = 0$ with multiplicities p, q,..... respectively.

Corollary: If α is an r-multiple root of $f(x) = 0$, it is then an $(r - 1)$ multiple root of $f'(x) = 0$, an $(r - 2)$ multiple root of $f''(x) = 0$, and so on.

➤ **Example 1**. Find the condition that the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may have two roots equal.

➤ ***Solution*** In this case the equations $f(x) = 0$, and $f'(x) = 0$, that is

$ax^3 + 3bx^2 + 3cx + d = 0$ ...(1)

$ax^2 + 2bx + c = 0$ ...(2)

must have a common root, and the condition required will be obtained by eliminating x between these two equations.

By combining (1) and (2), we have

$bx^2 + 2cx + d = 0$ ...(3)

www.pdfworld.net

www.pdfworld.net

From (2) and (3), we obtain

$$\frac{x^2}{2(bd-c^2)} = \frac{x}{bc-ad} = \frac{1}{2(ac-b^2)};$$

thus the required conditions is

$$(bc-ad)^2 = 4(ac-b^2)(bd-c^2).$$

➤ **Example 2.** Find the roots of $4x^3+20x^2-23x+6=0$ if two roots are equal.

➤ ***Solution*** Let roots be α, α and β

$$\therefore \quad \alpha+\alpha+\beta = -\frac{20}{4}$$

$$\Rightarrow \quad 2\alpha+\beta = -5 \qquad \ldots.(1)$$

$$\therefore \quad \alpha.\alpha+\alpha\beta+\alpha\beta = -\frac{23}{4}$$

$$\Rightarrow \quad \alpha^2+2\alpha\beta = -\frac{23}{4} \qquad \& \qquad \alpha^2\beta = \frac{6}{4}$$

from equation (1) $\alpha^2+2\alpha(-5-2\alpha) = -\frac{23}{4}$

$$\Rightarrow \quad \alpha^2-10\alpha-4\alpha^2 = -\frac{23}{4}$$

$$\Rightarrow \quad 12\alpha^2+40\alpha-23=0$$

$$\therefore \quad \alpha = 1/2, -\frac{23}{6} \text{ when } \alpha = \frac{1}{2}$$

from equation (1) $\alpha^2\beta = \frac{1}{4}(-5-1) = -\frac{3}{2}$

when $\alpha = \frac{23}{6}$

$$\alpha^2\beta = \frac{23\times 23}{36}\left(-5-5x\left(-\frac{23}{6}\right)\right) \neq \frac{3}{2}$$

$$\Rightarrow \quad \alpha = \frac{1}{2}, \qquad \beta = -6$$

Hence roots of equation $= \frac{1}{2}, \frac{1}{2}, -6$

Practice Problems [U]

1. The equations $2x^3+5x^2-6x-9=0$ and $3x^3+7x^2+7x^2-11x-15=0$ have two common roots. Find them.
2. Determine the common roots of the equation $6x^3+7x^2-x-2=0$, and $6x^4+19x^3+17x^2-2x-6=0$
3. If $x^3+px^2+qx+r=0$ and $x^3+ax^2+bx+c=0$ have two common roots, find the quadratic equation having these common roots as roots.
4. Find the roots common to the equations $x^5-x^3+x^2-1=0$, $x^4=1$.
5. Show that $x^3-2x^2-2x+1=0$ and $x^4-7x^2+1=0$ have two roots in common.
6. Find the solution of the following equations which have common roots :
   $2x^4-2x^3+x^2+3x-6=0$, $4x^4-2x^3+3x-9=0$
7. Find the ratio of b to a in order that the equations $ax^2+bx+a=0$ and $x^3-2x^2+2x-1=0$ may have (1) one, (2) two roots in common.
8. Find the condition that ax^3+bx+c and $a'x^3+b'x+c'$ may have a common linear factor.
9. If the two equations $ax^3+3bx^2+3cx+d=0$ and $ax^2+2bx+c=0$ have a common root and the system of equations $ax+by=0$, $cx+dy=0$ have a non-trivial solution then prove that either a, b, c are in G.P. or b, c, d are in G.P.
10. Solve the following equations each of which has equal roots :
    (i) $4x^3-12x^2-15x-4=0$,
    (ii) $x^4-6x^3+13x^2-24x+36=0$.
11. If x^3+3x^2-9x+c is the product of three factors, two of which are identical, show that c is either 5 or -27 and resolve the given expression into factors in each case.

2.28 INTERMEDIATE VALUE THEOREM

If f(x) is a polynomial function and if $f(a) \neq f(b)$, then f(x) takes on every value between f(a) and f(b).

If f(a) and f(b) are of opposite signs then one root of the equation $f(x)=0$ must lie between a and b.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

As x changes gradually from a to b, the function f(x) changes gradually from f(a) to f(b), and therefore must pass through all intermediate values ; but since f(a) and f(b) have contrary signs the value zero must lie between them ; that is, f(x) = 0 for some value of x between a and b.

It does not follow that f(x) = 0 has only one root between a and b ; neither does it follow that if f(a) and f(b) have the same sign f(x) = 0 has no root between a and b.

1. Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term, provided the leading coefficient is positive.

   In the function f(x) substitute for x the values ∞, 0, $-\infty$ successively, then

   $f(\infty) = \infty$, $f(0) = a_n$, $f(-\infty) = -\infty$;

   If a_n is positive, then f(x) = 0 has a root lying between 0 and $-\infty$, and if p_n is negative f(x) = 0 has a root lying between 0 and ∞

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2}$ ..... $+ a_n$, for sufficiently large values of x, f(x) has the same sign as a_0.

2. Every equation which is of an even degree and has its last term negative has at least two real roots, one positive and one negative, provided the leading coefficient is positive.

   In the function f(x) substitute for x the values ∞, 0, $-\infty$ successively, then

   $f(\infty) = \infty$, $f(0) = a_n$, $f(-\infty) = \infty$;

   Since a_n is negative, f(x) = 0 has a root lying between 0 and ∞ and a root lying between $-\infty$ and 0.

3. If the expressions f(a) and f(b) have contrary signs, an odd number of roots of f(x) = 0 will lie between a and b; and if f(a) and f(b) have the same sign, either no root or an even number of roots will lie between a and b

➤ **Example 1**. Show that the equation $10x^3 - 17x^2 + x + 6 = 0$ has atleast one root between –1 and 0.

➤ ***Solution*** Let $f(x) = 10x^3 - 17x^2 + x + 6 = 0$

$f(-1) = -10 - 17 - 1 + 6 = -22 < 0$

$f(0) = 6 > 0$

Since f(–1). f(0) < 0, f(x) = 0 has atleast one root between –1 and 0.

To determine the nature of roots of some equations the following statements are helpful:

(i) If the coefficients are all positive, the equation has no positive root ; thus the equation $x^5 + 4x^3 + 2x + 1 = 0$ cannot have a positive root.

(ii) If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers of x are all of opposite sign, the equation has no negative root; thus the equation $x^7 + x^5 - 3x^4 + x^3 - 3x^2 + 2x - 5 = 0$ cannot have a negative roots.

(iii) If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root ; thus the equation $x^8 + 3x^4 + 2x^2 + 1 = 0$ cannot have a real root.

(iv) If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root except x = 0 ; thus the equation $x^9 + 4x^5 + 5x^3 + 3x = 0$ has no real root except x = 0.

2.29 DESCARTES' RULE OF SIGNS

Consider a polynomial equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} ..... + a_n$$

with real coefficients and $a_n \neq 0$.

We have $f(-x) = a_0(-x)^n + a_1(-x)^{n-1} + a_2(-x)^{n-2} ..... + a_n$

By observing the sign variation in the coefficients of f(x) and f(–x) we can predict the following things about the nature of roots of f(x).

(i) The maximum number of positive roots of f(x) is equal to the number of sign changes in the coefficients of f(x). Let this number be 'p'.

www.pdfworld.net

www.pdfworld.net

(ii) The maximum number of negative roots of f(x) is equal to the number of sign changes in the coefficients of f(–x). Let this number be 'q'.

(iii) The minimum number of imaginary roots of f(x) is equal to n – (p + q), where n is the degree of f(x).

1. In this topic, we consider equations of the type f(x) = 0 where f(x) is a polynomial, and 'equation' will mean an equation of this kind.
2. The position of a real root is its position on the real number line, and is determined roughly for a non-integral root by finding two consecutive integers between which the root lies.

➤ **Example 1.** Show that $x^7 + 5x^4 - 3x + k = 0$ has at least four imaginary roots.

➤ ***Solution***

Case I. $k = 0$.

The equation is $x^7 + 5x^4 - 3x = 0 \Rightarrow x(x^6 + 5x^3 - 3) = 0$.

Therefore, x = 0 is a root and the other roots are the roots of $x^6 + 5x^3 - 3 = 0$.

Let $f(x) = x^6 + 5x^3 - 3$. Therefore, $f(-x) = x^6 - 5x^3 - 3$.

Each of f(x) and f(–x) has only one variation in sign. Also f(0) =– 3. Therefore, f(x) = 0 has one positive and one negative root. Thus the given equation possesses only three real roots. Consequently the equation has four imaginary roots.

Case II. $k > 0$.

Let $f(x) = x^7 + 5x^4 - 3x + k$.

$\therefore$ $f(-x) = -x^7 + 5x^4 + 3x + k$.

The number of variations in f(x) and f(–x) is two and one respectively. So the number of real roots of f(x) = 0 will not exceed three. Hence the equation has at least four imaginary roots.

Case III. $k < 0$.

Let $f(x) = x^7 + 5x^4 - 3x + k$.

$\therefore$ $f(-x) = -x^7 + 5x^4 + 3x + k$.

The number of variations in f(x) and f(–x) is one and two respectively. So the number of real roots of f(x) = 0 will not exceed three.

Hence the equation has at least four imaginary roots.

➤ **Example 2.** Find the number of positive and negative roots of the equation

$x^5 - x^4 + x^3 + 8x^2 + 2x - 2 = 0$.

➤ ***Solution*** Let $f(x) = x^5 - x^4 + x^3 + 8x^2 + 2x - 2$.

Then $f(-x) = -x^5 - x^4 - x^3 + 8x^2 - 2x - 2$.

f(x) has three variations and f(–x) has two variations. So the maximum number of positive and negative roots of f(x) = 0 may be three and two respectively.

Now $(x + 1) f(x) = x^6 + 9x^3 + 10x^2 - 2$.

Due to multiplication by (x + 1), the number of positive roots in f(x) = 0 will not be altered, but (x + 1)f(x) has one sign variation and changes sign in $(0, \infty)$. Therefore, f(x) = 0 has only one positive root.

Again f(0) < 0 , f(–1) > 0 and $f(-\infty) = -\infty$. These alternate signs definitely suggest that f(x) = 0 has two negative roots. Thus f(x) = 0 has one positive and two negative roots.

➤ **Example 3**. Find the number and position of the real roots of the equation

$x^4 - 41x^2 + 40x + 126 = 0$.

➤ ***Solution*** Substitute in f(x) the values 1, 2, 3, 4, 5, 6 in succession, and the signs will be +, +, –, –, –, +. Hence there is atleast one root between 2 and 3, and atleast one between 5 and 6; but by Descartes' Rule of Signs there cannot be more than two positive roots.

Hence there are two positive roots which lie between 2 and 3 and between 5 and 6 respectively.

We can find in a similar manner that there are two negative roots which lie between – 1 and –2 and between –6 and –7 respectively.

➤ **Example 4.** Find the number and position of the real roots of the equation

$x^4 - 14x^2 + 16x + 9 = 0$.

➤ ***Solution*** In this case we should easily find the two negative roots which lie between 0 and –1 and

www.pdfworld.net

www.pdfworld.net

between – 4 and –5 respectively. The positive roots would, however, probably escape notice as they both lie between 2 and 3; it will in fact be found that f(2) > 0, $f\left(2\frac{1}{4}\right) < 0$, and $f(3) > 0$.

Practice Problems [V]

1. Find the integral part of the greater root of the equation $x^3 + x^2 - 2x - 2 = 0$
2. Find the nature of the roots of the equation $3x^4 + 12x^2 + 5x - 4 = 0$
3. Show that the equation $2x^7 - x^4 + 4x^3 - 5 = 0$ has at least four imaginary roots.
4. What may be inferred respecting the roots of the equation $x^{10} - 4x^6 + x^4 - 2x - 3 = 0$?
5. Find the least possible number of imaginary roots of the equation $x^9 - x^5 + x^4 + x^2 + 1 = 0$.
6. Use Descartes' rule of signs to show that :
   (i) If q is positive, $x^3 + qx + r = 0$ has only one real root.
   (ii) $x^7 - 3x^4 + 2x^3 - 1 = 0$ has at least four imaginary roots.

2.30 ALGEBRAIC INTERPRETATION OF ROLLE'S THEOREM

Let f(x) be a polynomial having roots α and β where $\alpha < \beta$ so that we have $f(\alpha) = f(\beta) = 0$. Also, a polynomial function is continuous and differentiable everywhere. Thus f(x) satisfies the conditions of Rolle's theorem. Consequently, there exists atleast one number $\gamma \in (\alpha, \beta)$ such that $f'(\gamma) = 0$. In other words $x = \gamma$ is a root of $f'(x) = 0$. Thus, Rolle's theorem can be interpreted algebraically as follows :

Between any two roots of a polynomial f(x), there is a always a root of its derivative f '(x).

➤ **Example 1.** If a, b, c $\in$ R such that $2a + 3b + 6c = 0$, show that the quadratic equation $ax^2 + bx + c = 0$ has atleast one real root between 0 and 1.

➤ ***Solution*** Consider the polynomial

$$f(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$$

We have $f(0) = 0$

and $f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a + 3b + 6c}{6} = 0$

$[\because\ 2a + 3b + 6c = 0]$

So, 0 and 1 are two roots of f(x) = 0. Therefore, $f'(x) = 0$ i.e. $ax^2 + bx + c = 0$ has atleast one real root between 0 and 1.

Important Deductions

(1) If all the roots of f(x) = 0 are real, then all the roots of f′(x) = 0 are also real, and the roots of the latter equation separate those of the former. For if f(x) is of degree n, f′(x) is of degree n – 1, and a root of f′(x) = 0 exists in each of the n – 1 intervals between the n roots of f(x) = 0.

(2) If all the roots of f(x) = 0 are real, so also are those of f′(x) = 0, f″(x) = 0, f‴(x) = 0, ..., and the roots of any one of these equations separate those of the preceding equation.
This follows from (1).

(3) Not more than one root of f(x) = 0 can
   (i) lie between two consecutive roots of f′(x) = 0, or
   (ii) be less than the least of these, or
   (iii) be greater than the greatest of these.

For let $\beta_1, \beta_2, \ldots\ldots, \beta_r$ be the real roots of f′(x), any of which may be multiple roots, and suppose that $\beta_1 < \beta_2 < \ldots. < \beta_r$. Let α_1, α_2 be real roots of f(x) = 0.
If $\alpha_1 = \alpha_2$, then α_1 is one of the set $\beta_1, \beta_2, \ldots\ldots \beta_r$
If $\alpha_1 \neq \alpha_2$, by Rolle's theorem, a root of f′(x) lies between α_1 and α_2.
Hence
   (i) if $\beta_1 < \alpha_1 < \alpha_2 < \beta_2$, then β_1 and β_2 cannot be consecutive roots,
   (ii) if $\alpha_1 < \alpha_2 < \beta_2$, then β_1 cannot be the least root of f′(x) = 0;
   (iii) if $\beta_r < \alpha_1 < \alpha_2$, then β_r cannot be the greatest.

Thus, not more than one root of f(x) = 0 can lie in any one of the open intervals

www.pdfworld.net

www.pdfworld.net

$(-\infty, \beta_1), (\beta_1, \beta_2), \ldots\ldots (\beta_{r-1}, \beta_r), (\beta_r, \infty)$.

(4) If $f'(x) = 0$ has r real roots, then $f(x) = 0$ cannot have more than $(r + 1)$ real roots.
If $f(x) = 0$ has no multiple root, none of the roots of $f'(x) = 0$ is a root of $f(x) = 0$, and the theorem follows from (3).
If $f(x) = 0$ has an m-multiple root, we regard this as the limiting case in which m roots tend to equality. Thus the theorem is true in all cases.

(5) If $f^{(r)}(x)$ is the r^{th} derivative of $f(x)$ and the equation $f^{(r)}(x) = 0$ has some imaginary roots, then $f(x) = 0$ has atleast as many imaginary roots. If follows from (4) that $f(x) = 0$ has at least as many imaginary roots as $f'(x) = 0$.

(6) If all the real roots $\beta_1, \beta_2, \ldots$ of $f'(x) = 0$ are known, we can find the number of real roots of $f(x) = 0$ by considering the signs of $f(\beta_1), f(\beta_2), \ldots\ldots$. A single root of $f(x) = 0$, or no root, lies between β_1 and β_2. according as $f(\beta_1)$ and $f(\beta_2)$ have opposite signs, or the same sign.

➤ **Example 2.** Find the character of the roots of $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 1 = 0$.

➤ ***Solution*** $f'(x) = 12x^3 - 24x^2 - 12x + 24$
$= 12(x^2 - 1)(x - 2)$.
The roots of $f'(x) = 0$ are $-1, 1, 2$.

When $x = -\infty$	-1	1	2	∞
$f(x) = \infty$	$-$	$+$	$+$	∞

Therefore $f(x) = 0$ has two real roots, one lying between $-\infty$ and -1 and the other between -1 and 1. The other two roots are imaginary.

➤ **Example 3.** If $x^4 - 14x^2 + 24x - k = 0$ has four real and unequal roots, prove that k must lie between 8 and 11.

➤ ***Solution*** Let $f(x) = x^4 - 14x^2 + 24x - k$.
Then $f'(x) = 4x^3 - 28x + 24 = 4(x - 1)(x - 2)(x + 3)$.
$\therefore$ $f'(x) = 0$ has the roots $-3, 1, 2$.
By Rolle's theorem the position and sign of the roots of $f(x) = 0$ can be found out as follows.

$x =$	$-\infty$	-3	1	2	∞
$f(x) =$	∞	$-117 - k$	$11 - k$	$8 - k$	∞

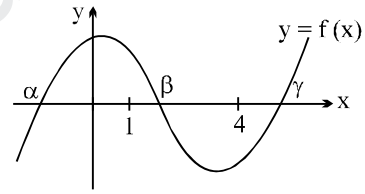
If $f(x) = 0$ has four real and unequal roots, we must have $-117 - k < 0$, $11 - k > 0$ and $8 - k < 0$, or, $k > -117, k < 11, k > 8$. Therefore $8 < k < 11$.

➤ **Example 4.** Show that the equation $(x - a)^3 + (x - b)^3 + (x - c)^3 + (x - d)^3 = 0$, where a, b, c, d are not all equal, has only one real root.

➤ ***Solution*** Let $f(x) = (x - a)^3 + (x - b)^3 + (x - c)^3 + (x - d)^3$.
Then $f'(x) = 3[(x - a)^2 + (x - b)^2 + (x - c)^2 + (x - d)^2]$.
$f'(x)$ remains positive for any real value of x. Therefore, $f'(x) = 0$ has only one real root.

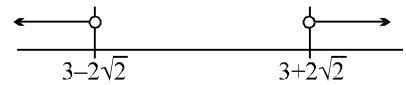
➤ **Example 5.** Given the cubic equation $x^3 - 2kx^2 - 4kx + k^2 = 0$. If one root of the equation is less than 1, other root is in the interval (1, 4) and the third root is greater than 4, then the value of k lies in the interval $\left(a + \sqrt{b}, b(a + \sqrt{b})\right)$ where $a, b \in N$. Find the value of a and b.

➤ ***Solution*** $f(x) = x^3 - 2kx^2 - 4kx + k^2 = 0$
note that $f(0) = k^2 > 0$
$f(1) > 0$
$\Rightarrow$ $1 - 2k - 4k + k^2 > 0$



$k^2 - 6k + 1 > 0$

$[k - (3 + 2\sqrt{2})][k - (3 - 2\sqrt{2})] > 0$

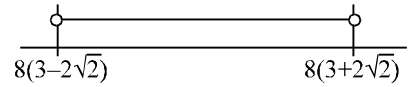


....(1)

Also $f(4) < 0$
$\Rightarrow$ $64 - 32k - 16k + k^2 < 0$ $\quad k^2 - 48k + 64 < 0$
$(k - 24)^2 < 512$

$(k - 24 + 16\sqrt{2})(k - 24 - 16\sqrt{2}) < 0$

$[k - 8(3 - 2\sqrt{2})][k - 8(3 + 2\sqrt{2})] < 0$



....(2)

$(1) \cap (2) \quad \Rightarrow \quad 3 + 2\sqrt{2} < k < 8(3 + 2\sqrt{2})$

$3 + \sqrt{8} < k < 8(3 + \sqrt{8})$

$\therefore$ $a = 3; \quad b = 8$

www.pdfworld.net

www.pdfworld.net

Practice Problems [W]

1. If a, b, c ∈ R and a + b + c = 0 then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has atleast one root in (0, 1).
2. If f(x) = (x – 1)(x – 2)(x – 3)(x – 4) , find the number of real roots of f′(x) = 0 and indicate the intervals in which they lie.
3. If $f(x) = x^2(1-x)^3$ then prove that the equation f′(x) = 0 has atleast one root in (0, 1).
4. Prove that the equation $x^4 - 4x - 2 = 0$ cannot have more than two real roots.
5. Prove that the equation $x^5 + x^3 + 4x + 1 = 0$ has atleast four imaginary roots.

Target Problems for JEE Advanced

➤ **Problem 1.** Find a quadratic equation whose roots x_1 and x_2 satisfy the condition

$$x_1^2 + x_2^2 = 5, 3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3).$$

(Assume that x_1, x_2 are real)

➤ ***Solution*** We have $3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3)$

$$\Rightarrow \quad \frac{x_1^5 + x_2^5}{x_1^3 + x_2^3} = \frac{11}{3}$$

$$\Rightarrow \quad \frac{(x_1^2 + x_2^2)(x_1^3 + x_2^3) - x_1^2x_2^2(x_1 + x_2)}{(x_1^3 + x_2^3)} = \frac{11}{3}$$

$$\Rightarrow \quad (x_1^2 + x_2^2) - \frac{x_1^2x_2^2(x_1 + x_2)}{(x_1 + x_2)(x_1^2 + x_2^2 - x_1x_2)} = \frac{11}{3}$$

$$\Rightarrow \quad 5 - \frac{x_1^2x_2^2}{5 - x_1x_2} = \frac{11}{3}$$

$$\Rightarrow \quad 3x_1^2x_2^2 + 4x_1x_2 - 20 = 0$$

$$\Rightarrow \quad 3x_1^2x_2^2 + 10x_1x_2 - 6x_1x_2 - 20 = 0$$

$$\Rightarrow \quad (x_1x_2 - 2)(3x_1x_2 + 10) = 0$$

$$\therefore \quad x_1x_2 = 2, -\frac{10}{3}$$

We have $(x_1 + x_2)^2 = 5 + 4$ (if $(x_1x_2 = 2) = 9$

$$\therefore \quad x_1 + x_2 = \pm 3$$

$$\therefore \quad (x_1 + x_2)^2 = 5 + 2(-10/3)$$

(if $x_1x_2 = -10/3$) $= 5/3$

which is not possible x_1, x_2 are real

Thus required quadratic equations are

$x^2 \pm 3x + 2 = 0$

➤ **Problem 2.** If the roots of the equation $ax^2 + 2bx + c = 0$ are real and distinct, then show that the roots of the equation $(a+c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$ are imaginary numbers and vice versa.

➤ ***Solution*** Given equations are

$ax^2 + 2bx + c = 0$ ...(1)

and $(a + c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$

or $(a^2 - ac + 2b^2)x^2 + 2(a + c)bx + c^2 - ac + 2b^2 = 0$ ...(2)

Let D_1 and D_2 be the discriminants of equations (1) and (2) respectively, then

$D_1 = 4(b^2 - ac) = 4k$, where $k = b^2 - ac$

According to question $D_1 > 0$ ...(3)

Now equation (2) becomes,

$(a^2 + b^2 + k)x^2 + 2(a + c)bx + b^2 + c^2 + k = 0$ ...(4)

Now $D_2 = 4(a + c)^2b^2 - 4(a^2 + b^2 + k)(b^2 + c^2 + k)$

$= 4[a^2b^2 + c^2b^2 + 2acb^2 - (a^2 + b^2)(b^2 + c^2) - k(a^2 + b^2 + b^2 + c^2) - k^2]$

$= 4[a^2b^2 + c^2b^2 + 2acb^2 - a^2b^2 - b^4 - a^2c^2 - b^2c^2 - k(a^2 + c^2 + 2b^2) - k^2]$

$= 4[2acb^2 - b^4 - a^2c^2 - k(a^2 + c^2 + 2b^2) - k^2]$

$= 4[-(b^4 + a^2c^2 - 2acb^2) - k(a^2 + c^2 + 2b^2) - k^2]$

$= 4[-k^2 - k(a^2 + c^2 + 2b^2) - k^2]$

$[\because b^4 + a^2c^2 - 2acb^2 = (b^2 - ac)^2 = k^2]$

$= -4k[2k + a^2 + c^2 + 2b^2) = -D_1[2(b^2 - ac) + a^2 + c^2 + 2b^2]$

$= -D_1[4b^2 + (a - c)^2]$

$\therefore \quad D_2 = -D_1$(a positive number) $< 0 \quad [\because D_1 > 0]$

$4b^2 + (a - c)^2 = 0 \Rightarrow b = 0$ and $a = c$

$\Rightarrow \quad D_1 = 4(0^2 - a.a) = -4a^2 < 0$ not possible from (3)

➤ **Problem 3.** The set of real parameter 'a' for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all real

www.pdfworld.net

www.pdfworld.net

solutions, is given by $\left[\frac{m}{n}, \infty\right)$ where m and n are relatively prime positive integers, find the value of $(m+n)$.

➤ ***Solution*** We have $a^2 - (2x^2 + 1)a + x^4 + x = 0$

$$\therefore \quad a = \frac{(2x^2+1) \pm \sqrt{(2x^2+1)^2 - 4(x^4+x)}}{2}$$

$$\therefore \quad 2a = (2x^2+1) \pm \sqrt{4x^2 - 4x + 1}$$
$$= (2x^2+1) \pm (2x-1)$$

+ ve sign $a = x^2 + x$ – ve sign
$2a = 2x^2 - 2x + 2$ $a = x^2 - x + 1$

$$\text{if } x^2 + x - a = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1+4a}}{2}$$

$$\text{if } x^2 - x + 1 - a \Rightarrow x = \frac{1 \pm \sqrt{1-4+4a}}{2} = \frac{1 \pm \sqrt{4a-3}}{2}$$

for x to be real $a \geq 3/4$ and $a \geq -1/4$
$\Rightarrow \quad a \geq 3/4$
$\Rightarrow \quad 3 + 4 = 7$

➤ **Problem 4.** Find all positive integers a, b such that each of the equations $x^2 - ax + b = 0$ and $x^2 - bx + a = 0$ has distinct positive integral roots.

➤ ***Solution*** Let integers $\alpha > \beta > 0$ be the roots of

(i) $x^2 - ax + b = 0$ and let integers $\gamma > \delta > 0$ be the roots of

(ii) $x^2 - bx + a = 0$. For definiteness, let $a \geq b$. Now
$\alpha + \beta = a$, $\alpha\beta = b$, and $\gamma + \delta = b$, $\gamma\delta = a$.

Hence $\alpha - \beta = 1 - (\alpha - 1)(\beta - 1)$.

Hence $0 \leq 1 - (a-1)(b-1) \leq 1$. So $\beta = 1$ since α, β are positive integers and $\beta \leq \alpha$. Thus $a - b = 1$.

Further, $a - b = (\gamma - 1)(\delta - 1) - 1$, so that $(\gamma - 1)(\delta - 1) = 2$. So since $\gamma > \delta > 0$ are integers, we see that $\gamma - 1 = 2$ and $\delta - 1 = 1$, so that $\gamma = 3$, $\delta = 2$.

Hence, $a = \gamma\delta = 6$ and $b = \gamma + \delta = 5$. Also, therefore, $\alpha = 5$, $\beta = 1$.

➤ **Problem 5.** Find all positive integers n for which $n^2 + 96$ is a perfect square.

➤ ***Solution*** Suppose m is a positive integer, such that

$$n^2 + 96 = m^2.$$

Then $m^2 - n^2 = 96$,

i.e., $(m-n)(m+n) = 96$.

Since $m - n < m + n$, and $m - n$, $m + n$ must be both even (because $m + n = (m - n) + 2n$ therefore $m - n$, $m + n$ must be both odd or both even; also if both of them are odd, then the product cannot be even).

Therefore the only possibilities are

$m - n = 2$, $m + n = 48$; $m - n = 4$, $m + n = 24$;
$m - n = 6$, $m + n = 16$; $m - n = 8$, $m + n = 12$.

The corresponding solutions are

$m = 25$, $n = 23$; $m = 14$, $n = 10$;
$m = 11$, $n = 5$; $m = 10$, $n = 2$.

➤ **Problem 6.** Find all integers values of a such that the quadratic expressions $(x + a)(x + 1991) + 1$ can be factored as $(x + b)(x + c)$, where b and c are integers.

➤ ***Solution*** $(x + a)(x + 1991) + 1 = (x + b)(x + c)$

$\Rightarrow \quad 1991 + a = b + c$

and $\quad 1991a + 1 = bc$

$$\therefore \quad (b-c)^2 = (b+c)^2 - 4bc = (1991+a)^2 - (1991a + 1)$$
$$= \underbrace{(1991+a)^2 - 4 \times 1991a} - 4$$
$$= (1991 - a)^2 - 4$$

or $\quad (1991 - a)^2 = (b - c)^2 = 4$

If the difference between two perfect square is 4, then one of them is 4 and the other is zero.

(Prove this)

Therefore, $1991 - a = \pm 2$, $(b - c)^2 = 0$

$\Rightarrow \quad a = 1991 + 2 = 1993$ and $b = c$

or $\quad a = 1991 - 2 = 1989$ and $b = c$

But $b + c = 2b = 1991 + a$

$= 1991 + 1993$ or $1991 + 1989$

$\Rightarrow \quad b = c = 1992$ or 1990

So, the only 2 values of a are 1993 and 1989.

➤ **Problem 7.** Let a & c be prime numbers and b an integer. Given that the quadratic equation $ax^2 + bx + c = 0$ has rational roots, show that one of the root is independent of the coefficients. Find the two roots .

➤ ***Solution*** $b^2 - 4ac = n^2 \Rightarrow (b - n)(b + n) = 4ac$

Case I $\quad b - n = 4a$ and $b + n = c$
Case II $\quad b - n = 4c$ and $b + n = a$
not possible as $2b = 4a + c$ = odd

$\Rightarrow \quad b$ is not an integer

Case III $\quad b - n = 2a$ and $b + n = 2c$

www.pdfworld.net

www.pdfworld.net

$\Rightarrow \quad b = a + c$

Now $\alpha\beta = c/a \quad \& \quad \alpha + \beta = -b/a$

$$= -\frac{a+c}{a} = -1 - c/a$$

$\Rightarrow \quad \alpha = -1 \quad$ and $\quad \beta = -c/a$

➤ **Problem 8.** If $x_1 \in N$ and x_1 satisfies the equation. If $x^2 + ax + b + 1 = 0$, where a, $b \neq -1$ are integers has a root in natural numbers then prove that $a^2 + b^2$ is a composite.

➤ ***Solution*** Let α and β be the two roots of the equation where $\alpha \in N$. Then

$\alpha + \beta = -a$ ...(1)

$\alpha . \beta = b + 1$ ...(2)

$\therefore \quad \beta = -a - \alpha$ is an integer. Also, since $b + 1 \neq 0$, $\beta \neq 0$.

From (1) and (2), we get

$a^2 + b^2 = (\alpha + \beta)^2 + (\alpha\beta - 1)^2 = \alpha^2 + \beta^2 + \alpha^2\beta^2 + 1$

$= (1 + \alpha^2)(1 + \beta^2)$

Now, as $\alpha \in N$ and β is a non–zero integer, $1 + \alpha^2 > 1$ and $1 + \beta^2 > 1$.

Hence $a^2 + b^2$ is composite number

If $b = -1$, then $a^2 + b^2$ can not be a composite number.

Consider $a = -6$, $b = -1$

$x^2 - 6x + (-1) + 1 = 0$, its are 6 and 0.

$a^2 + b^2 = 36 + 1 = 37$, a prime number.

➤ **Problem 9.** If a, b, c are odd integers, show that the roots of the equation $ax^2 + bx + c = 0$ cannot be rational.

➤ ***Solution*** Let $a = 2m + 1$, $b = 2n + 1$, $c = 2p + 1$

Now $D = (2n + 1)^2 - 4(2m + 1)(2p + 1)$ ...(1)

= (an odd integer) – (an even integer) = an odd integer

If possible let D be a perfect square of a rational number. Since D is an odd integer and is the square of a rational number therefore D must be the square of an odd integer.

Let $D = (2k + 1)^2$

Now from (1), $(2k + 1)^2 = (2n + 1)^2 - 4(2m + 1)(2p + 1)$

or $\quad 4(2m + 1)(2p + 1) = (2n + 1)^2 - (2k + 1)^2$

$= (2n + 1 + 2k + 1)(2n + 1 - 2k - 1)$

or $\quad (2m + 1)(2p + 1) = (n + k + 1)(n - k)$ ...(2)

If n and k are both odd or both even integers, then n – k will be an even integer and if one of n and k is an odd integer and other an even integer, then n + k + 1 will be an even integer. Thus R. H. S. of equation (2) is an even integer and L.H.S. is an odd integer which is not possible. Hence D cannot be a perfect square i.e. the roots of equation $ax^2 + bx + c = 0$ cannot be rational.

➤ **Problem 10.** Show that the quadratic equation $x^2 + 7x - 14(m^2 + 1) = 0$, where m is an integer, has no integral roots.

➤ ***Solution*** Given equation is $x^2 + 7x - 14(m^2 + 1) = 0$ ...(1)

Let α and β be the roots of equation (1).

then $\alpha + \beta = -7$ ...(2)

If possible, let α and β be integers, $\alpha\beta = 14(m^2 + 1)$ ...(3)

From (3), $\dfrac{\alpha . \beta}{7} = 2(m^2 + 1) =$ an integer

$\therefore \quad \beta\alpha$ is divisible by 7 and 7 is a prime number

$\therefore \quad$ at least one of α and β must be a multiple of 7.

Let $\alpha = 7k$, then from (2),

$= -7 - 7k = -7(k + 1) =$ a multiple of 7.

From (3), $7k\{-7(k + 1)\} = 14(m^2 + 1)$

or $\quad -\dfrac{2(m^2 + 1)}{7} = k(k + 1) =$ an integer ...(4)

If m = 0, L.H.S. of (4) is not an integer. Since m^2 occurs, therefore we may assume m to be a positive integer.

Now we claim that $m^2 + 1$ is not divisible by 7 for all $m \in N$.

Let $f(m) = m^2 + 1$

Let P(m) : f(m) is not divisible by 7

Now $f(1) = 1^2 + 1 = 2$, which is not divisible by 7.

Similarly, we can show that f(2), f(3), f(4), f(5), f(6), f(7) are not divisible by 7.

Hence P(1), P(2),....., P(7) are true ...(A)

Let P(m) be true $\Rightarrow$ f(m) i.e. $m^2 + 1$ is not divisible by 7. ...(5)

Now $f(m + 7) = (m + 7)^2 + 1 = m^2 + 1 + (14m + 49)$, which is not divisible by 7 as $(m^2 + 1)$ is not divisible by 7

www.pdfworld.net

www.pdfworld.net

Hence P(m + 7) is true whenever P(m) is true ...(B)
From (A) and (B) it follows that $m^2 + 1$ is not divisible by 7.
Thus from (4), we get a contradiction.

Hence α, β cannot be integers.

Since, $\alpha + \beta = -7$ = an integer, therefore exactly one of α and β cannot be non-integer.

➤ **Problem 11.** Let a, b and c be integers with $a > 1$, and let p be a prime number. Show that if $ax^2 + bx + c$ is equal to p for two distinct integral values of x, then it cannot be equal to 2p for any integral value of x.

➤ ***Solution*** Given, $ax^2 + bx + c - p = 0$ for two distinct integral values of x say α and β. Then α, β are the roots of equation $ax^2 + bx + c - p = 0$ ...(1)

$$\therefore \quad \alpha + \beta = -\frac{b}{a} \qquad ...(2)$$

$$\text{and} \quad \alpha\beta = \frac{c-p}{a} \qquad ...(3)$$

To prove $ax^2 + bx + c - 2p \neq 0$ for any integral value of x.
If possible, let $ax^2 + bx + c - 2p = 0$ for some integer k, then

$$ak^2 + bk + c - 2p = 0 \text{ or } ak^2 + bx + c - p = p$$

$$\text{or} \quad k^2 + \frac{b}{a}k + \frac{c-p}{a} = \frac{p}{a} \text{ or } k^2 - (\alpha+\beta)k + \alpha\beta = \frac{p}{a}$$

$$\text{or} \quad (k-\alpha)(k-\beta) = \frac{p}{a} \text{ or } \frac{p}{a} = (k-\alpha)(k-\beta)$$

= an integer ...(4)

Since p is a prime number and from (4), $\frac{p}{a}$ is an integer

$\therefore$ $a = p$ or $a = 1 \quad \Rightarrow \quad a = p \quad [\because a > 1]$
Now from (4), $(k-\alpha)(k-\beta) = 1 \quad [\because p = a]$
$\therefore$ either $k - \alpha = 1$ and $k - \beta = \Rightarrow \alpha = \beta$ (not acceptable as $\alpha \neq \beta$)
or $k - \alpha = -1$ and $k - \beta = -1 \Rightarrow \alpha = \beta$ (not possible)
Hence $p \neq a$.

Now from (4), $(k-\alpha)(k-\beta) = \frac{p}{a}$ (not possible)

Thus we get a contradiction.

Hence $ax^2 + bx + c \neq 2p$ for any integral value of x.

➤ **Problem 12.** If each pair of the following three equations

$x^2 + ax + b = 0, x^2 + cx + d = 0, x^2 + ex + f = 0$

has exactly one root in common, then show that

$(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$

➤ ***Solution*** $x^2 + ax + b = 0$ ...(1)

$x^2 + cx + d = 0$ ...(2)

$x^2 + ex + f = 0$ ...(3)

Let α, β be the roots of (1), β, γ be the roots of (2) and γ, δ be the roots of (3).

$\therefore$ $\alpha + \beta = -a, \alpha\beta = b$ ...(4)

$\beta + \gamma = -c, \beta\gamma = d$ ...(5)

$\gamma + \alpha = -e, \gamma\alpha = f$ ...(6)

$\therefore$ L.H.S. $= (a + c + e)^2 = (-\alpha - \beta - \gamma - \gamma - \alpha)^2$
{from (4), (5) & (6)}
$= 4(\alpha + \beta + \gamma)^2$ ...(7)

R.H.S. $= 4(ac + ce + ea - b - d - f)$
$= 4\{(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha)$
$+ (\gamma + \alpha)(\alpha + \beta) - \alpha\beta - \beta\gamma - \gamma\alpha)\}$
{from (4), (5) & (6)}
$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha)$
$= (\alpha + \beta + \gamma)^2$ ...(8)

From (7) & (8),
$(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$

➤ **Problem 13.** If the three equation $x^2 + ax + 12 = 0, x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common possible root then find a and b and the roots.

➤ ***Solution*** Let α be the common root of the three equations are their other roots be β, γ, δ respectively

$\therefore$ $\alpha + \beta = -a, \alpha\beta = 12$
$\alpha + \gamma = -b, \alpha\gamma = 15$
$\alpha + \delta = -(a + b), \alpha\delta = 36$
$\therefore$ $(\alpha + \beta) + (\alpha + \gamma) = -(a + b) = \alpha + \delta$
$\therefore$ $\alpha + \beta + \gamma = \delta$ ...(1)
Again $\alpha(\beta + \gamma + \delta) = 12 + 15 + 36 = 63$
or $\alpha(2\delta - \alpha) = 63$ by (1)
or $2\alpha\delta - \alpha^2 - 63$ or $72 - \alpha^2 = 63$
$\because$ $\alpha\delta = 36$
$\therefore$ $\alpha^2 = 9 \quad \therefore \quad \alpha = 3, -3$
$\alpha = 3 \quad \Rightarrow \quad \beta = 4, \gamma = 5, \delta = 12$

www.pdfworld.net

www.pdfworld.net

$\alpha = -3 \quad \Rightarrow \quad \beta = -4, \gamma = -5, \delta = -12$

$\therefore \quad a = -(\alpha + \beta) = 7, -7 \; ; \; b = -(\alpha + \gamma) = 8, -8$

➤ **Problem 14.** If α is a root of the equation $ax^2 + bx + c = 0$, β is a root of the equation $-ax^2 + bx + c = 0$ and γ is a root of the equation $\frac{a}{2}x^2 + bx + c = 0$ then prove that γ lies between α and β.

➤ ***Solution*** For a = 0, the three given equations become linear in x and has a single root so that $\alpha = \beta = \gamma$

Since $a\alpha^2 + b\alpha + c = 0$ ...(1)

we have $\frac{a}{2}\alpha^2 + b\alpha + c = (a\alpha^2 + b\alpha + c) - \frac{a\alpha^2}{2}$

$= -\frac{a\alpha^2}{2}$ {from (1)}

and also $-a\beta^2 + b\beta + c = 0$ ...(2)

then $\frac{a}{2}\beta^2 + b\beta + c = -a\beta^2 + b\beta + c + \frac{3}{2}a\beta^2$

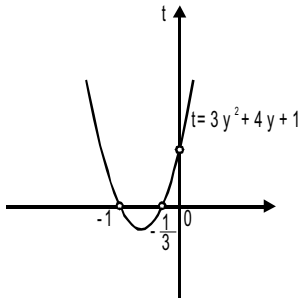
$= \frac{3}{2}a\beta^2$ {from (2)}

Consequently, the expressions $\frac{a}{2}\alpha^2 + b\alpha + c = -\frac{a\alpha^2}{2}$ and $\frac{a}{2}\beta^2 + b\beta + c = \frac{3}{2}a\beta^2$ are of different signs. Hence the graph of $f(x) = \frac{a}{2}x^2 + bx + c$ intersects the x-axis at $(\gamma, 0)$. Hence γ is a root where $\alpha < \gamma < \beta$ or $\alpha > \gamma > \beta$.

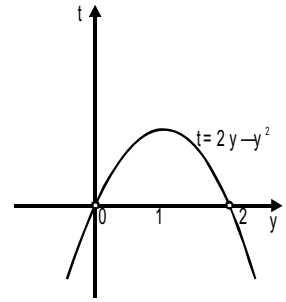
➤ **Problem 15.** Find the integral solutions of the equation $x^2 + x = y^4 + y^3 + y^2 + y$

➤ ***Solution***

(a)



(b)



Let us multiply both members of the equation by 4 and add 1 to them ; this results in the equivalent equation

$(2x + 1)^2 = 4y^4 + 4y^3 + 4y^2 + 4y + 1$

whose left-hand member is a perfect square. Further, we have

$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (4y^4 + 4y^3 + y^2) + (3y^2 + 4y + 1) = (2y^2 + y)^2 + (3y^2 + 4y + 1)$

$= (P(y))^2 + Q(y)$

Since the quadratic trinomial $Q(y) = 3y^2 + 4y + 1$ possesses (real) roots $y_1 = -1$ and $y_2 = -1/3$ it assumes positive values for all integral values of y different from y = –1 (see the graph of the function $t = 3y^2 + 4y + 1$ in figure (a).

Therefore $(2x + 1)^2 > (P(y))^2 = (2y^2 + y)^2$.

On the other hand,

$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (2y^2 + y + 1)^2 + (2y - y^2)$

$= (P_1(y))^2 + Q_1(y)$.

The graph of the function $Q_1(y) = 2y - y^2$ is shown in figure (b) ; the roots of the quadratic binomial $Q_1(y)$ are equal to 0 and 2.

Therefore $Q_1(y) < 0$ for all integral values of y different from 0, 1 and 2, whence

$(2x + 1)^2 < (P_1(y))^2 = (2y^2 + y + 1)^2$.

Thus, for all integral values of y different from –1, 0, 1 and 2 there hold the inequalities

$(2y^2 + y + 1)^2 > (2x + 1)^2 > (2y^2 + y)^2$

This means that for such y the number $(2x + 1)^2$ lies between the squares of the two consecutive whole numbers Q(y) and $Q_1(y)$, and therefore 2x + 1 cannot be equal to an integral number.

Thus, in case y is an integral number, the number x can be integral only when y is equal to –1, 0, 1 or 2, that is when the right-hand side of the original equation is equal to 0, 0, 4 or 30 respectively. It now remains to solve 3 quadratic equations of the form

www.pdfworld.net

www.pdfworld.net

$x^2 + x = c$ where c is equal to 0, 4 or 30 (1)

These equations have the following integral roots :

$x = 0$ and $x = -1$ for $c = 0$; $x = 5$ and $x = -6$ for $c = 30$;

for c = 4 equation (1) has no integral roots

Hence, finally, we arrive at the following set of integral solutions of the given equation :

$(0, -1), (-1, -1) ; (0, 0), (-1, 0) ; (5, 2), (-6, 2)$

➤ **Problem 16.** A quadratic trinomial $p(x) = ax^2 + bx + c$ is such that $|p(x)| \le 1$ for $|x| \le 1$. Prove that in this case from the condition $|x| \le 1$ it also follows that $|p_1(x)| \le 2$ where $p_1(x) = cx^2 + bx + a$.

➤ ***Solution*** Let us assume that $a \ge 0$ (if otherwise, we can replace the polynomial p(x) by the polynomial $-p(x) = -ax^2 - bx - c$ satisfying the same conditions). We shall also assume that $b \ge 0$ (if otherwise, we can replace p(x) by $p(-x) = ax^2 - bx + c$). Now we substitute the values x = 1, x = 0 and x = – 1 into the inequality $|p(x)| = ax^2 + bx + c| \le 1$, which results in

$|a + b + c| 1 \le$, $|c| \le 1$ and $|a - b + c| \le 1$,

that is $|c| \le 1$ and $|a + b| \le 2$, $|a - b| \le 2$

Further, if $c \ge 0$ then $0 \le cx^2 \le c$ for $|x| \le 1$; for $|x| \le 1$ we also have $-b \le bx \le b$. This means that for these values of x there hold the relations

$p_1(x) = cx^2 + bx + a \le c + b + a \le 1$

and $p_1(x) = cx^2 + bx + a \ge 0 + (-b) + a = a - b \ge -2$

whence it follows that $|p_1(x)| \le 2$. Similarly, if $c \le 0$ and $c \le cx^2 \le 0$ (and, as before, $-b \le bx \le b$) then

$p_1(x) = cx^2 + bx + a \le 0 + b + a = a + b \le 2$

and $p_1(x) = cx^2 + bx + a \ge c + (-b) + a = a - b + c \ge -1$

whence it follows that $|p_1(x)| \le 2$ in this case as well.

➤ **Problem 17.** Find out the values of 'a' for which any solution of the inequality, $\dfrac{\log_3 (x^2 - 3x + 7)}{\log_3 (3x + 2)} < 1$ is also a solution of the inequality, $x^2 + (5 - 2a) x \le 10a$.

➤ ***Solution*** Note that $x^2 - 3x + 7 > 0 \ \forall \ x \in R$

Also $x > -2/3$ and $x \ne -1/3$

Also $3x + 2 > 1 \Rightarrow x > -1/3$

and $0 < 3x + 2 < 1 \Rightarrow -2/3 < x < -1/3$

Hence for $x > -1/3$

$\log_{3x+2} (x^2 - 3x + 7) < 1$

$\Rightarrow x^2 - 3x + 7 < 3x + 2$

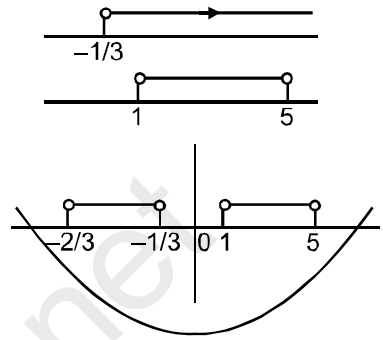
$x^2 - 6x + 5 < 0 \quad (x - 5)(x - 1) < 0$

$\Rightarrow x \in (1, 5)$

for $-2/3 < x < -1/3$

$$x \in \left(-\frac{2}{3}, -\frac{1}{3}\right)$$

Hence the solution of the first inequality is



$$x \in \left(-\frac{2}{3}, -\frac{1}{3}\right) \cup (1, 5)$$

Now if any solution of the inequality is also the solution of,

$f(x) = x^2 + (5 - 2a) x - 10a \le 0$ then

$f(5) \le 0$ and

$$f\left(-\frac{2}{3}\right) \le 0 \qquad \Rightarrow \quad a \ge 5/2$$

➤ **Problem 18.** Find the set of values of 'y' for which the inequality ,

$2 \log_{0.5} y^2 - 3 + 2x \log_{0.5} y^2 - x^2 > 0$

is valid for atleast one real value of 'x' .

➤ ***Solution*** $-4 \log_2 |y| - 3 - 4 \log_2 |y| \, x - x^2 > 0$

or $x^2 + 4 \log_2 |y| \, x + 3 + 4 \log_2 |y| < 0$

or $x^2 + 4tx + 3 + 4t < 0$

for this inequality to be true for atleast one x, $3 > 0$

$\Rightarrow 4t^2 - 4t - 3 > 0 \qquad (2t - 3)(2t + 1) > 0$

$$\Rightarrow \quad t > \frac{3}{2} \quad \text{or} \quad t < -\frac{1}{2} \quad \text{i.e.} \quad \log_2 |y| > \frac{1}{2}$$

$$\Rightarrow \quad |y| > 2\sqrt{2}$$

$$\Rightarrow \quad y > 2\sqrt{2} \quad \text{or} \quad y < 2\sqrt{2}$$

$$\text{Similarly} \quad \log_2 |y| < -\frac{1}{2} \qquad |y| < \frac{1}{\sqrt{2}}$$

www.pdfworld.net

www.pdfworld.net

$\Rightarrow \quad y < \frac{1}{\sqrt{2}} \quad \text{or} \quad y > -\frac{1}{\sqrt{2}}$

$y \in \left(-\infty, -2\sqrt{2}\right) \cup \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(0, \frac{1}{\sqrt{2}}\right) \cup \left(2\sqrt{2}, \infty\right)$

➤ **Problem 19.** Find the values of 'p' for which the inequality, $\left(2 - \log_2\left(\frac{p}{p+1}\right)\right)x^2 + 2x\left(1 + \log_2\frac{p}{p+1}\right) - 2\left(1 + \log_2\frac{p}{p+1}\right) > 0$ is valid for all real x.

➤ ***Solution*** $(2-t)x^2 + 2(1+t)x - 2(1+t) > 0$
when $t = 2$, $6x - 6 > 0$ which is not true $\forall \quad x \in R$.
Let $t \neq 2$; $t < 2$ ...(1)
and $4(1+t)^2 + 8(1+t)(2-t) < 0$
(for given inequality to be valid)
or $(t-5)(t+1) > 0$
$\Rightarrow \quad t > 5$ or $t < -1$ ...(2)
From (1) and (2) ; $t < -1$

$\Rightarrow \quad \log_2\frac{p}{p+1} < -1$

$\frac{p}{p+1} < \frac{1}{2} \quad \text{or} \quad \frac{p-1}{p+1} < 0$

$\Rightarrow \quad -1 < p < 1$

but $\frac{p}{p+1} > 0 \quad \Rightarrow \quad p > 0 \quad \text{or} \quad p < -1$.

common solution is $p \in (0, 1)$

➤ **Problem 20.** If $ax^2 - bx + c = 0$ have two distinct roots lying in the interval (0, 1), a, b, c ∈ N, then prove that $\log_5 abc \geq 2$.

➤ ***Solution*** Let roots are α and β
$\therefore \quad \alpha + \beta = b/a$ and $\alpha\beta = c/a$
and $0 < \alpha < 1, 0 < \beta < 1$
$\therefore \quad 0 < 1 - \alpha < 1 \;\&\; < 1 - \beta < 1$

then $\frac{\alpha + (1-\alpha)}{2} \geq \sqrt{\alpha(1-\alpha)}$ $\{\because$ A.M. ≥ G.M.$\}$

$\Rightarrow \quad \frac{1}{4} \geq \alpha(1-\alpha) > 0$

similarly $\frac{1}{4} \geq \beta(1-\beta) > 0$

$\therefore \quad \frac{1}{16} \geq \alpha\beta(1-\beta) > 0$

But α and β are distinct.

$\therefore \quad 0 < \alpha\beta(-\alpha)(1-\beta) < \frac{1}{16}$

$\Rightarrow \quad 0 < \alpha\beta(1 - (\alpha+\beta) + \alpha\beta) < \frac{1}{16}$

$\Rightarrow \quad 0 < \frac{c}{a}\left(1 - \frac{b}{a} + \frac{c}{a}\right) < \frac{1}{16}$

$\Rightarrow \quad 0 < c(a - b + c) < \frac{a^2}{16}$

$\because \quad c(a-b+c) =$ Natural Number $(\because a, b, c \in N)$
$\therefore \quad$ Minimum value of $c(a-b+c) = 1$

$\therefore \quad \frac{a^2}{16} > 1, \quad \therefore a \geq 5 \qquad (\because a \in N)$

and condition for real roots, $b^2 - 4ac \geq 0$
$\Rightarrow \quad b^2 \geq 4ac$
$\Rightarrow \quad b^2 \geq 2ac \qquad (\because a \geq 5)$
$\Rightarrow \quad b \geq 5 \qquad (\because b \in N)$
and minimum value of c = 1
Hence $abc \geq 25$
$\log_5(abc) \geq \log_5 5^2$
$\log_5(abc) \geq 2$

➤ **Problem 21.** Show that for all real values of x, then expression $\frac{2a(x-1)\sin^2\alpha}{x^2 - \sin^2\alpha}$ cannot lie between $2a\sin^2\frac{\alpha}{2}$ and $2a\cos^2\frac{\alpha}{2}$.

➤ ***Solution*** Let $y = \frac{2a(x-1)\sin^2\alpha}{x^2 - \sin^2\alpha}$

www.pdfworld.net

www.pdfworld.net

or $yx^2 - 2a\sin^2\alpha x + + (2a - y)\sin^2\alpha = 0$ ...(1)

Since x is real $\therefore$ $D \geq 0$

$\therefore$ $4a^2\sin^4\alpha - 4y(2a - y)\sin^2\alpha \geq 0$

or $a^2\sin^2\alpha - y(2a - y) \geq 0$ [Dividing by $4\sin^2\alpha$]

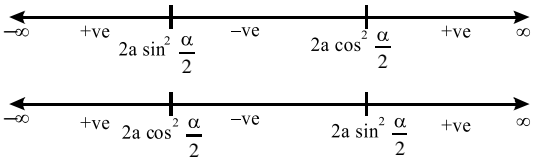
or $y^2 - 2ay + a^2\sin^2\alpha \geq 0$ ...(2)

Now $y^2 - 2ay + a^2\sin^2\alpha = 0$

$$\Rightarrow \quad y = \frac{2a \pm \sqrt{4a^2 - 4a^2\sin^2\alpha}}{2}$$

$$\Rightarrow \quad y = a \pm a\cos\alpha = a(1 \pm \cos\alpha) = 2a\sin^2\frac{\alpha}{2}, 2a\cos^2\frac{\alpha}{2}$$

sign scheme for $y^2 - 2ay + a^2\sin^2\alpha$ is as follows



$\therefore$ $y^2 - 2ay + a^2\sin^2\alpha \geq 0 \Leftrightarrow$ y does not lie between

$2a\sin^2\frac{\alpha}{2}$ and $2a\cos^2\frac{\alpha}{2}$

or, $2a\cos^2\frac{\alpha}{2}$ and $2a\sin^2\frac{\alpha}{2}$ as the case may be.

➤ **Problem 22.** Prove that if x is real, the minimum value of $\frac{(a+x)(b+x)}{(c+x)}$ $(x > -c)$, for $x > c$, $b > c$ is $(\sqrt{(a-c)} + \sqrt{(b-c)})^2$.

➤ ***Solution*** Let $y = \frac{(a+x)(b+x)}{(c+x)}$

$\Rightarrow$ $x^2 + (a+b)x + ab = cy + xy$

$\Rightarrow$ $x^2 + (a+b+y)x + ab - cy = 0$

for real x, $B^2 - 4AC \geq 0$

$\Rightarrow$ $(a+b+y)^2 - 4ab + 4cy \geq 0$

$\Rightarrow$ $(a+b)^2 y^2 - 2(a+b)y - 4ab + 4cy \geq 0$

$\Rightarrow$ $(a-b)^2 + y^2 - 2(a+b-2c)y \geq 0$

$\Rightarrow$ $y^2 - 2(a+b-2c)y + (a-b)^2 \geq 0$

$\Rightarrow$ $[y - (\sqrt{(a-c)} - \sqrt{(b-c)})^2][y - (\sqrt{(a-c)} + \sqrt{(b-c)})^2] \geq 0$

$\therefore$ $y \leq (\sqrt{(a-c)} - \sqrt{(b-c)})^2$

and $y \geq (\sqrt{(a-c)} - \sqrt{(b-c)})^2$

Hence minimum value of y is $(\sqrt{(a-c)} - \sqrt{(b-c)})^2$.

➤ **Problem 23.** If x be real, prove that the expression, $2(a-x)\left(x + \sqrt{x^2 + b^2}\right)$ cannot exceed $a^2 + b^2$.

➤ ***Solution*** Let $\sqrt{x^2 + b^2} + x = u$ ...(1)

then $\frac{1}{\sqrt{x^2 + b^2} + x} = \frac{1}{u}$

$$= \frac{\sqrt{x^2 + b^2} - x}{b^2}$$

or $\sqrt{x^2 + b^2} - x = b^2/u$ ...(2)

From (1) & (2), $2x = u + b^2/u$

Let $y = 2(a-x)\left(x + \sqrt{x^2 + b^2}\right)$

$$= \left[2a - \left(u + \frac{b^2}{u}\right)\right] u$$

$y = 2au - u^2 - b^2$ or $y^2 - 2au + y - b^2 = 0$

Since $u \in R \quad \Rightarrow \quad D \geq 0$

$4a^2 - 4(y - b^2) \geq 0 \quad a^2 - (y - b^2) \geq 0$

$a^2 + b^2 \geq y \quad \Rightarrow \quad y \leq a^2 + b^2$

➤ **Problem 24.** Find the greatest value of $\frac{\left\{\left(x + \frac{1}{x}\right)^4 - \left(x^4 + \frac{1}{x^4}\right) - 1\right\}}{\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)}$ for $\forall\, x \in R - \{0\}$.

➤ ***Solution*** Let $y = \frac{\left(x + \frac{1}{x}\right)^4 - \left(x^4 + \frac{1}{x^4}\right) - 1}{\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)}$.

www.pdfworld.net

www.pdfworld.net

Put $x + \frac{1}{x} = t$

$$\Rightarrow \quad y = \frac{t^4 - [(t^2-2)^2 - 2] - 1}{t^2 + t^2 - 2}$$

$$= \frac{t^4 - [t^4 - 4t^2 + 2] - 1}{2(t^2-1)}$$

$$\Rightarrow \quad y = \frac{4t^2 - 3}{2(t^2-1)} \Rightarrow y = \frac{4t^2 - 4 + 1}{2(t^2-1)}$$

$$= \left(2 + \frac{1}{2(t^2-1)}\right).$$

As $t = x + \frac{1}{x} \Rightarrow t^2 \geq 4 \Rightarrow t^2 - 1 \geq 3 \Rightarrow \frac{1}{t^2-1} \leq \frac{1}{3}$.

$\Rightarrow$ Maximum value of y is $2 + \frac{1}{3 \times 2} = \frac{13}{6}$

➤ **Problem 25.** Solve for integers x, y, z :
$x + y = 1 - z$, $x^3 + y^3 = 1 - z^2$.

➤ ***Solution*** Eliminating z from the given set of equations, we get
$x^3 + y^3 + \{1 - (x + y)\}^2 = 1$.
This factors to
$(x + y)(x^2 - xy + y^2 + x + y - 2) = 0$.

Case 1. Suppose $x + y = 0$. Then $z = 1$ and $(x, y, z) = (m, -m, 1)$, where m is an integer given one family of solutions.

Case 2. Suppose $x + y \neq 0$. Then we must have
$x^2 - xy + y^2 + x + y - 2 = 0$

This can be written in the form
$(2x - y + 1)^2 + 3(y + 1)^2 = 12$.

Here there are two possibilities :

$2x - y + 1 = 0$, $\quad y + 1 = \pm 2$;
or $\quad 2x - y + 1 = \pm 3$, $\quad y + 1 = \pm 1$.

Analysing all these case we get the following solutions :
$(0, 1, 0), (-2, -3, 6), (1, 0, 0),$
$(0, -2, 3), (-2, 0, 3), (-3, -2, 6)$.

➤ **Problem 26.** If $(ax^2 + bx + c)y + (a'x^2 + b'x + c') = 0$, find the condition so that x may be expressed as a rational function of y.

➤ ***Solution*** The given equation can be written as
$(a + a'y)x^2 + (b + b'y)x + (c + c'y) = 0$ ...(1)
Solving equation (1), we have

$$x = \frac{-(b + b'y) \pm \sqrt{(b + b'y)^2 - 4(a + a'y)(c + c'y)}}{2(a + a'y)}$$

For x to be a rational function of y, the expression under the radical sign must be a perfect square. The expression in y will be a perfect square if its discriminant is zero,

$\Rightarrow \quad (bb' - 2ac' - 2a'c)^2 - (b^2 - 4ac)(b'^2 - 4a'c') = 0$
$\Rightarrow \quad (ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$
which is the required condition.

➤ **Problem 27.** A quadratic equation with roots α & β where $\alpha\beta = 4$, satisfies the condition,

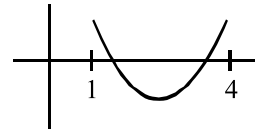
$$\frac{\alpha}{\alpha - 1} + \frac{\beta}{\beta - 1} = \frac{a^2 - 7}{a^2 - 4}.$$

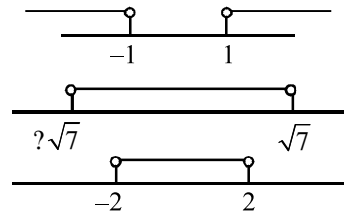
Find the set of values of 'a' for which $\alpha, \beta \in (1, 4)$.

➤ ***Solution*** $\frac{\alpha(\beta - 1) + \beta(\alpha - 1)}{(\alpha\beta + 1) - (\alpha + \beta)} = \frac{a^2 - 7}{a^2 - 4}$

$$\frac{8 - (\alpha + \beta)}{5 - (\alpha + \beta)} = \frac{a^2 - 7}{a^2 - 4}$$

$\Rightarrow \quad \alpha + \beta = a^2 + 1$
Hence the quadratic equation is
$x^2 - (\alpha + \beta)x + \alpha\beta = 0$
$x^2 - (a^2 + 1)x + 4 = 0$





Let $f(x) = x^2 - (a^2 + 1)x + 4 = 0$
$f(1) = 4 - a^2 > 0$ ...(1)
or $\quad f(4) = 16 - 4a^2 > 0$

www.pdfworld.net

www.pdfworld.net

$\Rightarrow \quad 4 - a^2 > 0$ ...(2)

Also $1 < \dfrac{a^2+1}{2} < 4$

$\Rightarrow \quad 1 < a^2 < 7$ ...(3)

and $D \geq 0$ ...(4)

From (1), (2), (3) & (4) we get the answer

$$a \in \left(-2, -\sqrt{3}\right] \cup \left[\sqrt{3}, 2\right)$$

➤ **Problem 28.** Find the set of values of 'a' for which the equation, $(1+a)\left(\dfrac{x^2}{x^2+1}\right)^2 - 3a\dfrac{x^2}{x^2+1} + 4a = 0$ have real roots.

➤ ***Solution*** Put $\dfrac{x^2}{1+x^2} = y \quad \Rightarrow \quad y \in [0, 1)$

$(a+1)y^2 - 3ay + 4a = 0$

Find 'a' for which the equation has atleast one solution, for $y = 0$; $a = 0$ ...(1)

Case I : Exactly one root between 0 and 1

for $a = -1$; $y = \dfrac{4}{3} \notin (0, 1)$

for $a \neq -1$ equation becomes

$$y^2 - \frac{3a}{a+1}y + \frac{4a}{a+1} = 0$$

(i) $D > 0$ & (ii) $f(0) \cdot f(1) < 0$

$\Rightarrow$ $(-1/2, 0)$ combining with (1)

$\Rightarrow$ $(-1/2, 0]$

Case II : When both roots lie in (0, 1)

(i) $f(a) > 0$; (ii) $f(1) > 0$;

(iii) $D \geq 0$; (iv) $0 < -b/2a < 1$

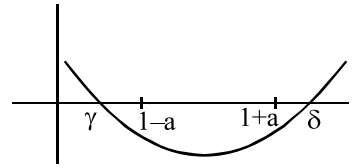
No solution in this case

Ans. : $-1/2 < x \leq 0$

➤ **Problem 29.** If α, β are the roots of the equation, $x^2 - 2x - a^2 + 1 = 0$ and γ, δ are the roots of the equation, $x^2 - 2(a+1)x + a(a-1) = 0$ such that $\alpha, \beta \in (\gamma, \delta)$ then find the values of 'a'.

➤ ***Solution*** $\alpha, \beta = \dfrac{2 \pm \sqrt{4 - 4(a^2-1)}}{2}$

$= 1 + a$ or $1 - a$



Let $f(x) = x^2 - 2(a+1)x + a(a-1)$

For $\alpha, \beta \in (\gamma, \delta)$ we should have

$f(1+a) < 0$ and

$f(1-a) < 0 \quad \Rightarrow \quad a \in \left(-\dfrac{1}{4}, 1\right)$

➤ **Problem 30.** Prove that $\dfrac{1}{3} \leq \dfrac{\sec^4\theta - 3\tan^2\theta}{\sec^4\theta - \tan^2\theta} < 1$.

➤ ***Solution*** Let $x = \dfrac{\sec^2\theta - 3\tan^2\theta}{\sec^4\theta - \tan^2\theta}$

$\Rightarrow \quad [(1+\tan^2\theta)^2 - \tan^2\theta]x = (1+\tan^2\theta)^2 - 3\tan^2\theta$

$\Rightarrow \quad (x-1)\tan^4\theta + (x+1)\tan^2\theta + (x-1) = 0$ ...(1)

Since $\tan^2\theta$ is real, therefore we have

$(x+1)^2 - 4(x-1) \geq 0$

$\Rightarrow \quad (x + 1 + 2x - 2)(x + 1 - 2x + 2) \geq 0$

$\Rightarrow \quad (3x-1)(-x+3) \geq 0$

$\Rightarrow \quad (3x-1)(x-3) \leq 0$

$\Rightarrow \quad \dfrac{1}{3} \leq x \leq 3$ ...(2)

Also, since $\tan^2\theta$ is positive, both the roots of equation (1) must be positive. Thus, we have

sum of the roots > 0

$\Rightarrow \quad -\dfrac{x+1}{x-1} > 0$ ...(3)

and product of the roots > 0 which is true

$\forall\, x \in R - \{1\}$ ...(4)

Intersection of inequalities (2), (3) and (4) gives $x \in [1/3, 1)$ which is the desired result.

➤ **Problem 31.** If p(x) is a polynomial with integer coefficients and a, b, c are three distinct integers, then show that it is impossible to have p(a) = b, p(b) = c and p(c) = a.

➤ ***Solution*** Suppose a, b, c are distinct integers such that

p(a) = b, p(b) = c and p(c) = a. Then

www.pdfworld.net

www.pdfworld.net

$p(a) - p(b) = b - c$,
$p(b) - p(c) = c - a$, $p(x) - p(a) = a - b$.
But for any two integers $m \neq n$, $m - n$ divides $p(m) - p(n)$. Thus we get,

$a - b \mid b - c$, $\quad b - c \mid c - a$, $\quad c - a \mid a - b$.

These force $a = b = c$, a contradiction, Hence there are no integers a, b, and c such that $p(a) = b$, $p(b) = c$ and $p(c) = a$.

➤ **Problem 32.** Prove that if a polynomial

$P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$

with integral coefficients assumes the value 7 for four integral values of x then it cannot take the value 14 for any integral value of x.

➤ ***Solution*** Let the polynomial P(x) take on the value 7 at the points $x = a$, $x = b$, $x = c$ and $x = d$. Then a, b, c and d are four integral roots of the equation $P(x) - 7 = 0$. This means that the polynomial $P(x) - 7$ is divisible by $x - a$, $x - b$, $x - c$ and $x - d$, that is

$P(x) - 7 = (x - a)(x - b)(x - c)(x - d)\,p(x)$

where p(x) may be equal to 1.
Now let us suppose that the polynomial P(x) assumes the value 14 for an integral value $x = A$. On substituting $x = A$ into the last equality we obtain

$7 = (A - a)(A - b)(A - c)(A - d)\,p(A)$

which is impossible because the integral numbers $A - a$, $A - b$, $A - c$ and $A - d$ are all distinct and the number 7 cannot be factored into five integers among which at least four are different.

➤ **Problem 33.** Solve in R the equation

$2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + ... + 2x + 3 = 0$.

➤ ***Solution*** $2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + ... + 2x + 3$

$= (2x + 3)(x^{98} + x^{96} + x^{94} + ... + 1)$,

$$= (2x + 3)\frac{(x^{100} - 1)}{x^2 - 1}.$$

The equation $x^{100} - 1 = 0$ has only two real roots, namely ± 1 which are not acceptable. Therefore the given equation has only one real root, namely $-3/2$.

➤ **Problem 34.** Prove that the equation $x^6 + 2x^3 + 5 + ax^3 + a = 0$ has atmost two real roots for all values of $a \in R - \{-5\}$.

➤ ***Solution*** The given expression is

$(x^3 + 1)^2 + a(x^3 + 1) + 4 = 0$

If discriminant of the above equation is less than zero i.e. $D < 0$
Then we have six complex roots and no real roots.
If $D \geq 0$, $\quad x^3 + 1 = t$, then the equation reduces to $f(t) = t^2 + at + 4 = 0$
we will get two real roots and other roots will be complex except when $t = 1$ is one of the roots
$\Rightarrow \quad f(1) = 0 \Rightarrow a = -5$.

➤ **Problem 35.** If α, β are the roots of equation $x^2 + px + q = 0$ and also of equation $x^{2n} + p^nx^n + q^n = 0$ and if $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of equation $x^n + 1 + (x + 1)^n = 0$, then prove that n must be an even integer.

➤ ***Solution*** $\alpha^n + \beta^n = -p^n$ ...(1)

Again since $\frac{\alpha}{\beta}$ is a root of equation

$x^n + 1 + (x + 1)^n = 0$

$$\therefore \quad \left(\frac{\alpha}{\beta}\right)^n + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n = 0$$

or $\quad \alpha^n + \beta^n + (\alpha + \beta)^n = 0$
or $\quad \alpha^n + \beta^n = -(\alpha + \beta)^n = -(-p)^n$ ...(2)
From (1) and (2), we have

$-p^n = -(-p)^n \Rightarrow p^n = (-p)^n$
$\Rightarrow \quad p^n = (-1)^n p^n \Rightarrow (-1)^n = 1$
$\Rightarrow \quad$ n is an even integer.

➤ **Problem 36.** Solve the equation

$6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$.

➤ ***Solution*** The expression on the left has the factor $x^2 - 1$ corresponding to the roots ± 1. Thus we have

$6(x^6 - 1) - 25x(x^4 - 1) + 31x^2(x^2 - 1) = 0$.

Hence the required roots are ± 1 and the roots of

$6x^4 - 25x^2 + 37x^2 - 25x + 6 = 0$.

Divide by x^2, then

$$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0.$$

www.pdfworld.net

www.pdfworld.net

Put $x + \frac{1}{x} = y$; $\therefore \quad x^2 + \frac{1}{x^2} = y^2 - 2$.

Hence $6y^2 - 25y + 25 = 0$;

$$y = \frac{5}{2} \text{ or } y = \frac{5}{3}.$$

From $x + \frac{1}{x} = \frac{5}{2}$, we have $x = 2$ or $\frac{1}{2}$.

From $x + \frac{1}{x} = \frac{5}{3}$, we have $x = \frac{1}{6}(5 \pm \sqrt{-11})$.

Thus the required roots are $\pm 1, 2, \frac{1}{2}, \frac{1}{6}(5 \pm \sqrt{-11})$.

➤ **Problem 37.** Solve the equation $x^3 - 15x = 126$.

➤ ***Solution*** Put $y + z$ for x, then

$$y^3 + z^3 + (3yz - 15)x = 126;$$

Put $3yz - 15 = 0$

then $y^3 + z^3 = 126$;

also $y^3z^3 = 125$,

hence y^3, z^3 are the roots of the equation

$$t^2 - 126t + 125 = 0;$$

$\therefore \quad y^3 = 125, z^3 = 1$;

$\therefore \quad y = 5, z = 1$

Thus $y + z = 5 + 1 = 6$;

$$\omega y^2 + \omega^2 z = -3 + 2\sqrt{-3};$$

$$\omega^2 y + \omega z = -3 - 2\sqrt{-3};$$

and the roots are $6, -3 + 2\sqrt{-3}, -3 - 2\sqrt{-3}$.

STUDY TIP The cube roots of unity are 1, ω, ω^2 where

$$\omega = \frac{-1 \pm \sqrt{-3}}{2}$$

We have $z^3 - 1 = (z - 1)(z^2 + z + 1)$.

The roots of $z^2 + z + 1 = 0$ are called ω, ω^2.

➤ **Problem 38.** Solve the equation

$$x^4 - 2x^2 + 8x - 3 = 0$$

➤ ***Solution*** Assume

$$x^4 - 2x^2 + 8x - 3 = (x^2 + kx + l)(x^2 - kx + m);$$

then by equating coefficients, we have

$$l + m - k^2 = -2, k(m - l) = 8, lm = -3;$$

we obtain $(k^3 - 2k + 8)(k^3 - 2k - 8) = -12k^2$,

or $\quad k^6 - 4k^4 + 16k^2 - 64 = 0$.

This equation is clearly satisfied when $k^2 - 4 = 0$, or $k = \pm 2$. It will be sufficient to consider one of the values of k; putting $k = 2$, we have

$$m + l = 2, m - l = 4;$$ that is, $l = -1, m = 3$.

Thus $x^4 - 2x^2 + 8x - 3 = (x^2 + 2x - 1)(x^2 - 2x + 3)$;

hence $x^2 + 2x - 1 = 0$, and $x^2 - 2x + 3 = 0$;

and therefore the roots are $-1 \pm \sqrt{2}$, $1 \pm \sqrt{-2}$.

➤ **Problem 39.** If the product of two roots of the equation

$$4x^4 - 24x^3 + 31x^2 + 6x - 8 = 0 \text{ is } 1,$$ find the roots.

➤ ***Solution*** Suppose the roots are α, β, γ, δ and $\alpha\beta = 1$.

Now, $\sigma_1 = (\alpha + \beta) + (\gamma + \delta) = -\frac{-24}{4} = 6$ ....(1)

$$\sigma_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{31}{4}$$

$$\Rightarrow \quad (\alpha + \beta)(\gamma + \delta) + \gamma\delta = \frac{31}{4} - 1 = \frac{27}{4} \quad ...(2)$$

$$\sigma_3 = \gamma\delta(\alpha + \beta) + \alpha\beta(\gamma + \delta) = \frac{-3}{2}$$

$$\Rightarrow \quad \gamma\delta(\alpha + \beta) + (\gamma + \delta) = \frac{-3}{2} \quad ...(3)$$

$$\sigma_4 = \alpha\beta\gamma\delta = -2$$

$$\Rightarrow \quad \gamma\delta = -2 \quad ...(4)$$

From Eq. (2) and Eq. (4), we get

$$(\alpha + \beta)(\gamma + \delta) = \frac{35}{4} \quad ...(5)$$

From Eq. (3) and Eq. (4), we get

$$-2(\alpha + \beta) + (\gamma + \delta) = \frac{-3}{2} \quad ...(6)$$

www.pdfworld.net

www.pdfworld.net

From Eq. (1) and Eq. (6), we get

$$3(\alpha+\beta)=\frac{15}{2} \text{ or } \alpha+\beta=\frac{5}{2} \qquad ...(7)$$

and $\alpha\beta=1$ ...(Given)

$\Rightarrow \quad \beta=1/\alpha$

Putting the value of β in Eq. (7), we get

$$a+\frac{1}{\alpha}=\frac{5}{2}$$

$\Rightarrow \quad 2\alpha^2-5\alpha+2=0 \Rightarrow (2\alpha-1)(\alpha-2)=0$

$\Rightarrow \quad \alpha=1/2 \text{ or } \alpha=2$ and hence $\beta=2$ or $\beta=1/2$

Taking $\alpha=1/2$ and $\beta=2$ and substituting in Eq. (5),

we get $\gamma+\delta=\frac{7}{2}$

and we know that $\gamma\delta=-2$

Again solving for γ and δ, we get

$$\gamma=\frac{-1}{2} \text{ and } \delta=4 \text{ or } \delta=\frac{-1}{2} \text{ and } \gamma=4$$

Thus the four roots are $\frac{1}{2}, \frac{-1}{2}$, 2 and 4.

➤ **Problem 40.** If a, b, c are the roots of the equation

$x^3+px^2+qx+r=0$,

form the equation whose roots are

$$a-\frac{1}{bc}, b-\frac{1}{ca}, c-\frac{1}{ab}$$

➤ ***Solution*** When $x=a$ in the given equation $y=a-\frac{1}{bc}$ in the transformed equation; but

$$a-\frac{1}{bc}=a-\frac{a}{abc}=a+\frac{a}{r} \ ;$$

and therefore the transformed equation will be obtained by the substitution

$$y=x+\frac{x}{r}, \text{ or } x=\frac{ry}{1+r} \ ;$$

thus the required equation is

$r^2y^3+pr(1+r)y^2+q(1+r)^2y+(1+r)^3=0$

➤ **Problem 41.** Form the equation whose roots are the squares of the differences of the roots of the cubic

$x^3+qx+r=0$

➤ ***Solution*** Let a, b, c be the roots of the cubic ; then the roots of the required equation are

$(b-c)^2, (c-a)^2, (a-b)^2$.

Now

$$(b-c)^2=b^2+c^2-2bc=a^2+b^2+c^2-a^2-\frac{2abc}{a}$$

$$=(a+b+c)^2-2(bc+ca+ab)-a^2-\frac{2abc}{a}$$

$$=-2q-a^2+\frac{2r}{a} \ ;$$

also when $x=a$ in the given equation, $y=(b-c)^2$ in the transformed equation ;

$$\therefore \quad y=-2q-x^2+\frac{2r}{x}$$

Thus we have to eliminate x between the equations

$x^3+qx+r=0$,

and $x^3+(2q+y)x-2r=0$

By subtraction $(q+y)x=3r$; or $x=\frac{3r}{q+y}$

Substituting and reducing, we obtain

$y^3+6qy^2+9q^2y+27yr^2+4q^3=0$

Conclusion : If a, b, c are real, $(b-c)^2, (c-a)^2, (a-b)^2$ are all positive, therefore $27r^2+4q^3$ is negative.

Hence in order that the equation $x^3+qx+r=0$ may have all its roots real $27r^2+4q^3$ must be negative.

If $27r^2+4q^3=0$, the transformed equation has one root zero, therefore the original equation has two equal roots.

If $27r^2+4q^3$ is positive, the transformed equation has a negative root, therefore the original equation

www.pdfworld.net

www.pdfworld.net

must have two imaginary roots, since it is only such a pair of roots which can produce a negative root in the transformed equation.

➤ **Problem 42.** Given that the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

has four positive roots, prove that

(a) $pr - 16s \geq 0$ (b) $q^2 - 36s \geq 0$

with equality in each case holding if and only if the four roots are equal.

➤ ***Solution*** Let the roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

be $\alpha, \beta, \gamma, \delta$, so that $\alpha > 0, \beta > 0, \gamma > 0, \delta > 0$

Now $\Sigma\alpha = -p$,

$\Sigma\alpha\beta = q$,

$\Sigma\alpha\beta\gamma = -r$,

$\alpha\beta\gamma\delta = s$,

(i) $pr = \Sigma\alpha \,.\, \Sigma\alpha\beta\gamma$.

By A.M.- G.M. inequality

$$\frac{1}{4}\Sigma\alpha \geq (\alpha\,\beta\,\gamma\,\delta)^{1/4}, \quad ...(1)$$

$$\frac{1}{4}\Sigma\alpha\,\beta\,\gamma \geq (\alpha\,\beta\,\gamma\,\delta)^{3/4}, \quad ...(2)$$

By multiplying corresponding sides of the above inequalities, we have

$$\frac{1}{16}\Sigma\alpha\,\Sigma\alpha\beta\gamma \geq \alpha\beta\gamma\delta,$$

i.e., $pr - 16s \geq 0.$ ...(3)

Equality holds in (3) $\Leftrightarrow$ equalities hold in both (1) and (2) $\Leftrightarrow \alpha, \beta, \gamma, \delta$ are all equal, and $\alpha\,\beta\,\gamma$, $\alpha\,\beta\,\delta$, $\alpha\,\gamma\,\delta$ are all equal $\Leftrightarrow \alpha = \beta = \gamma = \delta$.

(ii) $q^2 = (\Sigma\alpha\beta)^2 \geq [\,6\,(\alpha\beta \,.\, \alpha\gamma \,.\, \alpha\delta \,.\, \beta\gamma \,.\, \beta\delta \,.\, \gamma\delta\,)^{1/6}\,]^2,$

$= 36\alpha\beta\gamma\delta,$

$= 36s.$

Therefore $q^2 \geq 36s$.

Equality holds if and only if $\alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta$ are all equal,

i.e., if and only if $\alpha = \beta = \gamma = \delta$.

STUDY TIP Let $a_1, a_2, a_3, \ldots\ldots a_n$ be n positive real numbers, then we define their

Arithmetic Mean A.M. $= \dfrac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n}$ and

Geometric Mean G.M. $= (a_1 a_2 a_3 \ldots\ldots\ldots a_n)^{1/n}$

It can be shown that A.M. $\geq$ G.M. and equality holds places iff $a_1 = a_2 = a_3 = \ldots\ldots\ldots\ldots = a_n$.

➤ **Problem 43.** Find the number and position of the real roots of the equation

$$x^6 - 5x^5 - 7x^2 + 8x + 20 = 0.$$

➤ ***Solution*** By Descartes' Rule of Signs we see that there cannot be more than two positive roots and there cannot be more than two negative roots.

Now $f(1) > 0, f(2) < 0$; thus one real root lies between 1 and 2. Since $f(\infty)$ is ∞, there must be another positive root which is found to lie between 5 and 6.

Change x into $-x$, then the negative roots of the given equation are positive roots of

$$x^6 + 5x^5 - 7x^2 - 8x + 20 = 0.$$

Now f(x) must clearly be positive for all positive values between 0 and 1; and if $x > 1$.

$$f(x) > 6x^4 - 15x^2 + 20,$$

which is always positive since

$$4 \times 6 \times 20 - 15^2 > 0$$

Hence there can be no real negative roots.

Things to Remember

1. **The quadratic formula**

   The solution of the quadratic equation , $ax^2 + bx + c = 0$ is given by

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. **Sum and Product of roots**

   Vieta's Theorem : If a quadratic equation $ax^2 + bx + c = 0$ has roots α & β then their sum is equal to $-\dfrac{b}{a}$ while their product is $\dfrac{c}{a}$

www.pdfworld.net

www.pdfworld.net

If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, then we have
$ax^2 + bx + c = a(x - \alpha)(x - \beta)$

3. An **identity** in x is satisfied by all permissible values of x, where as an **equation** in x is satisfied by some particular value of x.

   If a quadratic equation is satisfied by three or more distinct values of 'x', then it is an identity.

   Two equations in x are **identical** if and only if the coefficient of similar powers of x in the two equations are proportional. Identical equations have the same roots.

4. **Newton's Theorem :**

   If α, β are roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ then for $n > 2, n \in N$, we have a $S_n + bS_{n-1} + cS_{n-2} = 0$

5. **Nature of Roots**

Consider the quadratic equation, $a x^2 + bx + c = 0$ having α, β as its roots; $D \equiv b^2 - 4ac$

- $D = 0$ — Roots are equal $\alpha = \beta = -b/2a$
- $D \neq 0$ — Roots are unequal
  - $a, b, c \in R$ & $D > 0$ — Roots are real; $\alpha = p + \sqrt{q}, \beta = p - \sqrt{q}$
    - $a, b, c \in Q$ & D is perfect square $\Rightarrow$ Roots are rational
      - $a = 1, b, c \in I$ & D is perfect square $\Rightarrow$ Roots are integral.
    - $a, b, c \in Q$ & D is not a perfect square $\Rightarrow$ Roots are irrational i.e. $\alpha = p + \sqrt{q}, \beta = p - \sqrt{q}$
  - $a, b, c \in R$ & $D < 0$ — Roots are imaginary $\alpha = p + Iq, \beta = p - iq$

6. **Common Roots of Quadratic Equations**

   Two Common Roots : $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have both roots common

   if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

   Atleast one Common Root : Let α be a common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$

   Then $\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$

www.pdfworld.net

www.pdfworld.net

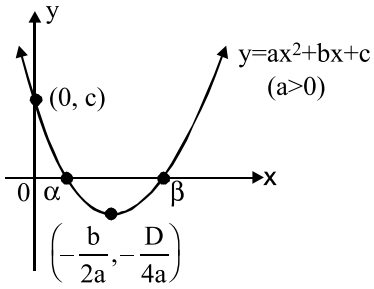
The condition for atleast one common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

If $f(x) = 0$ & $g(x) = 0$ are two polynomial equations having some common roots(s) then those common root(s) is/are also the root(s) of $h(x) = a f(x) + b g(x) = 0$.

7. Graph of a quadratic function

$$y = f(x) = a x^2 + b x + c$$

$$\left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$



(i) The graph between x, y is always a parabola.

(ii) If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

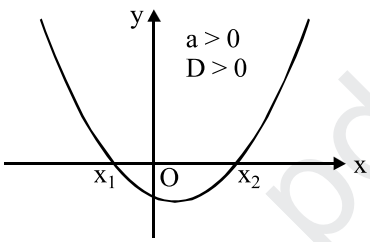
(iii) the co-ordinate of vertex are $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

(iv) the parabola intersect the y–axis at point $(0, c)$

(v) the x co-ordinate of point of intersection of parabola with x–axis are the real roots of the quadratic equation $f(x) = 0$. Hence the parabola may or may not intersect the x–axis at real points.

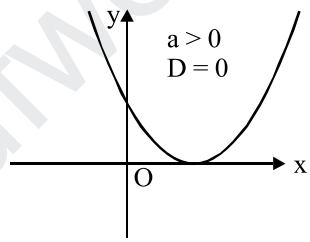
8. Position of graph based on a and D

Fig. 1



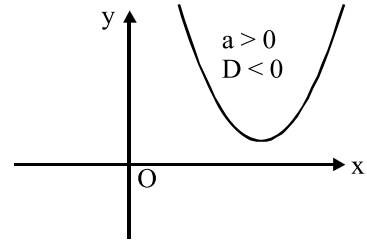
Roots are real & distinct

Fig. 2



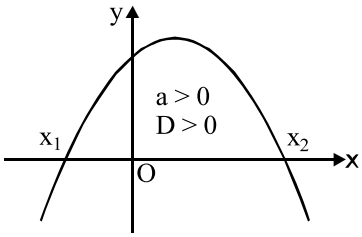
Roots are coincident

Fig. 3



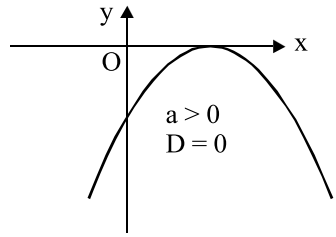
Roots are complex conjugate

Fig. 4



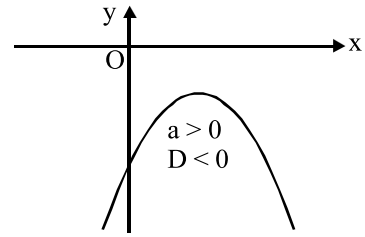
Roots are real & distinct

Fig. 5



Roots are coincident

Fig. 6



Roots are complex conjugate

pdfworld.net

www.pdfworld.net

www.pdfworld.net

9. Quadratic inequality

All the relevant properties of the quadratic polynomial $f(x) = ax^2 + bx + c$ are given in the table:

	a	D	Root of f(x) = 0	Sign of f(x)
1.	$a > 0$	$D > 0$	Real roots $\alpha < \beta$	$f(x) > 0$ for $x < \alpha$ < 0 for $\alpha < x < \beta$ > 0 for $x > \beta$
2.	$a > 0$	$D = 0$	Real roots $\alpha = \beta$	$f(x) > 0$ for $x < \alpha$ $= 0$ for $x = \alpha$ > 0 for $x > \alpha$
3.	$a > 0$	$D < 0$	Imaginary roots α, β	$f(x) > 0$ for all x
4.	$a < 0$	$D > 0$	Real roots $\alpha < \beta$	$f(x) < 0$ for $x < \alpha$ > 0 for $\alpha < x < \beta$ < 0 for $x > \beta$
5.	$a < 0$	$D = 0$	Real roots $\alpha = \beta$	$f(x) < 0$ for $x < \alpha$ $= 0$ for $x = \alpha$ < 0 for $x > \alpha$
6.	$a < 0$	$D < 0$	Imaginary roots α, β	$f(x) < 0$ for all x

10. Range of Quadratic Expression

$f(x) = a x^2 + b x + c$

(i) Range when $x \in R$

If $a > 0 \Rightarrow f(x) \in \left[-\frac{D}{4a}, \infty\right)$

$a < 0 \Rightarrow f(x) \in \left[-\infty, \frac{D}{4a}\right)$

Maximum & Minimum Value of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

(ii) Range in restricted domain

Given $x \in [x_1, x_2]$

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$ then,

$f(x) \in \lfloor \min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\} \rfloor$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$ then,

$f(x) \in \left[\min\left\{f(x_1), f(x_2) - \frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$

11. Resolution of a Second Degree Expression in X and Y

The condition that a quadratic function of x and y may be resolved into two linear factors is that

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

OR $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

and it can be written as a product of two linear factors with real coefficients if apart from the

www.pdfworld.net

www.pdfworld.net

above condition we have either (i) $h^2 - ab > 0$, or (ii) $h^2 - ab = 0, hg - af = 0, g^2 - ac \geq 0$

12. Location of roots

Necessary and Sufficient Conditions

1. roots are of opposite sign $\frac{c}{a} < 0$ $\alpha < 0, \beta > 0$
2. roots are of same sign

   $D \geq 0$ and $\frac{c}{a} > 0$ $\alpha < 0, \beta < 0$ or $\alpha > 0, \beta > 0$
3. both roots are positive

   $D \geq 0, -\frac{b}{a} > 0$ and $\frac{c}{a} > 0$ $\alpha > 0, \beta > 0$
4. both roots are negative

   $D \geq 0, -\frac{b}{a} < 0$ and $\frac{c}{a} > 0$ $\alpha < 0, \beta < 0$
5. both roots greater than k

   $a.f(k) > 0, -\frac{b}{2a} > k$, and $D \geq 0$ $\alpha > k, \beta > k$
6. both roots are less than k

   $a.f(k) > 0, -\frac{b}{2a} < k$, and $D \geq 0$ $\alpha < k, \beta < k$
7. k lies between the roots

   $a.f(k) < 0$ $\alpha < k, \beta > k$
8. one root less than d and other greater than e

   $a.f(d) < 0,\ a.f(e) < 0$ $\alpha < d, \beta > e$
9. both roots are lie in (d , e)

   $a.f(d) > 0, a.f(e) > 0, d < -\frac{b}{2a} < e$, and

   $\alpha < k, \beta < k$ $D \geq 0$
10. exactly one root lies in (d , e)

    $f(d).f(e) < 0^*$ $\alpha < k, \beta < k$

    * Refer theory since more conditions are required.

13. Theory of Equations

If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are the roots of the equation;

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ where $a_0, a_1, ... a_n$ are all real & $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0},\ \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0},$$

$$\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, ..., \alpha_1\alpha_2\alpha_3.....\alpha_n$$

$$= (-1)^n \frac{a_n}{a_0}$$

(i) Remainder Theorem : Let P(x) be any polynomial of degree greater than or equal to one and 'a' be any real number. If P(x) is divided (x – a), then the remainder is equal to P(a).

Factor Theorem : Let P(x) be polynomial of degree greater than of equal to 1 and 'a' be a real number such that P(a) = 0, then (x – a) is a factor of P(x). Conversely, if (x – a) is a factor of P(x), then P(a) = 0.

(ii) Every equation of nth degree ($n \geq 1$) has exactly n roots & if the equation has more than n roots, it is an identity .

(iii) If the coefficients of the equation f(x) = 0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs**.

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ and β is not a perfect square. Also if $\sqrt{p}$ and $\sqrt{q}$ are two dissimilar quadratic surds and $\sqrt{p} + \sqrt{q}$ is a root then $\pm\sqrt{p} \pm \sqrt{q}$ are also roots of the equation.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x) = 0 must have atleast one real root between 'a' and 'b'.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

(vi) Every equation f(x) = 0 of degree odd has atleast one real root of a sign opposite to that of its last term.

(vii) If α is an r-multiple root of f(x) = 0, it is then an (r – 1) multiple root of f′(x) = 0 , an (r – 2) multiple root of f″(x) = 0, and so on.

(viii) Descartes' rule of signs :

(a) The maximum number of positive roots of f(x) is equal to the number of sign changes in the coefficients of f(x). Let this number be 'p'.

(b) The maximum number of negative roots of f(x) is equal to the number of sign changes in the coefficients of f(–x). Let this number be 'q'.

(c) The minimum number of imaginary roots of f(x) is equal to n – (p + q).

(ix) Let f(x) be a polynomial having roots α and β where $\alpha < \beta$, then there exists atleast one number $\gamma \in (\alpha, \beta)$ such that f '(γ) = 0.

[Objective Exercises]

Single Correct Answer Type

1. The value of x satisfying the equation

$$\frac{6x + 2a + 3b + c}{6x + 2a - 3b - c} = \frac{2x + 6a + b + 3c}{2x + 6a - b - 3c}$$ is

(A) x = ab/c (B) 2ab/c

(C) ab/3c (D) ab/2c

2. If a < c < b then the roots of the equation $(a - b)^2 x^2 + 2(a + b - 2c)x + 1 = 0$ are

(A) imaginary (B) real

(C) one real and one imaginary

(D) equal and imaginary

3. If the ratio of the roots of $x^2 + bx + c = 0$, is same as that of $x^2 + qx + r = 0$, then

(A) $r^2b = qc^2$ (B) $r^2c = qb^2$

(C) $c^2r = q^2b$ (D) $b^2r = q^2c$

4. Let α, β be the roots of the equation $ax^2 + 2bx + c = 0$ and γ , δ be the roots of the equation $px^2 + 2qx + r = 0$. If α , β , γ , δ are in G.P. then

(A) $q^2ac = b^2pr$ (B) $qac = bpr$

(C) $c^2pq = r^2ab$ (D) $p^2ab = a^2qr$

5. If a > b > 0, then the value of

$$\sqrt{ab+(a-b)\sqrt{ab+(a-b)\sqrt{ab+(a-b)\sqrt{ab+...}}}}$$

is

(A) independent of only b

(B) independent of only a

(C) independent of both a & b

(D) dependent on both a & b .

6. If both the roots of the equations

$k(6x^2 + 3) + rx + 2x^2 - 1 = 0$

and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$

are common, then 2r – p is equal to

(A) 1 (B) –1 (C) 2 (D) 0

7. If the quadratic equations,

$ax^2 + 2cx + b = 0$ & $ax^2 + 2bx + c = 0$ $(b \neq c)$

have a common, root, then a + 4b + 4c is equal to

(A) – 2 (B) 2 (C) 0 (D) 1

8. If α, β be the roots of $x^2 - a(x - 1) - b = 0$, then the value of $\frac{1}{\alpha^2 - a\alpha} + \frac{1}{\beta^2 - a\beta} + \frac{2}{a + b}$ is

(A) $\frac{4}{(a + b)}$ (B) $\frac{1}{(a + b)}$

(C) 0 (D) $\frac{2}{(a + b)}$

9. If a, c $\in$ Q and the roots of the equation $cx^2 + (2 + \sqrt{2})x + 2a\left(1 + \frac{1}{\sqrt{2}}\right) = 0$ are real and distinct, then the roots of the equation $x^2 - 2cax + 1 = 0$ will be

www.pdfworld.net

www.pdfworld.net

(A) integer (B) rational
(C) irrational (D) imaginary

10. If α, β are the roots of the equation $ax^2+bx+c=0$, then the quadratic equation whose roots are $\frac{\alpha}{1+\alpha}$ and $\frac{\beta}{1+\beta}$ is

(A) $(a-b+c)x^2+(b-2c)x+c=0$
(B) $(a-b+c)x^2-(b-2c)x+c=0$
(C) $(a-b+c)x^2+(b-2c)-c=0$
(D) None of these

11. The greatest value of $\frac{4}{4x^2+4x+9}$ is

(A) 4/9 (B) 4 (C) 9/4 (D) 1/2

12. If roots of the equation $3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$ are of opposite signs, then a lies in the interval

(A) $(-\infty, 1)$ (B) $(-\infty, 0)$
(C) (1, 2) (D) (3/2, 2)

13. The number of integral values of a for which $x^2-(a-1)x+3=0$ has both roots positive and $x^2+3x+6-a=0$ has both roots negative is

(A) 0 (B) 1
(C) 2 (D) infinite

14. If the two roots of the equation, $x^3-px^2-qx-r=0$ are equal in magnitude but opposite in sign then

(A) pr = q (B) qr = p
(C) pq = r (D) None

15. If α, β & γ are the roots of the equation, $x^3-x-1=0$ then the value of $\frac{1+\alpha}{1-\alpha}+\frac{1+\beta}{1-\beta}+\frac{1+\gamma}{1-\gamma}$ is

(A) zero (B) – 1 (C) – 7 (D) 1

16. If roots of the equation $x^2 - bx + c = 0$ are two successive integers, then $b^2 - 4c$ equal

(A) 1 (B) 2 (C) 3 (D) 4

17. If the equation $\frac{a}{x-a}+\frac{b}{x-b}=1$ has roots equal in magnitude but opposite in sign, then the value of a + b is

(A) – 1 (B) 0
(C) 1 (D) None of these

18. The number of solutions of the equation $\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}\,x + 4$ is

(A) zero (B) one
(C) two (D) more than two

19. The set of values of 'p' for which the expression $x^2 - 2px + 3p + 4$ is negative for atleast one real x is

(A) ϕ (B) (–1, 4)
(C) $(-\infty, -1) \cup (4, \infty)$ (D) {–1, 4}

20. If one root of the equation
$x^2 + bx + a = 0$ & $x^2 + ax + b = 0$
is common and $a \neq b$ then

(A) a + b = 0 (B) a + b = – 1
(C) a – b = 1 (D) a + b = 1

21. If α_1, α_2, & β_1, β_2, are roots of the equations $ax^2+bx+c=0$ and $px^2+qx+r=0$ respectively and a non-zero solution of the system of equations $\alpha_1 y + \alpha_2 z = 0$, $\beta_1 y + \beta_2 z = 0$ exist then

(A) $p^2br = a^2qc$ (B) $b^2pr = q^2ac$
(C) $r^2pb = c^2ar$ (D) None of these

22. If α, β are the roots of the equation
$ax^2 + 3x + 2 = 0$ $(a > 0)$, then $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$
is greater than

(A) 0 (B) 1
(C) 2 (D) none

23. If $(\lambda^2 + \lambda - 2)x^2 + (\lambda + 2)x < 1$, $x \in R$, then λ belongs to the interval

(A) (–2, 1) (B) (–2, 2/5)
(C) (2/5, 1) (D) None of these

www.pdfworld.net

www.pdfworld.net

24. The number of positive integral values of k for which $(16x^2 + 12x + 39) + k(9x^2 - 2x + 11)$ is a perfect square is
(A) 2 (B) 0
(C) 1 (D) None

25. If x is real, then the least value the expression $\frac{x^2 - 6x + 5}{x^2 + 2x + 1}$ is
(A) -1 (B) $-1/2$
(C) $-1/3$ (D) none

26. If x is real, then $\frac{x^2 + 2x + c}{x^2 + 4x + 3c}$ can take all real values if
(A) $0 < c < 2$ (B) $0 < c < 1$
(C) $0 \le c \le 1$ (D) none

27. If the roots of $x^2 - p(x + 1) + c = 0$ are α & β then the value of $\alpha^2 + (2p)\alpha\beta + \beta^2$ is
(A) pc (B) $p^2 + pc$
(C) $p^2 - pc$ (D) $-pc$

28. The value(s) of 'b' for which the equation, $2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$ has coincident roots, is/are
(A) $b = -12$
(B) $b = 4$
(C) $b = 4$ or $b = -12$
(D) $b = -4$ or $b = 12$

29. If the quadratic equation, $ax^2 + bx + a^2 + b^2 + c^2 - ab - bc - ca = 0$ where a, b, c are distinct reals, has imaginary roots then
(A) $2(a-b) + (a-b)^2 + (b-c)^2 + (c-a)^2 > 0$
(B) $2(a-b) + (a-b)^2 + (b-c)^2 + (c-a)^2 < 0$
(C) $2(a-b) + (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$
(D) none of these

30. Three roots of the equation, $x^4 - px^3 + qx^2 - rx + s = 0$ are tan A, tan B & tan C where A, B, C are the angles of a triangle. The fourth root of the equation is
(A) $\frac{p - r}{1 - q + s}$ (B) $\frac{p - r}{1 + q - s}$
(C) $\frac{p + r}{1 - q + s}$ (D) $\frac{p + r}{1 + q - s}$

31. Number of values of 'p' for which the equation, $(p^2 - 3p + 2)x^2 - (p^2 - 5p + 4)x + p - p^2 = 0$ possess more than two roots, is
(A) 0 (B) 1
(C) 2 (D) none

32. The roots of the equation $(x - a)(x - b) = a^2 - 2b^2$ are real and distinct for all $a > 0$, provided
(A) $-a \le b < \frac{5}{7}a$ (B) $-a < b < \frac{5}{7}a$
(C) $-a < b \le \frac{5}{7}a$ (D) $-2a < b < \frac{7}{5}a$

33. If the roots of the quadratic equation $x^2 + 6x + b = 0$ are real and distinct and they differ by atmost 4 then the range of values of b is
(A) $[-3, 5]$ (B) $[5, 9)$
(C) $[6, 10]$ (D) none

34. If the equation $\sin^4 x - (k + 2)\sin^2 x - (k + 3) = 0$ has a solution then k must lie in the interval
(A) $(-4, -2)$ (B) $[-3, 2)$
(C) $(-4, -3)$ (D) $[-3, -2]$

35. For non-zero real distinct a, b, c, the equation, $(a^2 + b^2)x^2 - 2b(a + c)x + b^2 + c^2 = 0$ has non-zero real roots. One of these roots is also the root of the equation
(A) $a^2x^2 - a(b - c)x + bc = 0$
(B) $a^2x^2 + a(c - b)x - bc = 0$
(C) $(b^2 + c^2)x^2 - 2a(b + c)x + a^2 = 0$
(D) $(b^2 - c^2)a^2 + 2a(b - c)x - a^2 = 0$

36. The values of a, b, c such that $2x^2 - 5x - 1 \equiv a(x+1)(x-2) + b(x-2)(x-1) + c(x-1)(x+1)$ are respectively :
(A) –2, 1, –1 (B) 2, 1, –1
(C) 1, 2, –1 (D) None of these

www.pdfworld.net

www.pdfworld.net

37. If $a + b + c = 0$, then the roots of the equations $3ax^2 + 4bx + 5c = 0$ are
(A) Positive (B) Negative
(C) Real and distinct (D) Imaginary

38. The equation $(x - a)^3 + (x - b)^3 + (x - c)^3$ has
(A) three distinct real roots
(B) one real and two imaginary roots.
(C) three real roots, two of which are equal
(D) None of these

39. The equation $x^5 - 5ax + 4b = 0$ has
(A) three real roots if $a^5 > b^4$
(B) only one real root if $a^5 > b^4$
(C) five real roots
(D) None of these

40. The value of 'a' for which the equation $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$, have a common root is
(A) $a = 2$ (B) $a = -2$
(C) $a = 0$ (D) None

41. The necessary and sufficient condition for the equation $(1 - a^2)x^2 + 2ax - 1 = 0$ to have roots lying in the interval (0, 1) is
(A) $a > 0$ (B) $a < 0$
(C) $a > 2$ (D) None

42. If α and β are the roots of $ax^2 + bx + c = 0$, then the equation $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$ has roots.

(A) $\frac{\alpha}{1-\alpha}, \frac{\beta}{1-\beta}$ (B) $\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}$

(C) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (D) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$

43. If one root of the equation $ix^2 - 2(1 + i)x + 2 - i = 0$ is $(3 - i)$, then the other root is
(A) $3 + i$ (B) $3 + 2i$
(C) $-1 + i$ (D*) $-1 - i$

44. The equation

$$\frac{a^2}{x-\alpha} + \frac{b^2}{x-\beta} + \frac{c^2}{x-\gamma} = m - n^2x$$ (a, b, c, m, $n \in R$) has necessarily
(A) all the roots real
(B) all the roots imaginary
(C) two real and two imaginary roots
(D) two rational and two irrational roots.

45. The equation $x^5 - 2x^2 + 7 = 0$ has
(A) atleast two imaginary roots
(B) two negative real roots
(C) three positive real roots
(D) None of these

46. If the equation $ax^3 - 9x^2 + 12x - 5 = 0$ has two equal real roots then the 'a' equals
(A) 2 (B) $\frac{25}{54}$
(C) 1 (D) None

47. If the equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$; $a \neq c$ have a negative common root then the value of $(a - b + c)$ is
(A) 0 (B) 2
(C) 1 (D) None

48. The equation $ax^2 + bx + a = 0$, $x^3 - 2x^2 + 2x - 1 = 0$ have two roots common. Then $a + b$ must be equal to
(A) 1 (B) –1
(C) 0 (D) None

49. The equation $(x - 3)^9 + (x - 3^2)^9 + ... + (x - 3^9)^9 = 0$ has
(A) all the roots real
(B) one real & 8 imaginary roots
(C) real roots namely $x = 3, 3^2, ..., 3^9$
(D) None of these

50. If $f(x) = \sum_{r=0}^{100} a_r x^r$ and f(0) and f(1) are even numbers, then for any integer x
(A) f(x) is odd or even according as x is odd or even
(B) f(x) is even or odd according as x is odd or even
(C) f(x) is even for all integral values of x
(D) f(x) is odd for all integral values of x.

www.pdfworld.net

www.pdfworld.net

Multiple Correct Answer Type for JEE Advanced

51. The quadratic expression $(ax+b)(x-2)+c(x^2+3)$ is equal to 14 for all values of x. Then
(A) $a=-2, b=-4, c=2$
(B) $a=2, b=4, c=-2$
(C) $a=2, b=-4, c=2$
(D) None of these

52. The quadratic equation $ax^2 + bx + c = 0$ has real roots if :
(A) $a<-1, 0<c<1, b>0$
(B) $a<-1, -1<c<0, 0<b<1$
(C) $a<-1, c<0, b>1$
(D) none

53. The equation $x^2 - 4x + a \sin\alpha = 0$ has real roots
(A) for all values of x
(B) for all values of 'a' provided

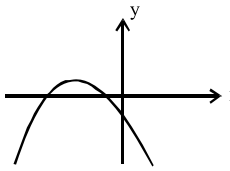
$$-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$$

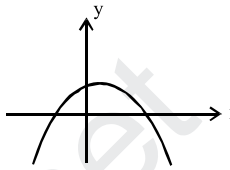
(C) for all values of $a \geq 4$ provided $\pi \leq \alpha \leq 2\pi$
(D) for all a provided $|a| \leq 4$

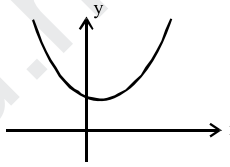
54. The integral values of m for which the roots of the equation $mx^2 + (2m-1)x + (m-2) = 0$ are rational are given by
(A) 15 (B) 12 (C) 6 (D) 4

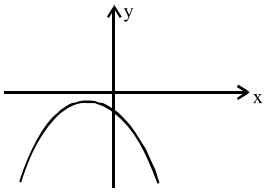
55. Let Δ^2 be the discriminant and α, β be the roots of the equation $ax^2 + bx + c = 0$. Then $2a\alpha + \Delta$ and $2a\beta - \Delta$ can be the roots of the equation:
(A) $x^2 + 2bx + b^2 = 0$
(B) $x^2 - 2bx + b^2 = 0$
(C) $x^2 + 2bx - 3b^2 + 16ac = 0$
(D) $x^2 - 2bx - 3b^2 + 16ac = 0$

56. For which of the following graphs of the quadratic expression $y = ax^2 + bx + c$, then product a b c is negative.

(A)


(B)


(C)


(D)


57. The values of k for which the equation, $x^2 + 2(k-1)x + k + 5 = 0$ has atleast one positive root, are
(A) $[4, \infty)$ (B) $(-\infty, -1] \cup [4, \infty)$
(C) $[-4, -1]$ (D) $(-\infty, -1]$

58. If $p \neq q$ then the equations, $x^2 - px + q = 0$ & $x^2 - qx + p = 0$ are such that :
(A) both can simultaneously have a zero root
(B) if the roots of the first are both positive so are the roots of the second
(C) if the roots of the first are both negative, those of the second have opposite signs

www.pdfworld.net

www.pdfworld.net

(D) both can have roots belonging to the set {1, 2, 3, 5}

59. Given the equation $(a-x)(b-x)-h^2=0$, $a<b$, the incorrect statement is
(A) both the roots are less than b
(B) both the roots lie between a and b
(C) the roots are separated by a and by b.
(D) None of these

60. Which of the following is/are false?
(A) $2x^3 + 7x^2 - 5x + 2x^{-1}$ is a polynomial in standard form of degree 3 and contains four terms.
(B) If two polynomials of degree 3 are added, their sum must be a polynomial of degree 3.
(C) There is no such thing as a polynomial in x having four terms, written in descending powers, and lacking a third-degree term.
(D) Suppose a square garden has area represented by $9x^2$ square metre. If one side is made 7 metre longer and the other side is made 2 metre shorter, then the trinomial that represents the area of the larger garden in $9x^2 + 15x - 14$ square metre.

61. Let a, b, c be the roots of $x^3 - 9x^2 + 11 - 1 = 0$ and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$ and $t = \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$, then
(A) $t^2 = 11 + 2s$
(B) $s^4 = 125 + 36t + 8s$
(C) $s^4 - 18s^2 - 8s = -37$
(D) $s^4 - 18s^2 + 8s = 37$

62. Let $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$ have roots $a_k + i\,b_k$ for $k = 1, 2, 3, 4$ where a_k, b_k are all integers. Then
(A) $(a_1^2 + b_1^2)(a_3^2 + b_3^2) = 65$
(B) $a_1 + a_3 = 3$
(C) $|b_1| + |b_2| + |b_3| + |b_4| = 10$
(D) $a_1^2 + b_3^2 = 12$

63. Let $P(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3$ where $a_1, a_2, a_3 \in I$ and $a_1 + a_2 + a_3$ is an even number, then
(A) P(–1) is an even number
(B) P(–1) is an odd number
(C) P(x) = 0 has no integral solutions
(D) P(x) = 0 has atleast one integral solution

64. The equation $x^3 - \frac{3}{4}x = -\frac{\sqrt{3}}{8}$ is satisfied by

(A) $x = \cos\left(\frac{5\pi}{18}\right)$ (B) $x = \cos\left(\frac{7\pi}{18}\right)$

(C) $x = \cos\left(\frac{17\pi}{18}\right)$ (D) $x = \cos\left(\frac{23\pi}{18}\right)$

65. The correct statements about the cubic equation $2x^3 + x^2 + 3x - 2 = 0$ are
(A) it has exactly one positive root
(B) it has either one or three negative roots
(C) it has a root between 0 and 1
(D) it has no imaginary roots

66. The integral value(s) of a for which the equation $x^2 - 2(a-1)x + (2a+1) = 0$ has both roots positive is / are
(A) 3 (B) 4 (C) 1 (D) 5

67. If $b^2 \geq 4ac$ for the equation $ax^4 + bx^2 + c = 0$ then all roots of the equation will be real when
(A) $b>0, a<0, c>0$ (B) $b<0, a>0, c>0$
(C) $b>0, a>0, c>0$ (D) $b>0, a<0, c<0$

68. If $x^2 + px + 1 = 0$ and $(a-b)x^2 + (b-c)x + (c-a) = 0$ have both roots common, then which of the following is true?
(A) $p = -2$ (B) b, a, c are in A.P.
(C) $2a - 3b + c = 1$ (D) $2c - 3b + c = 0$

69. The equation $3x^4 + 8x^3 - 6x^2 - 24x + r = 0$ has no real root if r is equal to
(A) 8 (B) 20
(C) 23 (D) None

70. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is

www.pdfworld.net

www.pdfworld.net

identical with this cubic, then the possible cases are

(A) $a = b = 0$

(B) a, b are roots of $x^2 + x + 2 = 0$

(C) $a = b = 3$

(D) None of these

Comprehension - 1

Let P(x) be a polynomial such that $P(1) = 1$ and $\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$ for all real x for which both sides are defined.

71. The degree of P(x) is

(A) 2 (B) 3 (C) 4 (D) 5

72. The value of P(–1) is

(A) $\frac{2}{7}$ (B) $-\frac{4}{21}$

(C) $-\frac{5}{21}$ (D) $-\frac{11}{21}$

73. The number of real roots of $P(x) = 0$ is

(A) 5 (B) 3

(C) 2 (D) None

Comprehension - 2

Consider a polynomial equation $x^4 - 63x^3 + ax^2 - 7! = 0$ whose roots are four distinct natural numbers of which three are odd.

74. The largest of the roots is

(A) 16 (B) 48 (C) 24 (D) 32

75. The value of a is

(A) 791 (B) 971

(C) 719 (D) None

76. The number of roots which are prime is

(A) 1 (B) 2

(C) 3 (D) None

Comprehension - 3

Consider the equation $x^4 + ax^3 - bx^2 + ax + 1 = 0$. After dividing the equation by x^2, we can rearrange it as $\left(x + \frac{1}{x}\right)^2 + a\left(x + \frac{1}{x}\right) - b - 2 = 0$. By subtitution, we can solve the equation as a quadratic equation. Let S be the set of points (a, b) in the a – b plane.

77. The equation has four real solutions if and only if

(A) $2a + b - 2 > 0$ and $2a - b + 2 > 0$

(B) $2a + b - 2 < 0$ and $2a - b + 2 < 0$

(C) $2a + b - 2 > 0$ and $2a - b + 2 < 0$

(D) None of these

78. If $b = 2$, then the equation has

(A) four real solutions

(B) two real solutions

(C) no real solution

(D) None of these

79. If $0 \le a, b \le 1$, then for the equation to have atleast one real root, the area of the region S is

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{1}{4}$ (D) None

Comprehension - 4

Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$ and the roots of g are square of the roots of f.

80. The equation $f(x) = 0$ has

(A) atleast one positive root

(B) atleast two negative roots

(C) exactly one negative root

(D) None of these

81. The polynomial $g(x^2)$ is identical with

(A) $f(x^2)$ (B) $(f(x))^2$

(C) $2f(x)\, f(-x)$ (D) $-f(x)\, f(-x)$

82. The value of g(9) is

(A) 889 (B) 899

(C) 961 (D) None

Comprehension - 5

A monic ineducible polynomial with integral coefficients is a polynomial with leading

www.pdfworld.net

www.pdfworld.net

coefficient 1 that cannot be factored, and the prime factorization of a polynomial with leading coefficient 1 is the factorization into monic irreducible polynomials. Consider two polynomials $f(x) = x^8 + x^4 + 1$ and $g(x) = x^8 + x + 1$.

83. The number of real roots of the equation $f(x) \cdot g(x) = 0$ is

(A) 2 (B) 4

(C) 0 (D) None

84. The highest common factor between f(x) and g(x) is of degree

(A) 1 (B) 2

(C) 4 (D) None

85. The number of distinct monic irreducible polynomials in the prime factorization of $f(x) \cdot g(x)$ is

(A) 2 (B) 3

(C) 4 (D) None

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

86. **Assertion (A) :** Let $f(x) = ax^2 + bx + c, a, b, c \in R$. If f(x) assumes real values for real values of x and non-real values for non-real values of x, then $a = 0$.

Reason (R) : If a, b, c are complex numbers, $a \neq 0$, then $\alpha + i\beta, \beta \neq 0$ is a root of $ax^2 + bx + c = 0$ if and only if $\alpha - i\beta$ is a root of $ax^2 + bx + c = 0$.

87. **Assertion (A) :** If $a, b, c, \in Q$ and $2^{1/3}$ satisfies $a + bx + cx^2 = 0$, then $a = 0, b = 0, c = 0$.

Reason (R) : A polynomial equation with rational coefficients cannot have irrational roots.

88. **Assertion (A) :** The sum of absolute value of the roots of

$P(x) = x^4 - 4x^3 - 4x^2 + 16x - 8$

$= 0$ is $2 + 2\sqrt{2} + 2\sqrt{3}$.

Reason (R) : $P(x) = (x^4 - 4x^3 + 4x^2) - (8x^2 - 16x + 8)$

$= (x^2 - (2 + 2\sqrt{2})x + 2\sqrt{2})$

$(x^2 - (2 - 2\sqrt{2})x - 2\sqrt{2})$

Sum of absolute value of the roots

$= 2 + 2\sqrt{2} + \sqrt{(2 - 2\sqrt{2})^2 + 4.2\sqrt{2}}$

89. **Assertion (A) :** If $x, y, z \in R$ and $2x^2 + y^2 + z^2 = 2x - 4y + 2xz - 5$, then the maximum possible value of $x - y + z$ is 4.

Reason (R) : The above equation rearranges as such of three squares equated to zero.

90. **Assertion (A) :** There are only two cubic polynomials $ax^3 + bx^2 + cx + d$, $a \neq 0$ whose coefficients a, b, c, d are a permutation of numbers 0, 1, 2, 3 (without repetition) and having all rational roots.

Reason (R) : The polynomials $x^3 + 3x^2 + 2x$ and $2x^3 + 3x^2 + x$ have three rational roots each.

91. **Assertion (A) :** There is no function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$ whose coefficients are a permutation of 0, 1, 2, 3, 4 (without repetition) and whose all roots are rational.

Reason (R) : $f(1) = a + b + c + d + e = 10$ cannot be written as a product of 3 integers greater than 1.

92. **Assertion (A) :** If two quadratic function $p(x) = a_1x^2 + b_1x + c_1$ and $q(x) = a_2x^2 + b_2x + c_2$ satisfy $p(w) = p(z)$ and $q(w) = q(z)$ for same $w, z \in R$, $w \neq z$ then $b_1a_1 = b_2a_1$

www.pdfworld.net

www.pdfworld.net

Reason (R) : $\frac{p(w)-p(z)}{w-z} = a_1(w+z)+b_1 = 0;$

$\frac{q(w)-q(z)}{w-z} = a_2(w+z)+b_2 = 0;$

$w+z = \frac{-b_1}{a_1} = \frac{-b_2}{a_2} \Rightarrow b_1a_1 = b_2a_1.$

93. **Assertion (A) :** The function $f(x) = x^4 + 4x^2 - x + 6$ cannot be resolved into two quadratic factors with real coefficients.

Reason (R) : In the equation $(x^2 + 2)^2 + 2 = x$ we observe that a solution to $x^2 + 2 = x$ is a solution of the quadratic equation by substitution of the left hand side into itself.

94. **Assertion (A) :** There is only one ordered pair (a, b) of real numbers for which $x^2 + ax + b = 0$ has a non real root whose cube is 343.

Reason (R) : $x^3 - 343 = (x - 7)(x^2 + 7x + 49)$. Therefore the orderd pair (7, 49) is suitable.

95. **Assertion (A) :** If P(x) is a quadratic polynomial with real coefficients such that

$P(x + 1) \cdot P(x^2 - x + 1)$ real coefficients such that $P(x+1) \cdot P(x^2 - x + 1) = P(x^3 + 1)$ for all real x, then $P(2) = 4$.

Reason (R) : Put $P(x) = ax^2 + bx + c$ into the identity.

$[a(x + 1)^2 + b(x + 1) + c][a(x^2 - x + 1) + b(x^2 - x + 1) + c] \equiv [a(x^3+1)^2 + b(x^3 + 1) + c]$

On comparision of coefficients we get

$P(x) = x^2 - \frac{1}{4}x + 1.$

Match the Columns for JEE Advanced

96. Match equations on left with the properties on right.

Column - I	**Column - II**
(A) $a < b < c < d$ and equation is $(x - a)(x - c) + \pi(x - b)(x - d) = 0$	(P) real roots
(B) $a > 0$, $a + b + c < 0$ and equation is $ax^2 + bx + c = 0$	(Q) distinct real roots
(C) b, c, $\in$ I and the equation $x^2 + bx + c = 0$ has rational roots	(R) integral roots
(D) a, b, c, d $\in$ R are in G.P. and equation is $(a^2 + b^2 + c^2)x^2 + 2(ab + bc + cd)x + b^2 + c^2 + d^2 = 0$	(S) discriminant ≥ 0

97.

Column-I	**Column-II**
(A) If a, b and c are positive real numbers, then $ax^3 + bx + c = 0$ has	(P) $4a^2 < (2b - c)a$
(B) If $c > 0$ and the quadratic equation $ax^2 + bx + c = 0$ has no real root, then (a, b, c $\in$ R)	(Q) $a^2 + a(b + c) > 0$
(C) If the quadratic equation $ax^2 + bx + c = 0$ has real roots and –2 lies between the roots, then (a, b, c $\in$ R)	(R) $a^2 < 2a(b - 2c)$
(D) If the quadratic equation $ax^2 + bx + c = 0$ has roots α, β such $\alpha < -2$ and $\beta > 2$ then	(S) $a^2 + a(b + c) < 0$

www.pdfworld.net

www.pdfworld.net

98. The set of value(s) of $k \in R$ for which

	Column - I		**Column - II**
(A)	$kx^2 - (k+1)x + 2k - 1 = 0$ has no real roots	(P)	$\{1, -2\}$
(B)	$x^2 - 2(4k-1)x + 15k^2 - 2k - 7 > 0$ for each x	(Q)	$(-\infty, -1/7) \cup (1, \infty)$
(C)	Sum of the roots of $x^2 + (2 - k - k^2)x - k^2 = 0$ is zero	(R)	$\{-4\}$
(D)	The roots of $x^2 + (2k-1)x + k^2 + 2 = 0$ are in the ratio 1 : 2	(S)	(2,4)

99.

	Column-I		**Column-II**
(A)	The value of m for which $\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$ have roots equal in magnitude and opposite in sign is/are	(P)	3
(B)	The value of m for which $x^2 - 15 - m(2x - 8) = 0$ has equal roots, is/are	(Q)	$\frac{a-b}{a+b}$
(C)	If α, β are roots of the equation $x^2 - ax + b = 0$, then $(\alpha - a)^{-4} + (\beta - a)^{-4}$ equals	(R)	5
(D)	Reciprocal of greatest value of $\frac{x+2}{2x^2 + 3x + 6}$ is	(S)	$\frac{a^4 - 4a^2b + 2b^2}{b^4}$

100. Let $a, b, c \in R$ be such $b^2 - ac \geq 0$. the equation $ax^2 + 2bx + c = 0$ has

	Column - I		**Column - II**
(A)	two positive roots	(P)	a, –c are of the same sign
(B)	two negative roots	(Q)	a, –b, c are of the same sign
(C)	one positive and one negative roots	(R)	a, b, c are of the same sign
(D)	equal roots	(S)	a, b, c are in G. P.

Review Exercises for JEE Advanced

1. If the equation $ax^2 + 2bx + c = 0$ has real roots, a, b and c being real numbers and if m and n are real numbers such that $m^2 > n > 0$ then prove that the equation $ax^2 + 2mbx + nc = 0$ has real roots.

2. If the roots of the equation $(a^4 + b^4)x^2 + 4abcdx + c^4 + d^4 = 0$ are real then prove that they must be equal.

3. Prove that if $x^2 + px - q$ and $x^2 - px + q$ both factorise into linear factors with integral coefficients, then the positive integers p and q are respectively the hypotenuse and area of a right triangle with sides of integer length.

4. If $2(a + b + c) = \alpha^2 + \beta^2 + \gamma^2$, and the roots of $x^2 + \alpha x - a = 0$ are β, γ and the roots of $x^2 + \beta x - b = 0$ are γ, α, show that the equation whose roots are α, β is $x^2 + \gamma x - c = 0$.

5. If α, β are the roots of equation $x^2 - p(x+1) - c = 0$, show that $(\alpha + 1)(\beta + 1) = 1 - c$. Hence prove that $\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c} = 1$

6. If $\beta + \cos^2\alpha, \beta + \sin^2\alpha$ are the roots of $x^2 + 2bx + c = 0$ and $\gamma + \cos^4\alpha, \gamma + \sin^4\alpha$ are the roots of $x^2 + 2Bx + C = 0$ then prove that $b^2 - B^2 = c - C$.

www.pdfworld.net

www.pdfworld.net

7. Show that the roots of the equation $(a^2 - bc)\,x^2 + 2(b^2 - ac)\,x + c^2 - ab = 0$ are equal if either $b = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$

8. If the equation $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and the other root of the second equation is the reciprocal of the other root of the first then prove that $(q - b)^2 = bq\,(p - a)^2$.

9. If the equation $x^2 - 2px + q = 0$ has two equal roots, then prove that the equation $(1 + y)\,x^2 - 2(p + y)x + (q + y) = 0$ will have roots real and distinct only when y is negative and p is not unity.

10. If one root of the equation $x^2 + ax + b = 0$ is also a root of $x^2 + mx + n = 0$, then show that its other root is a root of $x^2 + (2a - m)x + a^2 - am + n = 0$.

11. Find all real values of parameter p for which the least value of the quadratic expression $4x^2 - 4px + p^2 - 2p + 2$ on the interval $0 \le x \le 2$ is equal to 3.

12. If $\dfrac{x^2 + ax + 3}{x^2 + x + a}$ takes all real values for possible real values of x then prove that $4a^3 + 39 < 0$.

13. Find the range of values of a for which all the roots of the equation $(a - 1)(1 + x + x^2)^2 = (a + 1)(1 + x^2 + x^4)$ are imaginary.

14. If the quadratic function, $y = (\cot \alpha)\,x^2 + 2(\sqrt{\sin \alpha})\,x + \dfrac{1}{2}\tan \alpha$, $\alpha \in (0, 2\pi) - \{\pi/2, \pi\}$, can take negative values for all $x \in R$, then find the interval in which α lies.

15. Find the value of a for which the range of the function $y = \dfrac{x - 1}{a - x^2 + 1}$ does not contain any value belonging to the interval $\left[-1, -\dfrac{1}{3}\right]$.

16. Find the values of a for which the inequality is satisfied $25^x + (a + 2)\,5^x - (a + 3) < 0$ for atleast one x.

17. Find the integral values of 'a' for which the equation $x^4 - (a^2 - 5a + 6)\,x^2 - (a^2 - 3a + 2) = 0$ has real roots only.

18. Find the condition on a, b, c such that equations $2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.

19. For what values of $a \in R$ does the equation $ax^2 + x + a - 1 = 0$ possess two distinct real roots x_1 and x_2 satisfying the inequality $\left|\dfrac{1}{x_1} - \dfrac{1}{x_2}\right| > 1$?

20. Find all the values of the parameter a for which the inequality $4^x - a \cdot 2^x - a + 3 \le 0$ is satisfied by atleast one real x.

21. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and α^4 and β^4 are the roots of the equation $\ell x^2 + mx + n = 0$, then prove that the roots of the equation $a^2\ell x^2 - 4ac\ell x + 2c^2\ell + a^2 m = 0$ are always real and opposite in sign. (α, β are real and different)

22. If the roots of the equation $\dfrac{1}{x + p} + \dfrac{1}{x + q} = \dfrac{1}{r}$ are equal in magnitude but opposite in sign, show that $p + 1 = 2r$ and that the product of the roots is equal to $-\dfrac{p^2 + q^2}{2}$.

23. Find all values of a for which the inequation $4^{x^2} + 2(2a + 1)\,2^{x^2} + 4a^2 - 3 > 0$ is satisfied for any x.

24. For what values of a, the equation $2x^2 - 2(2a + 1)\,x + a(a + 1) = 0$ has $(a \in R)$

(a) both roots smaller than 2
(b) both roots greater than 2
(c) both roots lying in the interval (2, 3)
(d) exactly one root lying in the interval (2, 3)
(e) one root smaller than 1, and other root greater than 1
(f) atleast one root lying in the interval (2, 3)
(g) one root less than 2 and other greater than 3.

www.pdfworld.net

www.pdfworld.net

25. For what values of $a \in R$ does the equation $x^2 + 1 = x/a$ possess two distinct real roots x_1 and x_2 satisfying the inequality $|x_1^2 - x_2^2| > 1/a$?

26. For what values of the parameter a is the inequality

$[a^3 + (1 - \sqrt{2})\, a^2 - (3 + \sqrt{2})\, a + 3\sqrt{2}\,]\, x^2$

$+ 2(a^2 - 2)x + a > -\sqrt{2}$ satisfied for any $x > 0$?

27. For what values of the parameter p the equation $x^4 + 2px^3 + x^2 + 2px + 1 = 0$ has atleast two distinct negative roots.

28. If α, β, γ are the roots of $x^3 + qx + r = 0$, prove that the equation whose roots are

$$\frac{\beta}{\gamma} + \frac{\gamma}{\beta},\ \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma},\ \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

is $r^2(z + 1)^3 + q^3(z + 1) + q^3 = 0$.

29. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with this cubic, prove that either $a = b = 0$ or $a = b = 3$, or a, b are the roots of $z^2 + z + 2 = 0$.

30. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of equation $x^n + nax - b = 0$, show that

$(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots (\alpha_1 - \alpha_n) = n(\alpha_1^{n-1} + a)$

31. Find the integral part of the greatest root of equation $x^3 - 10x^2 - 11x - 100 = 0$

32. Two roots of the equation

$x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$

are of the form $\alpha + i\beta$, $\beta + i\alpha$. Find all the roots of the equation.

33. If α, β, γ are the roots of the cubic $x^3 - px^2 + qx - r = 0$ find the cubic equations whose roots are

(i) $\beta\gamma + \frac{1}{\alpha}, \gamma\alpha + \frac{1}{\beta}, \alpha\beta + \frac{1}{\gamma}$

(ii) $(\beta + \gamma - \alpha), (\gamma + \alpha - \beta), (\alpha + \beta - \gamma)$

Also find the value of

$(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma)$

34. If the equation $x^4 - 4x^3 + ax^3 + bx + 1 = 0$ has four positive roots, then find a and b.

35. If the equations $x^2 - ax + b = 0$ & $x^3 - px^2 + qx = 0$, where $b \neq 0, q \neq 0$ have common roots & the second equation has two equal roots, then prove that $2(q + b) = ap$.

Target Exercises for JEE Advanced

1. Let $p(x) = ax^2 + bx + c$ be such that $p(x)$ takes real values for real values of x and non-real values for non-real values of x. Prove that $a = 0$.

2. For what values of the quantity h is the polynomial

$x^4 - 2^{\tanh} \cdot x^2 + (\cos h + \cos 2h)\, x + 2^{\tanh - 2}$

the square of the quadratic trinomial with respect to x ?

3. Find the values of a for which the equation $(x^2 + x + 2)^2 - (a - 3)(x^2 + x + 2)(x^2 + x + 1) + (a - 4)(x^2 + x + 1)^2 = 0$ has atleast one real root.

4. Find all values of α for which the solutions of the system of inequalities

$x^2 + 6x + 7 + \alpha \leq 0$, $x^2 + 4x + 7 \leq 4\alpha$ form an interval of length unity on the number axis.

5. For real values of x, if the expression $\frac{(ax - b)(dx - c)}{(bx - a)(cx - d)}$ assumes all real values then prove that $(a^2 - b^2)$ and $(c^2 - d^2)$ must have the same sign.

6. Find a for which the range of the function

$y = \frac{x - 1}{1 - x^2 - a}$ does not contain any value from the interval $[-1, 1]$

7. For what values of 'k', the inequality $\frac{x^2 + k^2}{k(x + 6)} \geq 1$ satisfied for all $x \in (-1, 1)$.

8. Let A, B, C be three angles such that $A = \pi/4$ and tan B. tan C = p find all possible values of p such that A, B, C are the angles of a triangle.

9. Find all values of α for which the system of inequalities

www.pdfworld.net

www.pdfworld.net

$x^2 + 4x + 3 \le \alpha$, $x^2 - 2x \le 3 - 6a$ has a unique solution.

10. Find all real values of a for which the equation $\sqrt{(x-a)}\,[x^2 + (1 + 2a^2)x + 2a^2] = 0$ has only two distinct roots.

11. Find the value of a for which the equation $x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$
(a) has no solution
(b) has one solution
(c) has two solutions
(d) has three solutions

12. Find all values of k for which all solutions of the inequation $kx^2 - 2(k^2 - 3)x - 12 \ge 0$ are the solutions of the inequation $x^2 - 49 \ge 0$

13. Find all the values of a for which from the inequality $x^2 - a(1 + a^2)\,x + a^4 < 0$ follows the inequality $x^2 + 4x + 3 > 0$.

14. Find all the values of a for which from the inequality $0 \le x \le 1$ follows the inequality $(a^2 + a - 2)x^2 - (a + 5)\,x - 2 \le 0$

15. Find the values of a for which the range of function $y = \dfrac{x-1}{a+x^2}$ contain the interval [0, 1].

16. Find the least possible value of the expression $\dfrac{2x^4 + 4x^3 + 9x^2 - 4x + 2}{(x^2+1)^2}$ for real values of x.

17. Find the set of values of k for which any solution of the inequality $\dfrac{\log_2\left(x^2 - 5x + 6\right)}{\log_2(2x)} < 1$ is also a solution of the inequality $x^2 - 7kx + 2k - 6 \le 0$.

18. If $f(x) = x^3 + bx^2 + cx + d$ and f(0), f(–1) are odd integers, prove that f(x) = 0 cannot have all integral roots.

19. If the equation $x^4 + px^3 + qx^2 + rx + 5 = 0$ has four positive roots, then find the minimum value of pr.

20. If equation $ax^3 + 2bx^2 + 3cx + 4d + 0$ and $ax^2 + bx + c = 0$ have a non-zero common root, then prove that $(c^2 - 2bd)\,(b^2 - 2ac) \ge 0$.

21. Prove that the roots of the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ cannot all be real if $2a^2 < 5b$.

22. If $(n-1)\,x^2 - 2\,(a_n - a_1)\,x + a_1^2 + 2a_3^2 + \ldots. + 2a_{n-1}^2 + a_n^2 = 2\,(a_1a_2 + a_2a_3 + \ldots. + a_{n-1}\,a_n)$. show that $a_1, a_2 \ldots\ldots, a_n$ are in A.P. with common difference x.

23. If a, b, c be the three possible values of $n \in N$ for which the polynomial $1 + x^2 + x^4 \ldots\ldots + x^{2n-2}$ is divisible by $1 + x + x^2 + \ldots\ldots + x^{n-1}$, then show that the roots of the quadratic equation $ax^2 + bx + c = 0$ are not rational.

24. If the equations $ax^3 + bx + c = 0$ and , $a'x^3 + b'x + c' = 0$ have a common root then prove that $a(ca'-c'a)^3 + b(ab'-a'b)^2\,(ca'-c'a) + c(ab'-a'b)^3 = 0$.

25. If $(x^2 - 7x + 12).f(x) = (x^2 + 7x + 12).g(x)$ then prove that $h(x) = f(x)\,.\,g(x) + x^4 - 25x^2 + 144$ has four real roots. Also find them.

26. Show that $ax^4 + bx^3 + \left(a + \dfrac{b^2}{4a}\right)x^2 + dx + e$ will have the same sign as a if $4ae > d^2$.

27. Show that $\dfrac{x^3 - 3x + 1}{x(x-1)}$ is unaltered by substituting either $\dfrac{1}{1-x}$ or $1 - \dfrac{1}{x}$ for x. Hence or otherwise find all the roots of $\dfrac{x^3 - 3x + 1}{x(x-1)} = \dfrac{c^3 - 3c^2 + 1}{c(1-c)}$.

28. Prove that the necessary and sufficient conditions that the roots of $\lambda\,(ax^2 + bx + c) + \mu(a'x^2 + b'x + c') = 0$ may be real all real vaues of λ and μ are that $b^2 - 4ac > 0$ and $(bc' - b'c)\,(ab' - a'b) - (ca' - c'a)^2 > 0$.

www.pdfworld.net

www.pdfworld.net

29. Find all values of k for each of which there is atleast one common solution of the inequalities $x^2 + 4kx + 3k^2 > 1 + 2k$, $x^2 + 2kx \le 3k^2 - 8k + 4$.

30. Find all real values of the quantity h for which the equation $x^4 + (h-1)x^2 + x^2 + (h-1)x + 1 = 0$ possesses not less than two distinct negative roots.

31. For what values of a does the equation $\log_3(9^x + 9a^3) = x$ possess two solutions?

32. Find the real values of the parameter a for which every solution of the inequality $\log_{1/2} x^2 \ge \log_{1/2}(x+2)$ is a solution of the inequality $49x^2 - 4a^4 \le 0$.

33. For what values of 'p' does the equation $p\,2^x + 2^{-x} = 5$ possess a unique solution.

34. For what real values of a is the sum of roots of the equation $\left(\frac{1}{x} + \frac{1}{a} - \frac{1}{a^2} = \frac{1}{-a^2 + a + x}\right)$ smaller than $a^3/10$?

35. Prove that for every $a > 0$ the inequality $\sqrt{a + \sqrt{a + \sqrt{a + \ldots + \sqrt{a}}}} < \frac{1 + \sqrt{4a+1}}{2}$ (the left side contains an infinite number of radicals) holds true.

Previous Years' Questions (JEE Advanced)

A. Fill in the blanks

1. If : $2 + i\sqrt{3}$ a root of the equation $x^2 + px + q = 0$ where p and q are real, then (p, q) = (.....). **[IIT - 1982]**

2. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \ne b)$ have a common root, then the numerical value of a + b is..... **[IIT - 1986]**

B. True / False

3. If $a < b < c < d$, then the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and distinct. **[IIT - 1984]**

4. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$, where $ac \ne 0$, then $P(x)\,Q(x) = 0$ has at least two real roots. **[IIT - 1985]**

C. Multiple Choice Questions with ONE correct answer

5. If l, m, n are real, $l \ne m$, then the roots by the equation: $(l-m)x^2 - 5(l+m)x - 2(l-m) = 0$ are **[IIT - 1979]**

(A) Real and equal (B) Complex
(C) Real and unequal (D) None of these

6. If x, y and z are real and different and $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$, then u is always. **[IIT - 1979]**

(A) non negative (B) zero
(C) non positive (D) None of these

7. Let $a > 0$, $b > 0$ and $c > 0$. Then the roots of the equation $ax^2 + bx + c = 0$ **[IIT - 1979]**

(A) are real and negative
(B) have negative real parts
(C) are imaginary
(D) none of these

8. Both the roots of the equation $(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) = 0$ are always **[IIT - 1980]**

(A) positive (B) real
(C) negative (D) None of these

9. If $(x^2 + px + 1)$ is a factor of $(ax^3 + bx + c)$, then **[IIT - 1980]**

(A) $a^2 + c^2 = -ab$ (B) $a^2 - c^2 = -ab$
(C) $a^2 - c^2 = ab$ (D) None of these

10. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is **[IIT - 1982]**

(A) 4 (B) 1 (C) 3 (D) 2

11. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is **[IIT - 1982]**

(A) $-4 < x \le 0$ (B) $0 < x < 1$
(C) $-100 < x < 100$ (D) $-\infty < x < \infty$

12. If $a^2 + b^2 + c^2 = 1$, then ab + bc + ca lies in the interval **[IIT - 1984]**

www.pdfworld.net

www.pdfworld.net

(A) [1/2, 2] (B) [–1, 2]
(C) [–1/2, 1] (D) [–1, 1/2]

13. If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always **[IIT - 1989]**

(A) two real roots
(B) two positive roots
(C) two negative roots
(D) one positive and one negative roots

14. Let a, b, c be real numbers $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$. β is the root of $a^2x^2 - bx - c = 0$ then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies **[IIT - 1989]**

(A) $\gamma = \frac{\alpha + \beta}{2}$ (B) $\gamma = \alpha + \frac{\beta}{2}$

(C) $\gamma = \alpha$ (D) $\alpha < \gamma < \beta$.

15. Let α, β be the roots of the equation $(x - a)(x - b) = c$, $c \neq 0$. Then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are **[IIT - 1992]**
(A) a, c (B) b, c (C) a,b (D) a + c, b + c

16. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real less than 3; then **[IIT - 1999]**
(A) $a < 2$ (B) $0 \leq a \leq 3$
(C) $3 < a \leq 4$ (D) $a > 4$

17. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then **[IIT - 2000]**
(A) $0 < \alpha < \beta$ (B) $\alpha < 0 < \beta < |\alpha|$
(C) $\alpha < \beta < 0$ (D) $\alpha < 0 < |\alpha| < \beta$

18. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has **[IIT - 2000]**
(A) both roots in (a, b)
(B) both roots in $(-\infty, a)$
(C) both roots in $(a, +\infty)$
(D) one root in $(-\infty, a)$ and the other in (b, ∞)

19. For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the root is square of the other, then p is equal to **[IIT - 2000]**
(A) 1/3 (B) 1 (C) 3 (D) 2/3

20. For all 'x', $x^2 + 2ax + 10 - 3a > 0$, then the interval in which 'a' lies is **[IIT - 2004]**
(A) $a < -5$ (B) $-5 < a < 2$
(C) $a > 5$ (D) $2 < a < 5$

21. If one root is square of the other root of the equation $x^2 + px + q = 0$, then the relation between p and q is **[IIT - 2004]**
(A) $p^3 - q(3p - 1) + q^2 = 0$
(B) $p^3 - q(3p + 1) + q^2 = 0$
(C) $p^3 + q(3p - 1) + q^2 = 0$
(D) $p^3 + q(3p + 1) + q^2 = 0$

22. If p, q, r are +ve and are in A.P., the roots of quadratic equation $px^2 + qx + r = 0$ are all real for **[IIT - 1994]**

(A) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (B) $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$

(C) all p and r (D) no p and r

23. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots is **[IIT - 1994]**
(A) 15 (B) 9 (C) 7 (D) 8

24. Let a, b, c be the sides of a triangle where $a \neq b \neq c$ and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ real, then **[IIT - 2006]**

(A) $\lambda < \frac{4}{3}$ (B) $\lambda < \frac{5}{3}$

(C) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

25. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}$; 2β be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is **[IIT - 2007]**

www.pdfworld.net

www.pdfworld.net

(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$

(C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

26. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ as its roots is **[IIT - 2010]**

(A) $(p^3+q)x^2 - (p^3+2q)x + (p^3+q) = 0$

(B) $(p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$

(C) $(p^3-q)x^2 - (5p^3-2q)x + (p^3-q) = 0$

(D) $(p^3-q)x^2 - (5p^3+2q)x + (p^3-q) = 0$

27. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[IIT - 2011]**

(A) 1 (B) 2

(C) 3 (D) 4

28. A Value of b for which the equations

$$x^2 + bx - 1 = 0$$
$$x^2 + x + b = 0$$

have one root in common is **[IIT - 2011]**

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$

(C) $i\sqrt{5}$ (D) $-\sqrt{2}$

29. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2 + \left(\sqrt{1+a}-1\right) + \left(\sqrt[6]{1+a}-1\right) = 0$ where $a > -1$.

Then $\lim_{a\to 0+} \alpha(a)$ and $\lim_{a\to 0+} \beta(a)$ are **[IIT - 2012]**

(A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and 1

(C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

D. Multiple Choice Questions with ONE or MORE THAN ONE correct answer

30. For real x, the function $\frac{(x-a)(x-b)}{x-c}$ will assume all real values provided **[IIT - 1984]**

(A) $a > b > c$ (B) $a < b < c$

(C) $a > c > b$ (D) $a < c < b$

31. If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \leq 0$ then a, b, c, d **[IIT - 1987]**

(A) are in A.P. (B) are in G.P.

(C) are in H.P. (D) satisfy ab = cd

E. Subjective Problems

32. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root. **[IIT - 1979]**

33. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the n-th power of the other, then show that $(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$ **[IIT - 1983]**

34. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$. **[IIT - 1995]**

35. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP. Find the intervals in which β and γ lie. **[IIT - 1996]**

36. Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that $2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$. **[IIT - 1997]**

37. If α, β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$ and $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$ $(A \neq 0)$ for some constant δ, then prove that

$$\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$$ **[IIT - 2000]**

38. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$.

www.pdfworld.net

www.pdfworld.net

Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β. **[IIT - 2001]**

39. If $x^2 + (a - b)x + (1 - a - b) = 0$ where $a, b \in R$ then find the values of a for which equation has unequal real roots for all values of b. **[IIT - 2003]**

40. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d then the value of $a + b + c + d$, when $a \neq b \neq c \neq d$, is. **[IIT - 2006]**

F. Assertion (A) and Reason (R)

41. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, 1/\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$. **[IIT - 2008]**
Assertion (A): $(p^2 - q)(b^2 - ac) \geq 0$
Reasons (R): $b \neq pa$ or $c \neq qa$

G. Integer Answer Type

42. The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have value at least 4, is. **[IIT - 2009]**

ANSWERS

Practice Problems—A

1. m, $\frac{1-m^2}{2m}$ **2.** 1, $\frac{a-b}{b-c}$
3. –1, 4 **4.** –1/3, 1/2
5. –9, 2

Practice Problems—B

1. $a = -2, c = -4$ **2.** $x^2 + (31/36)x - (449/216) = 0$
3. $4x^2 - 5x - 6$ **4.** (a) ϕ (b) $-\frac{8}{9}$ (c) $\frac{-3}{11}$
5. (a) 15 (b) – 22 (c) 127
7. $a_1 = -3/2, a_2 = 6$ **8.** $a_1 = 3/2, a_2 = 3$
10. –3, –4 **12.** $b = -13$
14. $a^2l^2x^2 - ablmx + (b^2 - 2ac)\ln + (m^2 - 2ln)\,ac = 0$

Practice Problems—C

1. 1 **2.** $p(x) = x^2 + c$
3. $a = 3, b = -3, c = 1$. **4.** $a = 5, b = -3, c = 1$.
5. $x \in R$
6. $x = ab + bc + ca$; If $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 0$ the given equation becomes an identity & is true for all $x \in R$
7. $8x^2 - 10x - 61 = 0$
9. (i) $x^2 - 6x + 11 = 0$ (ii) $3x^2 - 2x + 1 = 0$
10. (i) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$, (ii) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$

Practice Problems—D

1. – 43, – 29/14, 14
2. $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$
3. $x^2 - 4mnx - (m^2 - n^2)^2 = 0$
4. (i) $\frac{p(p^2-4q)(p^2-q)}{q}$ (ii) $\frac{p^4 - 4p^2q + 2q^2}{q^4}$
5. $\frac{b^3 - 3abc}{a^3c^3}$ **7.** – 160
8. $x^2 - 3x + 2 = 0$ **9.** $x^2 + 31x + 112 = 0$

Practice Problems—E

1. $a = 4$ **2.** 1, – 1/3
4. (a) $a \geq -81/4$, (b) $a \leq 2$, (c) all real a
9. real and distinct.

Practice Problems—F

1. $k^2 + k, k \in W$ **5.** $\{-2, 0\}$
6. –4, –3

Practice Problems—G

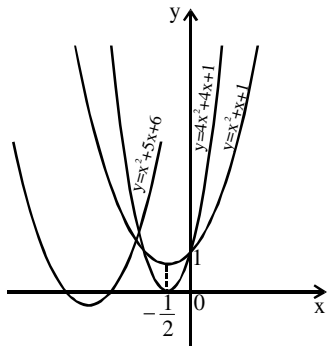
7. $n\pi + (-1)^n \frac{\pi}{6}, n \in I$ **8.** $\lambda = 3/2, \mu = -6$

www.pdfworld.net

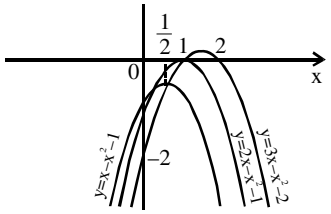
www.pdfworld.net

Practice Problems—H

1.



2.



3. {–4, –3, 3, 4} **4.** (a) (–∞ , 1.5), (b) (–2 , ∞)

5. (a) (–1.5 , ∞), (b) (–∞ , 2)

6. [5 , ∞) **7.** 4

8. (i) $a < 0, b < 0, c > 0, D > 0$,

(ii) $a > 0, b > 0, c > 0, D = 0$,

(iii) $a < 0, b > 0, \quad c < 0, \ D < 0$.

9. Negative **12.** positive

Practice Problems—I

1. (i) [–1, 4], (ii) {–1/2}

2. (A) All reals,

(B) $x > 3$ or $x < 1$

(C) $-1-\sqrt{13} < x < -1+\sqrt{13}$,

(D) $-1 \le x \le 8/3$

3. (i) $[-4,-1)\cup(3,\infty)$, (ii) {4}, (iii) (6, 8]

4. $\left(-\infty, -\dfrac{a-\sqrt{a^2-4a}}{2}\right)\cup\left(\dfrac{\sqrt{a^2-4a}-a}{2}, \infty\right)$,

$a \in (-\infty, 0)\cup(4,\infty)$; R , $a \in (0,4)$; R $-\left\{-\dfrac{a}{2}\right\}$,

$a \in \{0, 4\}$

5. 12, 13 **6.** $k \le -7/9$, $k \ge 1$

7. [0, 1/2)

8. $k = 3$ **9.** For all $a \in (5/3, +\infty)$.

10. (–∞, 6)

11. (–6, 3) **12.** 3

13. $(-\infty, 0)\cup\left(\dfrac{9}{2}, \infty\right)$

15. (a) Not for any a, (b) for any a, (c) $a \ne 0$, (d) for any a, (e) $a = 0$, (f) $a \ne 0$

Practice Problems—J

1. (i) $-\dfrac{3}{2}, \infty$, (ii) $-1, \infty$ (iii) $-\dfrac{3}{2}, 23$

2. (a) $y_{min} = y(1) = 7$, $y_{max} = y(2) = 15$

(b) $y_{max} = y(2) = -14$, $y_{min} = y(3) = -29$

(c) $y_{max} = y(-1) = 8$, $y_{min} = y(1) = 4$

(d) $y_{min} = y(0) = -1$, $y_{max} = y(3) = 8$.

3. $\dfrac{25}{8}$. **4.** – 3/8, 33/8

5. – 2 **6.** $y \in \left(-\infty, -\dfrac{7}{8}\right]$

7. –7, 1

Practice Problems—K

1. $y \in (-\infty, -2]\cup(1, \infty)$ **9.** $-12 < a < 2$

10. $m < \dfrac{11}{4}$

Practice Problems—L

1.(i) $(3x+4y-1)(x-y+3)$ (ii) $(3x-4y+2)(x+2y-3)$

2.(i) $\lambda = 1$, $(2x + y - 2)(x - y + 1)$; $\lambda = -7/8$, $(x - 7y + 4)(x - y - 4)/8$

(ii) $\lambda = -1$, $(y - 1)(y - 2)$; $\lambda = -2$, $-(x + y)(x - y)$;

www.pdfworld.net

www.pdfworld.net

$\lambda = -\frac{10}{9}, -(x+3y+4)(x-3y-4)/9.$

3. $-5/2$ **4.** $0, -17/36$

Practice Problems—M

1. $\left[3, \frac{15}{4}\right]$ **2.** $(0, 4)$

3. $\frac{a-b}{a+b}$ **4.** $2 < m < 8/3$

5. (i) (a) $(0, 3)$, (b) $\{0, 3\}$,
(c) $(-\infty, 0) \cup (3, \infty)$, (d) $(-\infty, -1) \cup [3, \infty)$, (e) ϕ

(ii) $\left(-1, -\frac{1}{8}\right)$ (iii) $\left\{-\frac{1}{3}, 0, 3\right\}$ (iv) $\left\{-\frac{1}{3}\right\}$

(v) $(-\infty, -1) \cup \left[-1, -\frac{1}{2}\right] \cup [3, \infty)$

(vi) $\left(-1, -\frac{1}{8}\right)$

(vii) $\{0\}$ (viii) $\left(-1, -\frac{1}{3}\right)$

(ix) $\left(-\frac{1}{3}, -\frac{1}{8}\right)$

6. ϕ **7.** ϕ

8. $a < -2$ **9.** $\left(0, \frac{1}{8}\right)$

10. $2\sqrt{2} \le a < \frac{11}{3}$

11. (i) $[4, \infty)$ (ii) $(-1, 0)$ (iii) $(-1, 0]$
(iv) $[3, \infty)$ (v) $(-\infty, -1) \cup [1, \infty)$ (vi) ϕ (vii) ϕ

13. $\left[\frac{1}{2}, \infty\right)$

Practice Problems—N

2. $(1, \infty)$ **3.** $\left(\frac{5}{2}, \infty\right)$

4. $a < 3$

6. (a) $(0, 3)$, (b) ϕ, (c) $(-\infty, -1)$, (d) $(1, \infty)$

7. $[1, \infty)$

Practice Problems—O

5. $p = -2, q = -4$ **6.** $7x^2 - 11x - 6$

Practice Problems—P

1. $-\sqrt{5}, -3, 2, 3$ **2.** $x^4 - 16x^2 + 8x - 1 = 0$

3. $1 \pm 2i$ and $2 \pm i$ **4.** $2 \pm \sqrt{3}, -3 \pm \sqrt{2}$

5. $x^4 - 16x^2 + 4 = 0$

8. $(x^2+x+1)(x^2-x+1)(x^2+\sqrt{3}x+1)$
$(x^2-\sqrt{3}x+1)$

9. $-2, 1/2, 3$ **10.** 2

Practice Problems—Q

1. $2 \pm \sqrt{2}$ **2.** $-2, -\frac{1}{2}, \frac{3 \pm \sqrt{5}}{2}$

3. $\frac{-11 \pm \sqrt{97}}{6}$ **4.** $4, 2$

5. $-\frac{1}{12}, \frac{1}{2}$ **6.** $\frac{1 \pm \sqrt{5}}{2}$

Practice Problems—R

1. $-1, 2, -3/2$ **2.** $-1, 1, 3, 5$

3. $1, 3, 5, 7$ **4.** $-\frac{3}{2}, -2, 4$

5. $\pm\sqrt{3}, \frac{3}{4}, -\frac{1}{2}$ **6.** $4/3, 3/2, 1 \pm \sqrt{2}$

8. $6, 2, 2/3$

Practice Problems—S

1. $y^3 - 8y^2 + 197 - 15 = 0$

2. $4, 2, 4/3$ **3.** $x^3 - 2x^2 + 5x - 11 = 0$

4. **6**

Practice Problems—T

1. (i) $\frac{1}{a^2}(b^2 - 2ac)$ (ii) $\frac{-c}{a}$ (iii) $\frac{3ad - bc}{a^2}$

www.pdfworld.net

www.pdfworld.net

2. (i) $3p^3-16pq+64r$.
(ii) $(q^3-4pqr+8r^2)/r^3$.
(iii) $(q^3-p^3r)/r^4$.

3. $2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}$

4. $\frac{3}{2}, -\frac{3}{4}\left(1\pm\sqrt{5}\right)$

5. (a) $\frac{q^2-2pr}{r^2}$ (b) $\frac{p^2-2q}{r^2}$

6. (a) $-6q$ (b) q/r

Objective Exercises

1. A **2.** A **3.** D
4. A **5.** A **6.** D
7. C **8.** C **9.** D
10. A **11.** D **12.** C
13. D **14.** C **15.** C
16. A **17.** B **18.** B
19. C **20.** B **21.** B
22. D **23.** B **24.** B
25. C **26.** B **27.** A
28. B **29.** A **30.** A
31. B **32.** B **33.** B
34. D **35.** B **36.** B
37. C **38.** B **39.** A
40. B **41.** C **42.** C
43. D **44.** A **45.** A
46. A **47.** A **48.** C
49. B **50.** C **51.** A
52. ABC **53.** CD **54.** BC
55. AC **56.** ABCD **57.** CD
58. BCD **59.** AB **60.** ABC
61. ABC **62.** ABC **63.** BC
64. ABC **65.** AC **66.** BD
67. BD **68.** BD **69.** BC
70. ABC **71.** B **72.** C
73. B **74.** B **75.** A
76. C **77.** C **78.** B
79. C **80.** C **81.** D
82. B **83.** C **84.** B
85. C **86.** C **87.** C
88. A **89.** B **90.** B
91. A **92.** A **93.** D
94. A **95.** C
96. (A) – PQS, (B) – PQS, (C) – PRS, (D) – PS
97. (A) – Q, (B) – Q, (C) – P, (D) – PRS
98. (A) – QRS, (B) – S, (C) – P, (D) – R
99. (A) – Q, (B) – PR, (C) – S, (D) – P
100. (A) – Q, (B) – R, (C) – P, (D) – S

Review Exercises for JEE Advanced

1. 0 **5.** 1
11. $1-\sqrt{2}, 5+\sqrt{10}$ **13.** $(-2, 2)$
14. $\left(\frac{5\pi}{6}, \pi\right)$ **15.** $a<-\frac{1}{4}$
16. $R-\{-4\}$ **17.** $a=1, 2$
18. $(4bc+ad)^2=\frac{9}{2}(bd+4c^2)(b^2-ac)$
19. $a\in(0, 1)\cup(1, 6/5)$ **20.** $a\geq 2$
23. $a\in(-\infty,-1)\cup[\sqrt{3}/2,\infty)$
24. (a) $a\in\left(-\infty,\frac{7-\sqrt{33}}{2}\right)$
(b) $a\in\left(\frac{7+\sqrt{33}}{2},\infty\right)$
(c) $a\in\left(-\frac{7}{2},-\frac{5}{2}\right)$
(d) $a\in\left(\frac{7-\sqrt{33}}{2},\frac{11-\sqrt{73}}{2}\right)\cup\left(\frac{7+\sqrt{33}}{2},\frac{11+\sqrt{73}}{2}\right)$
(e) $a\in(0, 3)$
(f) $a\in\left(-\frac{7}{2},-\frac{5}{2}\right)\cup\left(\frac{7-\sqrt{33}}{2},\frac{11-\sqrt{73}}{2}\right)\cup\left(\frac{7+\sqrt{33}}{2},\frac{11+\sqrt{73}}{2}\right)$
(g) $a\in\left(\frac{11-\sqrt{73}}{2},\frac{7+\sqrt{33}}{2}\right)$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

25. $a \in (-1/2, 0) \cup (0, \sqrt{5}/5)$.

26. $(-\sqrt{2}, 1) \cup [\sqrt{2}, \infty)$

27. $p \in \left(\frac{3}{4}, \infty\right)$ **31.** 11

32. $2 \pm i, 1 \pm 2i$

33. (i) $ry^3 - q(r+1)y^2 + p(r+1)^2 y - (r+1)^3 = 0$;
(ii) $y^3 - py^2 + (4q - p^{2)}y + (8r - 4pq + p^3) = 0$;
and $4qp - p^3 - 8r$

34. $a = 6, b = -4$

Target Exercises for JEE Advanced

2. $h = (2k+1)\pi, k \in I$ **3.** $5 < a \le 19/3$

4. $\alpha = 1, \alpha = \frac{7}{4}$ **6.** no value of a

7. $k \in \left[\frac{7+3\sqrt{5}}{2}, \infty\right)$

8. $p \in (-\infty, 3 - 2\sqrt{2}\,] \cup [3 + 2\sqrt{2}, \infty)$

9. $\alpha = 1, \alpha = 0$

10. $x_1 = a, x_2 = -1$ for $a \in (-\infty, -1)$; $x_1 = a, x_3 = -2a^2$ for $a \in (-1/2, 0)$

11. (a) $a \in (-\infty, -1) \cup (5/4, \infty)$,
(b) $a = -1$,
(c) $a \in (-1, 1) \cup \{5/4\}$,
(d) $a = 1$

12. $k \in (0, 7/6]$ **13.** $a \le -3, a \ge -1$.

14. $-3 \le a \le 3$ **15.** $-\infty < a < -1$ or $-1 < a < \frac{5}{4}$

16. $\frac{6}{5}$ **17.** [3/4, 3]

19. 80 **25.** $x = \pm 3, \pm 4$

27. $\frac{1}{c}, \frac{c}{c-1}, 1 - c$ **29.** $k < \frac{1}{2}, k > \frac{3}{2}$

30. $h > \frac{5}{2}$ **31.** $0 < a < \frac{1}{\sqrt[3]{36}}$

32. $a \le -\sqrt{7}, a \ge \sqrt{7}$ **33.** $p \in (-\infty, 0] \cup \left\{\frac{25}{4}\right\}$

34. $a \in (0, 1) \cup (1, 5 - \sqrt{15}) \cup (5 + \sqrt{15}, \infty)$

Previous Years' Questions (JEE Advanced)

1. –4, 7	**2.** 1	**3.** T
4. T	**5.** C	**6.** A
7. B	**8.** B	**9.** D
10. A	**11.** D	**12.** C
13. A	**14.** D	**15.** C
16. A	**17.** B	**18.** D
19. C	**20.** B	**21.** A
22. AB	**23.** C	**24.** A
25. D	**26. B**	**27.** C
28. B	**29.** B	**30.** CD

31. B

32. $q(r-p)^2 - p(r-p)(s-q) + (s-q)^2$; $(q-s)^2 = (r-p)(ps-qr)$

33. $\beta \in \left(-\infty, \frac{1}{3}\right], \in \left[-\frac{1}{27}, \infty\right)$

34. $\alpha^2\beta, \alpha\beta^2$ **35.** $a > 1$ **36.** 1210

37. B **38.** 2

pdfworld.net

www.pdfworld.net

www.pdfworld.net

3 Sequence and Series

3.1 INTRODUCTION

Defined informally, a sequence is a list of numbers. We are most interested in lists of numbers that satisfy some pattern. For example 2, 4, 6, 8, 10, ... is a list of the natural number multiples of 2. This can be written as 2n, where n is a natural number, so a sequence may be defined like a function by a variable expression. More formally a sequence is defined as follows.

Definition of Sequence

A sequence is a function of natural numbers with co-domain as the set of real numbers (complex numbers). If range is a subset of real numbers (complex numbers) then it is called a real sequence (complex sequence).

A sequence whose range is a subset of real numbers is called a **real sequence**.

Examples of Real Sequence

A. 4, 7, 10, 13, ... (Here each term of the sequence can be obtained by adding 3 to the preceding term)

B. 5, 3, 1, –1, ... (Here each term of the sequence can be obtained by subtracting 2 from the preceding terms)

C. 3, – 6, 12, –24, ... (Here each term can be obtained by multiplying the preceding term by –2)

Instead of using f(x) notation to indicate a sequence, it is customary to use a_n, where n represents an element in the domain of the sequence. Thus, $a_n = f(n)$. The sequence itself will be denoted by the symbol (a_n) or $\{a_n\}$.

The letter n is used instead of x as a reminder that n represents a natural number. The elements in the range of a sequence, called the terms of the sequence, are $a_1, a_2, a_3, ..., a_n, ...$

The general term, or nth term, of the sequence is a_n

A sequence is often given by the formula of its nth (or general) term :

$$a_n = f(n) \quad (n = 1, 2, 3, ...)$$

enabling us to compute a certain term of the sequence by its number.

The first term is found by letting n = 1, the second term is found by letting n =2, and so on.

For instance, if it is known that $a_n = n^2$ for any n, then $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, and so on.

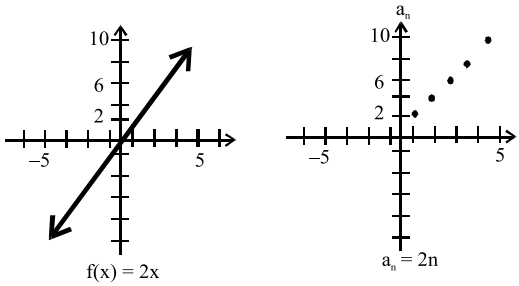
www.pdfworld.net

www.pdfworld.net

$a_n = 2n - 1$ denotes the sequence 1, 2, 3, ... (a sequence of odd integers).

$a_n = \frac{1}{n}$ denotes the sequence 1, $\frac{1}{2}, \frac{1}{3}$, ... (a harmonic sequence).

STUDY TIP The figure shows graphs of $f(x) = 2x$ and $a_n = 2n$. Notice that f(x) defines a "continuous" function, while a_n is discontinuous."



➤ **Example 1.** Write the first five terms for each of the following sequences:

(a) $a_n = \frac{n+1}{n+2}$ (b) $a_n = (-1)^n \cdot n$ (c) $b_n = \frac{(-1)^n}{2^n}$

➤ ***Solution***

(a) Replacing n, in turn, with 1, 2, 3, 4 and 5 gives

$\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$

(b) Replace n with 1, 2, 3, 4 and 5 to get

$a_1 = (-1)^1 \cdot 1 = -1$

$a_2 = (-1)^2 \cdot 2 = 2$

$a_3 = (-1)^3 \cdot 3 = -3$

$a_4 = (-1)^5 \cdot 4 = 4$

$a_5 = (-1)^5 \cdot 1 = -5$

(c) Here, we have $b_1 = -1/2$, $b_2 = 1/4$, $b_3 = -1/8$, $b_4 = 1/16$, and $b_5 = -1/32$.

The formula of the nth term can be more complicated.

For instance, the formula

$$a_n = \begin{cases} n & \text{if} \quad n = 2k \\ \frac{1}{n} & \text{if} \quad n = 2k-1 \end{cases} \quad (k = 1,2.....)$$

specifies the sequence

$$1, 2, \frac{1}{3}, 4, ..., \frac{1}{2k-1}, 2k, \frac{1}{2k+1} ....$$

STUDY TIP The factor $(-1)^n$ in the general term of a sequence causes the signs of the terms to alternate between positive and negative, depending on whether n is even or odd.

If you start with n = 1 and use consecutive integers, the factor $(-1)^n$ will result in terms with the alternating signs. $- + - + - +$ etc.

The factor $(-1)^{n+1}$ will result in terms with the alternating signs $+ - + - + -$ etc.

➤ **Example 2.** Write the first four terms of the sequence whose nth term is

(i) $(-1)^n \left(\frac{3n+2}{5}\right)$ (ii) $\frac{1}{n^2}\sin\left(\frac{n\pi}{3}\right)$

➤ ***Solution***

(i) Let $a_n = (-1)^n \left(\frac{3n+2}{5}\right)$

Putting = 1, 2, 3, 4,..... successively, we get

$a_1 = (-1)^1 \left(\frac{3.1+2}{5}\right) = -1$

$a_2 = (-1)^2 \left(\frac{3.2+2}{5}\right) = 8/5$

$a_3 = (-1)^3 \left(\frac{3.3+2}{5}\right) = -11/5$

$a_4 = (-1)^4 \left(\frac{3.4+2}{5}\right) = 14/5$

.....

Hence we obtain the sequence

$-1, 8/5, -11/5, 14/5, ...$

(ii) Let $a_n = \frac{1}{n^2}\sin\left(\frac{n\pi}{3}\right)$

Putting n = 1, 2, 3, 4,... successively, we get

$a_1 = \frac{1}{1^2}\sin\left(\frac{\pi}{3}\right) - \sqrt{3}/2$

$a_2 = \frac{1}{2^2}\sin\left(\frac{2\pi}{3}\right) - \sqrt{3}/8$

www.pdfworld.net

www.pdfworld.net

$$a_3 = \frac{1}{3^2}\sin\left(\frac{3\pi}{3}\right) = 0$$

$$a_4 = \frac{1}{4^2}\sin\left(\frac{4\pi}{3}\right) = -\sqrt{3}/32$$

.....

Hence we obtain the sequence

$\sqrt{3}/2,\ \sqrt{3}/8,\ 0,\ -\sqrt{3}/32,...$

Progressions

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula of the n^{th} term. Those sequences whose terms follow certain patterns are called **progressions**.

Recursion Formula

Sequences can also be defined using recursion formulae. A recursion formula defines the nth term of a sequence as a function of the previous term(s). A sequence given by a recurrence formula uses the knowledge of the previous terms to find its terms.

Those sequences which are defined explicitly, with a formula for a_n does not depend on any previous term for finding a term.

When a sequence is specified by a recurrence formula, the following is usually indicated:

A. The first term of the sequence (or first several terms);

B. The formula enabling us to determine any term of the sequence from a knowledge of its previous terms.

For example, let us consider the sequence (a_n) whose first term is equal to 1, the second to 2, and each subsequent term, beginning with the third, equals the sum of two previous terms:

$a_1 = 1,\ \ a_2 = 2,\ a_{n+2} = a_n + a_{n+1}$

then

$a_3 = 1 + 2 = 3,\ a_4 = 2 + 3 = 5,\ a_5 = 3 + 5 = 8,$ etc.

Hence, a sequence (a_n) is defined.

It should be noted that it may not be possible to represent a_n by some algebraic formula or by a recurrence formula.

For instance, it is possible to form a sequence of approximate values (with deficit) of the number $\sqrt{2}$

1.4, 1.41, 1.414, 1.4142,.....

or a sequence of prime numbers (in increasing order):

2, 3, 5, 7, 11, 13,.....,

Although in both cases we do not have the formula of the nth term or a recurrence formula. For every sequence a rule must be given, enabling us to obtain any of its terms. The form in which this rule is given is of no importance.

➤ **Example 3.** Find the first four terms for the sequences defined as follows :

(a) $a_1 = 4$; for $n > 1$, $a_n = 2.\ a_{n-1} + 1$

(b) $a_1 = 2$; for $n > 1$, $a_n = a_{n-1} + n - 1$

➤ ***Solution***

(a) This is an example of a recursive definition. Since

$a_1 = 4$, and $a_n = 2.\ a_{n-1} + 1$, we have

$a_2 = 2\ .\ a_1 + 1 = 2\ .\ 4 + 1 = 9$

$a_3 = 2\ .\ a_2 + 1 = 2\ .\ 9 + 1 = 19$

$a_4 = 2\ .\ a_3 + 1 = 2\ .\ 19 + 1 = 39$

(b) $a_1 = 2,\ a_n = a_{n-1} + n - 1$

$a_2 = a_1 + 2 - 1 = 2 + 1 = 3$

$a_3 = a_2 + 3 - 1 = 3 + 2 = 5$

$a_4 = a_3 + 4 - 1 = 5 + 3 = 8$

➤ **Example 4.** A sequence of numbers $a_1, a_2, a_3,...$ satisfies the relation $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$. Find a_4 given $a_1 = a_2 = 1$. Note that this sequence is called Fibonacci sequence.

➤ ***Solution*** Putting $n = 2$ in the given relation, then we obtain $a_3 = a_2 + a_1 = 1 + 1 = 2$

Again putting $n = 3$, then we obtain

$a_4 = a_3 + a_2 \qquad (\because a_3 = 2)$

$= 2 + 1 = 3$

Hence $a_4 = 3$

www.pdfworld.net

www.pdfworld.net

➤ **Example 5.** A sequence of numbers $a_1, a_2, a_3,...$ satisfies the relation $a^2_{n+1} = a_n\, a_{n+2} + (-1)^n$ then find a_3 if $a_1 = 2$ and $a_2 = 5$

➤ ***Solution*** Putting n = 1 in the given relation, we obtain

$a_2^2 = a_1a_3 + (-1)^1$

$5^2 = 2a_3 - 1 \qquad (\because a_1 = 2, a_2 = 5)$

$2a_3 = 26$

$a_3 = 13$

Finding the nth Term of a Sequence

So far we have listed terms in a sequence with the general term given. In contrast, listing the first few terms of a sequence is not enough to say for certain what the general term is. However, we can make a prediction by looking for a pattern. For example, the sequence, 1, 4, 9,16 ..... has terms that are squares of consecutive even integers.

Thus, a possible formula for the general term is $a_n = n^2$. It is important to realize that simply listing the first few terms is not sufficient to define a sequence – the nth term must be given. To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},.....\frac{1}{2^n},.....$$

$$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{15},.....,\frac{6}{(n+1)(n^2-n+6)},.....$$

When given the first few terms of a sequence, you can, however, be asked to find the apparent nth term.

CAUTION Write the first five terms for the following sequences whose general term is given.

A. $a_n = n^2$

B. $a_n = n^2 + (n-1)(n-2)(n-3)(n-4)\left(\frac{\pi - 25}{1.2.3.4}\right)$

What do you observe? Can you see why listing the first few terms of a sequence is not enough to define the general term in only one way ?

If we are given the first few terms of a sequence, the best we can do is to find one possible general term, while realizing that more than one formula for a_n may be correct.

➤ **Example 6.** Find a possible formula for the general term of the sequences:

(i) 3, 5, 7, 9. 11,.....

(ii) –1, 4, –9, 16, –25,.....

(iii) 3, 12, 27, 48, 75,....

➤ ***Solution***

(i) n : 1, 2, 3, 4, 5, ...

Terms : 3, 5, 7, 9, 11, ...

2.1 + 1,2.2 + 12.3 + 12.4 + 12.5 + 1 ...

One pattern is that each term is 1 more than twice n. A possible formula for the general term is $a_n = 2n + 1$

(ii) n : 1, 2, 3, 4, 5, ...

Terms : –1, 4, –9, 16, –25, ...

$(-1)^1 . 1^2, (-1)^2 . 2^2 (-1)^3 . 3^2 \quad (-1)^4 . 4^2$

$(-1)^5 . 5^2$ ...

The terms of the sequence are the squares of consecutive positive integers with alternating signs. A possible formula for the general term is $a_n = (-1)^n n^2$.

(iii) n : 1, 2, 3, 4, 5, ...

Terms : 3, 12, 27, 48, 75, ...

$3 . 1^2,\ 3 . 2^2,\ 3 . 3^2,\ 3 . 4^2,\ 3 . 5^2,$ ...

One pattern is that each term is 3 times the square of n. A possible formula for the general term is $a_n = 3n^2$.

STUDY TIP When we replace n in the general term by 2n – 1, $n \in N$ we generate the odd-positioned terms. Similarly, when we replace n in the general term by 2n, $n \in N$ we generate the even-positioned terms.

Finite and Infinite Sequences

A sequence is said to be finite or infinite according as the number of elements is finite or infinite.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

A sequence is a finite sequence if the domain is the set {1, 2, 3, 4,...., n}, where n is a natural number. An infinite sequence has the set of all natural numbers as its domain.

The sequence of natural number multiples of 2,

2, 4, 6, 8, 10, 12, 14,....., is infinite,

but the sequence of days in the month of June

1, 2, 3, 4, ..., 29, 30 is finite.

Increasing and Decreasing Sequences

A sequence (a_n) is called increasing if each subsequent term (or successor) is greater than its preceding term (or predecessor), i.e., $a_{n+1} > a_n$ for any n.

Examples of increasing sequences :

(i) $1, 4, 9, 16, \ldots\ldots\ldots n^2, \ldots..; \; a_n = n^2$;

(ii) $-2, 0, 2, 4, \ldots\ldots\ldots, 2n-4, \ldots\ldots..; \; a_n = 2n-4$;

(iii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots\ldots, \frac{n}{n+1}, \ldots..; \quad a_n = \frac{n}{n+1}$

A sequence (a_n) is called decreasing if each successor is less than its predecessor,

i.e. $a_{n+1} < a_n$ for any n.

Examples of decreasing sequences :

(i) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots, \frac{1}{n}, \ldots..; \; a_n = \frac{1}{n}$;

(ii) $-1, -2, -3, -4, \ldots., -n, \ldots.$; $a_n = -n$.

Increasing and decreasing sequences are called monotonic. Not every sequence is monotonic.

For instance, the finite sequence 7, –5, –6, 0, 1, the infinite sequence

$1, -1, 1, -1, \ldots., (-1)^{n-1}, \ldots.$; $a_n = (-1)^{n-1}$,

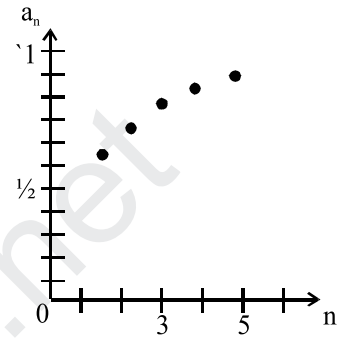
are neither increasing nor decreasing.

Note that a sequence whose terms are equal is said to be constant.

For instance, the sequence 5, 5, 5, ..., 5, ... ; $a_n = 5$ is constant.

Convergent and Divergent Sequences

If the terms of an infinite sequence get closer and closer to some real number, the sequence is said to be **convergent** and to converge to that real number. Graphs of sequences illustrate this property. For instance, the sequence $a_n = \frac{n+1}{n+2}$ is graphed in the figure. We notice that the sequence converges to 1. A sequence that does not converge to some number is **divergent**.



Practice Problems [A]

1. Write the sequence whose nth term is

(i) $3n + 1$ (ii) $n^2 + 1$

(iii) $\frac{n}{n+1}$ (iv) $\frac{n^2}{n+1}$

2. Find the 5th term of the sequence whose first three terms are 3, 3, 6 and each term after the second is the sum of two preceding terms.

3. The Fibonacci sequence is defined by $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$, $n > 2$. Find $\frac{a_{n+1}}{a_n}$, for n = 1, 2, 3, 4, 5.

4. Write the first five terms of the sequences:

(i) $a_n = \left(\frac{1}{3}\right)^n (n-1)$ (ii) $a_n = (-1)^n(2n)$

(iii) $a_n = \frac{4n-1}{n^2+2}$

5. Decide whether the given sequence is finite or infinite.

(i) The sequence of days of the week

www.pdfworld.net

www.pdfworld.net

(ii) 1, 2, 3, 4,.....

(iii) $a_1 = 3$; for $2 \le n \le 10$, $a_n = 3 \cdot a_{n-1}$

6. Find the first four terms for each sequence :

(i) $a_1 = -2$, $a_n = a_{n-1} + 3$, for $n > 1$.

(ii) $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$, for $n \ge 3$

7. Find the third, the sixth and the eighth terms of the sequence

$$a_n = \begin{cases} 2, \text{ where n is an odd number} \\ \dfrac{n+1}{n}, \text{ where n is an even number} \end{cases}$$

8. Write the formula of the nth term for each of the given sequences :

(a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots;$ (b) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\ldots;$

(c) $\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \ldots;$

(d) 1, 4, 9, 16, 25,.....;

(e) $\frac{10}{3}, \frac{20}{9}, \frac{30}{27}, \ldots\ldots;$ (f) $\frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \ldots\ldots;$

3.2 SERIES

Suppose a sequence has terms $a_1, a_2, a_3, \ldots$. Then S_n is defined as the sum of the first n terms. That is, $S_n = a_1 + a_2 + a_3 + \ldots\ldots a_n$. The sum of the first n terms of a sequence is called a **series**. Special notation is used to represent a series. The symbol Σ, the Greek capital letter sigma, is used to indicate a sum.

Thus, the series $a_1 + a_2 + \ldots + a_n$ is abbreviated as $\sum_{i=1}^{n} a_i$ where i is the index of summation, n is the upper limit of summation, and 1 is the lower limit of summation.

When the term series is used, it refers to the indicated sum not to the sum itself. For example, 1 + 3 + 5 + 7 is a finite series with four terms. When we use the phrase "sum of a series," we will mean the number that results from adding the terms, the sum of the series is 16.

➤ Examples:

(i) $1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots + n$

(ii) $2 + 4 + 8 + 16 + \ldots\ldots\ldots\ldots$

A **finite series** is an expression of the form

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{t=1}^{n} a_i$$

and **infinite series** is an expression of the form

$$S_n = a_1 + a_2 + a_3 + \ldots\ldots + a_n + \ldots = \sum_{i=1}^{\infty} a_i$$

the letter i is called the index of summation.

Do not confuse this use of i with the use of i to represent an imaginary number. Other letters may be used for the index of summation.

Using Summation Notation

Polynomial functions, defined by expressions of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0$, can be written in compact form, using summation notation, as

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

➤ **Example 1.** Evaluate the series $\sum_{k=1}^{6} (2^k + 1)$.

➤ ***Solution*** Writing each of the six terms, we can evaluate the sum.

$$\sum_{k=1}^{6} (2^k + 1) = (2^1 + 1) + (2^2 + 1) + (2^3 + 1) + (2^4 + 1) + (2^5 + 1) + (2^6 + 1)$$
$$= (2 + 1) + (4 + 1) + (8 + 1) + (16 + 1) + (32 + 1) + (64 + 1)$$
$$= 3 + 5 + 9 + 17 + 33 + 65 = 132.$$

Using Summation Notation with Subscripts

➤ **Example 2.** Write out the terms for each of the following series and evaluate each sum.

(a) $\sum_{i=1}^{3} (6x_i - 2)$ if $x_1 = 2, x_2 = 4, x_3 = 6$

(b) $\sum_{i=1}^{4} (6x_i - 2)\ f(x_i)\ \Delta x$ if $f(x) = x^2$, $x_1 = 0$, $x_2 = 2$, $x_3 = 4$, $x_4 = 6$ and $\Delta x = 2$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution***

(a) Let i = 1, 2, and 3 respectively, to get

$$\sum_{i=1}^{3}(6x_i - 2) = (6x_1 - 2) + (6x_2 - 2) + (6x_3 - 2)$$

Now, substitute the given values for x_1, x_2 and x_3

$$\sum_{i=1}^{3}(6x_i - 2) = (6 \cdot 2 - 2) + (6 \cdot 4 - 2) + (6 \cdot 6 - 2)$$
$$= 10 + 22 + 34 = 66$$

(b) $$\sum_{i=1}^{4}(6x_i - 2)\ f(x_i)\,\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$
$$= x_1^2\,\Delta x + x_2^2\,\Delta x + x_3^2\,\Delta x + x_1^2\,\Delta x$$
$$= 0^2(2) + 2^2(2) + 4^2(2) + 6^2(2)$$
$$= 0 + 8 + 32 + 72 = 112$$

➤ **Example 3.** Use summation notation to rewrite each series with the index of summation starting at the indicated number :

(a) $\sum_{i=1}^{8}(3i-4);\ 0$ (b) $\sum_{i=2}^{10} i^2;\ -1$

➤ ***Solution*** (a) Let the new index be j. Since the new index is to start at 0, which is j = i – 1, or i = j + 1. Substitute j + 1 for i in the summation.

$$\sum_{i=1}^{8}(3i-4);\ 0 = \sum_{j+1=1}^{j+1=8}[3(j+1)-4] = \sum_{j=0}^{j=7}(3j-1)$$

or $$\sum_{j=0}^{7}(3j-1)$$

(b) Here, if the new index is j, then i = j + 3 and

$$\sum_{i=2}^{10} i^2 = \sum_{j+3=2}^{j+3=10}(j+3)^2 = \sum_{j=-1}^{j=7}(j+3)^2 \text{ or } \sum_{j=-1}^{7}(j+3)^2$$

Finding General Term from Summation Formula

We have $S_n = a_1 + a_2 + a_3 + ... + a_n$

Also $S_{n-1} = a_1 + a_2 + a_3 + ... + a_{n-1}$

Hence, $a_n = S_n - S_{n-1}$

➤ **Example 4.** If the sum of n terms of a series is given by

$S_n = 2n^2 + 3n$, find its 50th term.

➤ ***Solution*** Let a_n be the n^{th} term of the series so $a_n = S_n - S_{n-1}$.

$= 2n^2 + 3n - 2(n-1)^2 - 3(n-1) = 4n + 1$

so the 50th term is $a_{50} = 201$.

Summation Properties

Several properties of summation are given below. These provide useful shortcuts for evaluating series. If $a_1, a_2, a_3, ... a_n$ and $b_1, b_2, b_3, ... b_n$ are two sequences and c is a constant, then for every positive integer n,

(a) $$\sum_{i=1}^{n} c = nc$$

(b) $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

(c) $$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

(d) $$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

To prove Property (a), expand the series to get

$$c + c + c + c + ... + c,$$

where there are n terms of c, so the sum is nc.

Property (c) also can be proved by first expanding the series

$$\sum_{i=1}^{n}(a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n).$$

Now, use the commutative and associative properties to rearrange the terms.

$$\sum_{i=1}^{n}(a_i + b_i) = (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n)$$
$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Proofs of the other two properties are similar.

➤ **Example 5.** Use the properties of series to evaluate

$$\sum_{i=1}^{6}(i^2 + 3i + 5).$$

➤ ***Solution*** $$\sum_{i=1}^{6}(i^2 + 3i + 5) = \sum_{i=1}^{6} i^2 + \sum_{i=1}^{6} 3i + \sum_{i=1}^{6} 5$$
$$= \sum_{i=1}^{6} i^2 + 3\sum_{i=1}^{6} i + \sum_{i=1}^{6} 5$$
$$= \sum_{i=1}^{6} i^2 + 3\sum_{i=1}^{6} i + 6(5)$$

www.pdfworld.net

www.pdfworld.net

$$= \frac{6(6+1)(2.6+1)}{6} + 3\left[\frac{6(6+1)}{2}\right] + 6(5)$$

$$= 91 + 3(21) + 6(5) = 184$$

Writing Sums in Summation Notation

➤ **Example 6.** Write the sum using summation notation

(i) $1 + 8 + 27 + 64 + 125 + 216 + 343$

(ii) $-1 + 8 - 27 + 64 - 125 + 216 - 343$

(iii) $3 + 9 + 27 + 81 + ...$

➤ ***Solution*** (i) $1 + 8 + 27 + 64 + 125 + 216 + 343$

This is the sum of cubes, $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$. Using i for the index of summation, a possible general term is i^3. Thus,

$$1 + 8 + 27 + 64 + 125 + 216 + 343 = \sum_{i=1}^{7} i^3$$

(ii) $-1 + 8 - 27 + 64 - 125 + 216 - 343$

This is almost like the series in part (a) except that the signs alternate. Since $(-1)^i = 1$ when i is even and $(-1)^i = -1$ when i is odd, we write the series in summation notation as follows :

$$-1 + 8 - 27 + 64 - 125 + 216 - 343 = \sum_{i=1}^{7} (-1)^i i^3$$

(iii) $3 + 9 + 27 + 81 + ...$

This infinite series consists of powers of 3. We use the symbol ∞ to represent infinity and write the series in summation notation as follows :

$$3 + 9 + 27 + 81 + ... = \sum_{i=1}^{\infty} 3^i$$

Infinite Series

The expression $a_1 + a_2 + a_3 + ... + a_n + ...$ is called an infinite series. It is, in short, written as $\sum_{i=1}^{\infty} a_i$ where $\sum$ is the notation of summation.

We deduce a sequence (S_n), called the sequence of partial sums, from the series $\sum a_n$ by taking its first term, the sum of its first two terms, the sum of its first three terms, etc., as the 1st, 2nd, 3rd, ... terms of the sequence. Thus

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

.....

$S_n = a_1 + a_2 + a_3 + ... + a_n$

.....

The infinite series, $\sum a_n$ is said to be convergent, divergent or oscillatory according as (S_n) is convergent, divergent or oscillatory.

If in the case of convergence, $S_n \to s$ as $n \to \infty$, where s is finite, we say that $\sum a_n$ is convergent and its sum is equal to s.

If $S_n \to \infty$ as $n \to \infty$, we say that $\sum a_n$ is divergent and diverges to ∞.

If $S_n \to -\infty$ as $n \to \infty$, we say that the series is divergent and diverges to $-\infty$.

The series is said to oscillate finitely and infinitely according as (S_n) oscillates finitely or infinitely.

Let us consider $a_r = \frac{1}{r(r+1)}$.

Here $S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)}$.

We further note that $a_r = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$.

Hence putting r = 1, 2, ... , n we have

$a_1 = 1 - \frac{1}{2}$

$a_2 = \frac{1}{2} - \frac{1}{3}$

$a_3 = \frac{1}{3} - \frac{1}{4}$

.....

$a_n = \frac{1}{n} - \frac{1}{n+1}$

Adding, we get $S_n = 1 - \frac{1}{n+1}$

www.pdfworld.net

www.pdfworld.net

Therefore, $S_n \to 1$ as $n \to \infty$.
The series is, therefore, convergent and its sum is 1.

➤ **Example 7.** Evaluate $\sum_{i=1}^{\infty} a_i$ if $a_r = r$.

➤ ***Solution*** Here $S_n = 1 + 2 + 3 + ... + n = \frac{1}{2} n (n+ 1)$.
Obviously, $S_n \to \infty$ as $n \to \infty$.
Hence Σr is divergent and diverges to ∞.
If $a_r = - r$, it can be easily shown that Σa_n is divergent and diverges to $-\infty$.

➤ **Example 8.** Evaluate $\sum_{i=1}^{\infty} a_i$ if $a_r = (-1)^{r-1}$.

➤ ***Solution*** Here $S_n = 1 - 1 + 1 - 1 + 1 - .....$ to n terms.
Therefore, $S_n = 0$ or 1 according as n is even or odd.
Hence $S_n \to 0$ or 1 as $n \to \infty$.

Moreover $|S_n| \le 1$.
Thus (S_n) oscillates finitely.
Therefore, Σa_n is oscillatory and oscillates finitely.

➤ **Example 9** Evaluate $\sum_{i=1}^{\infty} a_i$ if $a_r = (-1)^{r-1} r$.

➤ ***Solution*** Here $S_n = 1 - 2 + 3 - 4 + 5 - 6 + ...$ to n terms.

Therefore, $S_n = -\frac{n}{2}$ or $\frac{1}{2}(n + 1)$ according as n is even or odd.
Thus $S_n \to \infty$ or ∞ according as n is even or odd.
Hence (S_n) oscillates infinitely.

Practice Problems [B]

1. Evaluate each series :

(i) $\sum_{j=1}^{4} \frac{1}{j}$ (ii) $\sum_{i=1}^{4} i^i$ (iii) $\sum_{k=1}^{6} (-1)^k . k$

2. Evaluate the terms for each sum, where $x_1 = -2, x_2 = -1, x_3 = 0, x_4 = 1$ and $x_5 = 2$:

(i) $\sum_{i=1}^{5} (2x_i + 3)$ (ii) $\sum_{i=2}^{5} \frac{x_i + 1}{x_i + 2}$

3. Use summation notation to rewrite each series with the index of summation starting at the indicated number:

(i) $\sum_{i=1}^{5} (6 - 3i)$; 3 (ii) $\sum_{i=1}^{10} 2(3)^i$; 0

(iii) $\sum_{i=-1}^{9} (i^2 - 2i)$; 0

4. Evaluate

(i) $\sum_{i=1}^{5} (4i^2 - 2i + 6)$ (ii) $\sum_{i=1}^{4} (3i^3 + 2i - 4)$

5. Write each series in summation notation. Use the index i and let i begin at 1.

(i) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... - \frac{1}{128}$

(ii) $\frac{x}{x+1} + \frac{x}{x+2} + \frac{x}{x+3} + \frac{x}{x+4}$

(iii) $8 + 16 + 24 + 32 + 40 + ...$

6. If f is a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.

3.3 ARITHMETIC PROGRESSION (A.P.)

A sequence of numbers each term of which, beginning with the second, is formed by adding a constant number to the predecessor is called an **arithmetic progression**.

If a sequence (a_n) is an arithmetic progression, then, by definition,

$$a_2 - a_1 = a_3 - a_2 = ... = a_{n+1} - a_n = ...,$$

that is, the difference between any term and its predecessor is equal to one and the same number. This constant number is called the **common difference** of an arithmetic progression and is denoted by d.

Thus, an arithmetic progression (a_n) is defined by the following conditions :

www.pdfworld.net

www.pdfworld.net

(i) $a_1 = a$, where a is called the first term

(ii) $a_{n+1} = a_n + d$ for any $n \geq 1$.

If a is the first term and d the common difference, then AP can be written as

$$a, a+d, a+2d, \ldots\ldots a+(n-1)d, \ldots\ldots .$$

Each of the following series forms an arithmetical progression :

3, 7, 11, 15,....

8, 2, –4, –10,.....

p, p – q, p – 2q, p – 3q,.....

The common difference is found by substracting any term of the series from that which follows it. In the first of the above examples the common difference is 4; in the second it is –6; in the third it is – q.

Let (a_n) be an arithmetic progression, a_1 its first term, and d its common difference.

Let us derive the formula for the nth term of an arithmetic progression.

By the definition of the arithmetic progression,

$a_2 = a_1 + d$,
$a_3 = a_2 + d$,
$a_4 = a_3 + d$,
.....
$a_{n-1} = a_{n-2} + d$,
$a_n = a_{n-1} + d$.

Adding these n – 1 equalities termwise, we obtain

$(a_2 + a_3 + a_4 + \ldots\ldots + a_{n-1}) + a_n$
$= a_1 + (a_2 + a_3 + \ldots\ldots + a_{n-2} + a_{n-1}) + (n-1)d$,

$\Rightarrow \quad a_n = a_1 + (n-1)d$.

This formula makes it possible to find any term of an arithmetic progression if its first term and common difference are known. Therefore it is called the formula of the general term of an arithmetic progression. Let us now derive the formula for the sum of (the first) n terms of an arithmetic progression.

Denoting the sum of n terms of an arithmetic progression (a_n) by S_n, we write this sum twice, arranging the terms in the second line in a reverse order :

$S_n = a_1 + a_2 + a_3 + \ldots\ldots + a_{n-1} + a_n$,
$S_n = a_n + a_{n-1} + a_{n-2} + \ldots\ldots + a_2 + a_1$,

Adding these equalities termwise, we obtain

$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2})$
$+ \ldots\ldots + (a_{n-1} + a_2) + (a_n + a_1)$

On the right hand side of the equality the sum of two numbers in each parentheses is equal to $a_1 + a_n$. Indeed,

$a_2 + a_{n-1} = (a_1 + d) + (a_n - d) = a_1 + a_n$,
$a_3 + a_{n-2} = (a_2 + d) + (a_{n-1} - d) = a_2 + a_{n-1}$
$= a_1 + a_n$, etc.

The number of terms in the parentheses is equal to n. Therefore,

$$2S_n = (a_1 + a_n)n, \quad \Rightarrow \quad S_n = \frac{n(a_1 + a_n)}{2},$$

which is the formula for the sum of n terms of an arithmetic progression.

Replacing in this formula the terms a_n by the expression $a_1 + d(n-1)$, we obtain

$$S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$$

This formula expresses the sum of n terms of an arithmetic progression (a_n) in terms of the first term, common difference and the number of terms.

If a is the first term, n be the number of terms, and if ℓ denotes the last, or nth term, we have

$$\ell = a + (n-1)d.$$

$$\therefore \quad S_n = \frac{n}{2}(a + \ell)$$

and $S_n = \frac{n}{2}\{2a + (n-1)d\}$

Here we have three useful formulae; in each of these any one of the letters may denote the unknown quantity when the three others are known.

(i) If a, b, c are in A.P. $\Rightarrow 2b = a + c$.

(ii) The common difference of an A.P. $= \frac{\ell - a}{n-1}$

where a = first term of A.P., ℓ = last term of A.P. and n = number of terms of A.P.

www.pdfworld.net

www.pdfworld.net

The common difference can be zero, positive or negative.

(iii) Three numbers in A.P. can be taken as a – d, a, a + d; five numbers in A.P. can be taken as
a – 2d, a – d, a, a + d, a + 2d.
In general : (2r + 1) numbers in A.P. can be taken as (r ∈ N)
a – rd, a – (r – 1) d, ..., a – d, a, a + d,...., a + (r – 1)d, a + rd

(iv) Four numbers in A.P. can be taken as
a – 3d, a – d, a + d, a + 3d; six numbers in A.P. can be taken as a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d.
In general : 2r numbers in A.P. can be taken as (r ∈ N)

a – (2r – 1) d, a – (2r – 3) d,...., a – 3d, a – d, a + d, a + 3d,....,a + (2r – 3) d, a +(2r – 1) d

(v) If nth term of any sequence is linear expression in n then the sequence is in A.P.

Since $a_n = a_1 + (n-1)d$ we have
$a_n = d\,n + (a_1 - d)$, which is a linear expression in n.

Further if $a_n = pn + q$, then
$a_n - a_{n-1} = pn + q - \{p(n-1) + q\} = p$ (constant)

Note that the coefficient of n represents the common difference.

(vi) Any sum of the form $\sum_{i=1}^{n}(mi + p)$, where m and p are real numbers, represents the sum of the terms of an arithmetic sequence having first term $a_1 = m + p$ and common difference d = m.

(vii) If sum of n terms of any sequence is a quadratic expression in n without the constant term (i.e., $S_n = pn^2 + qn$), then the sequence is in A.P.

Since $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$= \frac{d}{2}n^2 + (a - \frac{d}{2})n$, we observe that S_n is a quadratic expression in n without the constant term.

Further if
$S_n = pn^2 + qn$, then $a_n = S_n - S_{n-1}$
$= pn^2 + qn - \{p(n-1)^2 + q(n-1)\} = 2pn + (q - p)$
which is a linear expression in n and hence the sequence is in A.P. Note that common difference is twice the coefficient of n^2.

(viii) If it is to be determined whether a set of given numbers are terms of an A.P. then let these numbers be the pth, qth and rth terms of an A.P. Then find a relation in p, q, r. If the relation in p,q, r is true for distinct positive integral values of p, q, r, then those numbers are terms of an A.P.

(ix) If number of identical terms in two A.P.'s is to be found, then let r be the number of identical terms in the two A.P.'s. Form the sequence of identical terms, it will be an A.P. Find the rth term of this A.P. make $t_r \leq$ the smaller of the last term of the two A.P.'s and find the positive integral value of r.

➤ **Example 1.** Is 55 a term of the sequence 1, 3, 5, 7,... ? If yes, find which term is it.

➤ ***Solution*** If possible let nth term of the sequence be 55.
Now $t_n = a + (n-1)d$. Here $t_n = 55$, a = 1, d = 2
$\therefore \quad 55 = 1 + (n-1)2$ or $2n = 56 \quad \therefore \; n = 28$
Hence 55 is 28th term of the given sequence.

If n does not come out to be an integer, then 55 will not be term of the given sequence.

➤ **Example 2.** Suppose that an arithmetic sequence has $a_8 = -16$ and $a_{16} = -40$, find a_1

➤ ***Solution*** We must find d first. Since
$a_8 = a_1 + (8-1)d$,
replacing a_8 with –16 gives
$-16 = a_1 + 7d$ or $a_1 = -16 - 7d$.
Similarly, $-40 = a_1 + 15d$ or $a_1 = -40 - 15d$. From these two equations, we get
$-16 - 7d = -40 - 15\,d \Rightarrow d = -3$.
To find a_1, substitute –3 for d in $-16 = a_1 + 7d$
$-16 = a_1 + 7(-3) \Rightarrow a_1 = 5$

www.pdfworld.net

www.pdfworld.net

➤ **Example 3.** A sequence is given by the formula of its nth term: $a_n = 10 - 3n$. prove that a_n is an arithmetic progression.

➤ ***Solution*** We find $a_{n+1} = 10 - 3(n + 1) = 7 - 3n$ and consider the difference

$a_{n+1} - a_n = 7 - 3n - (10 - 3n) = -3$ and $d = -3$.

➤ **Example 4.** If nth term of a sequence is $2n^2 + 1$, find the sequence. Is this sequence an A.P. ?

➤ ***Solution*** Given $t_n = 2n^2 + 1$

Putting $n = 1$, we get $t_1 = 2.1^2 + 1 = 3$

Putting $n = 2$, we get $t_2 = 2.2^2 + 1 = 9$

Putting $n = 3$, we get $t_3 = 2.3^2 + 1 = 19$

Putting $n = 4$, we get $t_4 = 2.4^2 + 1 = 33$

and so on.

Hence given sequence is 3, 9, 19, 33....

$t_{n-1} = 2(n - 1)^2 + 1$

From (i) and (ii), we get

$t_n - t_{n-1} = 2n^2 + 1 - \{2(n - 1)^2 + 1)$

$= 2n^2 + 1 - \{2(n^2 - 2n + 1) + 1\} = 4n - 2$,

which is not independent of n i.e. it is not a constant, hence the given sequence is not an A.P.

➤ **Example 5.** Let $a_n = n^2 + 1$ and b_n is defined

$b_n = a_{n+1} - a_n$.

Show that $\{b_n\}$ is an arithmetic sequence.

➤ ***Solution*** The first formula determine the series :

$2 + 5 + 10 + 17 + 26 + ...$

The second determines the series :

$(5 - 2) + (10 - 5) + (17 - 10) + ...$

or $3 + 5 + 7 + 9 + ...$ which is clearly an A.P.

Alternatively,

$b_n = a_{n+1} - a_n = [(n + 1)^2 + 1] - (n^2 + 1) = 2n + 1$

which is a linear polynomial in n.

➤ **Example 6.** If the numbers a^2, b^2, c^2 form an arithmetic progression prove that the numbers $\frac{1}{b+c}$, $\frac{1}{a+c}$, $\frac{1}{a+b}$ also form an arithmetic progression.

➤ ***Solution*** Given $b^2 - a^2 = c^2 - b^2$, Consider

$$\frac{1}{a+c} - \frac{1}{b+c} = \frac{b-a}{(a+c)(b+c)}$$

$$= \frac{b^2 - a^2}{(a+c)(b+c)(a+b)}$$

$$\frac{1}{a+b} - \frac{1}{a+c} = \frac{c-b}{(a+b)(a+c)}$$

$$= \frac{c^2 - b^2}{(a+b)(a+c)(b+c)}$$

Hence it follows that

$$\frac{1}{a+c} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{a+c} \text{ and}$$

consequently, the numbers $\frac{1}{b+c}$, $\frac{1}{a+c}$, $\frac{1}{a+b}$ form an arithmetic progression.

➤ **Example 7.** Prove that if the numbers $\log_k x$, $\log_m x$ and $\log_n x$ $(x \neq 1)$ form an arithmetic progression then $n^2 = (kn)^{\log_k m}$.

➤ ***Solution*** $2 \log_m x = \log_n x + \log_k x$

$$\Rightarrow \frac{2}{\log_x m} = \frac{1}{\log_x n} + \frac{1}{\log_x k} \text{ and consequently,}$$

$$\Rightarrow 2 = \frac{\log_x m}{\log_x n} + \frac{\log_x m}{\log_x k}$$

$$\Rightarrow 2 = \log_n m + \log_k m$$

Let us rewrite this equality as

$$\log_n n^2 = \log_n m + \log_n \left(n^{\log_k m}\right)$$

Now, raising we obtain

$$n^2 = mn^{\log_k m} \text{ or } n^2 = ((kn)^{\log_k m}$$

➤ **Example 8.** For what values of the parameter a are there real values of x such that

$$5^{1+x} + 5^{1-x}, \frac{a}{2}, 25^x + 25^{-x}$$

are three consecutive terms of an AP?

➤ ***Solution*** We should have

$$2 \cdot \frac{a}{2} = (5^{1+x} + 5^{1-x}) + (25^x + 25^{-x})$$

$$\Rightarrow a = 5(5^x + 5^{-x}) + (5^{2x} + 5^{-2x})^2 = 5\{(5^{x/2} - 5^{-x/2})^2 + 2.\ 5^{x/2} .\ 5^{-x/2}\} + \{(5^x - 5^{-x})^2 + 2.5^x .\ 5^{-x}\}$$

$$= 5\{(5^{x/2} - 5^{-x/2})^2 + 2\} + (5^x - 5^{-x})^2 + 2$$

$$\therefore \quad a \geq 5.2 + 2 = 12$$

www.pdfworld.net

www.pdfworld.net

➤ **Example 9** Show that cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an A.P.

➤ ***Solution*** If possible let $p^{1/3}\, q^{1/3}$, $r^{1/3}$ be three terms of an A.P. where p, q, r are three distinct prime numbers.

Then $p^{1/3} = a$, $q^{1/3} = a + md$, $r^{1/3} = a + nd$; where m, n must be positive integers. On subtracting these equations pairwise and dividing, we get

$$\frac{m}{n} = \frac{q^{1/3} - p^{1/3}}{r^{1/3} - p^{1/3}} \Rightarrow mr^{1/3} - nq^{1/3} = (m-n)p^{1/3}$$

$\Rightarrow$ $(mr^{1/3} - nq^{1/3})^3 = (m-n)^3\, p$

$\Rightarrow$ $m^3r - n^3q - 3mn\, r^{1/3}q^{1/3}\,(m-n)p^{1/3} = (m-n)^3p$

$$\Rightarrow \quad p^{1/3}q^{1/3}r^{1/3} = \frac{m^3r - n^3q - (m-n)^3 p}{3mn(m-n)}$$

$\Rightarrow$ irrational number = rational number ($\because$ p, q, r are distinct prime, $p^{1/3}q^{1/3}r^{1/3}$ cannot become rational) which is a contradiction.

➤ **Example 10.** When we divide the ninth term of an arithmetic progression by its second term, we get 5 as a quotient, and when we divide the thirteenth term of that progression by the sixth term, we get 2 as a quotient and 5 as a remainder. Find the first term and the common difference.

➤ ***Solution*** From the given conditions :

$a_9 = a_2 \,.\, 5 \quad \Rightarrow \quad a_1 + 8d = (a_1 + d)\,.\,5,$

$a_{13} = 2a_6 + 5 \Rightarrow \quad a_1 + 12\,d = 2(a_1 + 5d) + 5$

containing only two unknowns, a_1 and d whose is

$a_1 = 3,\ d = 4.$

➤ **Example 11.** The sum of three numbers in A.P. is 27 and the sum of their squares is 293. Find the numbers.

➤ ***Solution*** Let the three numbers in A.P. be $\alpha - \beta$, α, $\alpha+\beta$.

Given, $\alpha - \beta + \alpha + \alpha + \beta = 27$

or $\quad 3\alpha = 27 \qquad \therefore\ \alpha = 9$

and $(\alpha - \beta)^2 + \alpha^2 + (\alpha + \beta)^2 = 293$

or $\quad \alpha^2 + \beta^2 - 2\alpha\beta + \alpha^2 + \alpha^2 + \beta^2 + 2\alpha\beta = 293.$

or $\quad 3\alpha^2 + 2\beta^2 = 293 \qquad$ or $\qquad 3.9^2 + 2\beta^2 = 293$

or $\quad 2\beta^2 = 293 - 243 = 50$ or $\ \beta^2 = 25$

$\therefore \quad \beta = \pm 5$

If $\quad \beta = -5$, the three numbers are 14, 9, 4

If $\quad \beta = 5$, the three numbers are 4, 9, 14.

➤ **Example 12.** Find four numbers in A.P. such that their sum is 50 and the greatest of them is 4 times the least.

➤ ***Solution*** Let the four numbers in A.P. be

$\alpha - 3\beta,\ \alpha - \beta\ ,\ \alpha + \beta,\ \alpha + 3\beta.$

Given, $\alpha - 3\beta + \alpha - \beta + \alpha + \beta + \alpha + 3\beta = 50$

or $\quad 4\alpha = 50 \qquad \therefore\ \alpha = \frac{25}{2}$

and $\quad \alpha + 3\beta = 4(\alpha - 3\beta)$ or $\qquad 3\alpha = 15\,\beta$

or $\quad \beta = \frac{\alpha}{5} = \frac{25}{5 \times 2} = \frac{5}{2}$

Hence the four numbers are 5, 10, 15, 20.

➤ **Example 13.** How many terms are identical in the two arithmetic progressions 2, 4, 6, 8,.... up to 100 terms and 3, 6, 9,... up to 80 terms.

➤ ***Solution*** Let r terms be identical.

Now the sequence of identical terms is 6, 12, 18, ...

Its rth term = 6 + (r – 1)6 = 6r

The 100th term of the sequence 2, 4, 6, 8, ...

= 2 + (100 – 1)2 = 200

and 80th term of the sequence 3, 6, 9, ...

= 3 + (80 – 1)3 = 240

Since, last term i.e. rth term of the sequence of identical terms cannot be greater than 200.

$\therefore \quad 6r \le 200$ or $\ r \le \frac{200}{6} \qquad$ or $\ r \le 33\frac{1}{3}$

$\therefore \quad r = 33$. Hence 33 terms are identical.

If we get $r \le k$, then $r = k$, if k is an integer; and r = integer just less than k if k is not an integer.

➤ **Example 14.** (a) Evaluate S_{12} for the arithmetic sequence –9, –5, –1, 3, 7 ,.....

www.pdfworld.net

www.pdfworld.net

(b) Use a formula for S_n to evaluate the sum of the first 60 positive integers.

➤ ***Solution***

(a) We want the sum of the first 12 terms. Using $a_1 = -9$, $n = 12$, and $d = 4$ in the formula

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$S_{12} = \frac{12}{2}[2(-9) + 11(4)] = 156$$

(b) Here $n = 60$, $a_1 = 1$ and $a_{60} = 60$, so it is convenient to use the formula

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{60} = \frac{60}{2}(1 + 60) = 1830$$

➤ **Example 15.** Find the sum of n terms of the series

$$\frac{2a^2 - 1}{a},\ 4a - \frac{3}{a},\ \frac{6a^2 - 5}{a}, \ldots..$$

➤ ***Solution*** $u_1 = \dfrac{2a^2 - 1}{a}$, $u_2 = 4a - \dfrac{3}{a}$.

$\therefore$ Common difference

$$= \left(4a - \frac{3}{a}\right) - \left(2a - \frac{1}{a}\right) = 2\left(a - \frac{1}{a}\right) = 2\frac{a^2 - 1}{a}.$$

$$\therefore \quad S_n = \frac{n}{2}\left\{2 \cdot \frac{2a^2 - 1}{a} + (n-1)\frac{2(a^2 - 1)}{a}\right\}$$

➤ **Example 16.** If the sum of n terms of a series be $5n^2 + 3n$, find its nth term. Are the terms of this series in A.P.?

➤ ***Solution*** Given $S_n = 5n^2 + 3n$.

$\therefore$ $S_{n-1} = 5(n-1)^2 + 3(n-1)$ [Putting n–1 in place of n]

$\because$ $S_n = (t_1 + t_2 + \ldots + t_{n-1}) + t_n = S_{n-1} + t_n$

$\therefore$ $t_n = S_n - S_{n-1}$

$= 5n^2 + 3n - 5(n-1)^2 - 3(n-1)$

$= 5[n^2 - (n-1)^2] + 3[n - n + 1]$

$= 5(2n - 1) + 3 = 10n - 2$

$\therefore$ $t_1 = 10.1 - 2 = 8$ [Putting n = 1]

$t_2 = 10.2 - 2 = 18$ [Putting n = 2]

$t_3 = 10.1 - 2 = 28$ [Putting n = 3]

and so on.

Thus terms of the given series are 8, 18, 28.... which are in A.P. whose c.d. is 10.

➤ **Example 17.** Find $1 - 3 + 5 - 7 + 9 - 11 + \ldots.$ to n terms.

➤ ***Solution*** Here alternate terms of the series are in A.P.

Case I : When n is even, let $n = 2m$

$1 - 3 + 5 - 7 + 9 - 11 + \ldots$ to n terms.

$= 1 - 3 + 5 - 7 + 9 - 11 + \ldots$ to 2m terms.

$= [1 + 5 + 9 + \ldots$ to m terms$] - [3 + 7 + 11 + \ldots$ to m terms$]$

$$= \frac{m}{2}\{2.1 + (m-1)4\} - \frac{m}{2}\{2.3 + (m-1)4\}$$

$$= m(2m - 1 - 2m - 1) = -2m = -n$$

Case II : When n is odd, let $n = 2m + 1$

$1 - 3 + 5 - 7 + 9 - 11 + \ldots$ to n terms.

$= 1 - 3 + 5 - 7 + 9 - 11 + \ldots$ to $2m + 1$ terms.

$= [1 + 5 + 9 + \ldots$ to $(m + 1)$ terms$] - [3 + 7 + 11 + \ldots$ to m terms$]$

$$= \frac{m+1}{2}\{2.1 + (m+1-1)4\} - \frac{m}{2}\{2.3 + (m-1)4\}$$

$$= \frac{m+1}{2}(4m + 2) - \frac{m}{2}(4m + 2)$$

$= (2m + 1)(m + 1 - m) = 2m + 1 = n$

$[\because n = 2m + 1]$

Second Method :

Case I : When $n = 2m$

$1 - 3 + 5 - 7 + 9 - 11 + \ldots\ldots$ to n terms.

$= 1 - 3 + 5 - 7 + 9 - 11 + \ldots..$ to 2m terms.

$= (1 - 3) + (5 - 7) + (9 - 11) + \ldots.$ to m brackets

$= -2 - 2 - 2 - \ldots$ to m terms $= -2m = -n[\because n = 2m]$

Case II : When $n = 2m + 1$

$1 - 3 + 5 - 7 + 9 - 11 + \ldots\ldots$ to n terms

$= 1 - 3 + 5 - 7 + 9 - 11 + \ldots.$ to $(2m + 1)$ terms.

$= 1 - [3 - 5 + 7 - 9 + \ldots\ldots$ to 2m terms$]$

$= 1 - [(3 - 5) + (7 - 9) + \ldots..$ to m brackets$]$

$= 1 - [-2 - 2 \ldots$ to m terms$] = 1 - (-2m) = 2m + 1 = n$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

➤ **Example 18.** Let a_n be the n^{th} term of an arithmetic progression. Let S_n be the sum of the first n terms of the arithmetic progression with $a_1 = 1$ and $a_3 = 3a_8$. Find the largest possible value of S_n.

➤ ***Solution*** From $a_3 = 3a_8$ we obtain

$1 + 2d = 3(1 + 7d)$

$\Rightarrow \quad d = -\frac{2}{19}$.

Then $S_n = \frac{n}{2}\left(2-(n-1)\frac{2}{19}\right)$

$= \frac{n}{19}[19-(n-1)] = \frac{(20-n)n}{19}$.

Now consider $20n-n^2 = -[n^2-20n]$

$= -[(n-10)^2 - 100]$

$\therefore \quad S_n = \frac{100-(n-10)^2}{19}$

now, S_n will be maximum if n = 10 and

$(S_n)_{max} = \frac{100}{19}$

➤ **Example 19.** The sum of n terms of two A.P.'s are in the ratio (3n – 13) : (5n + 21). Find the ratio of their 24th terms.

➤ ***Solution*** Let the two A.P.'s be a, a + d, a + 2d, ...and a′, a′ + d′, a′ + 2d′,.....

Let their sum be S_n and S'_n respectively.

$\therefore \quad S_n = \frac{n}{2}[2a + (n-1)d]$

and $S'_n = \frac{n}{2}[2a' + (n-1)d']$

$\because \quad \frac{S_n}{S'_n} = \frac{3n-13}{5n+21}$

$\therefore \quad \frac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}[2a'+(n-1)d']} = \frac{3n-13}{5n+21}$

$\Rightarrow \quad \frac{a+\left(\frac{n-1}{2}\right)d}{a'+\left(\frac{n-1}{2}\right)d'} = \frac{3n-13}{3n+21}$ ...(1)

The 24th terms of two A.P.'s are respectively,

$T_{24} = a + 23d$ and $T'_{24} = a' + 23d'$

$\therefore \quad \frac{T_{24}}{T'_{24}} = \frac{a+23d}{a'+23d'}$ ...(2)

Comparing (1) and (2), we get

$\therefore \quad n = 47$

from (1), $\frac{a+23d}{a'+23d} = \frac{3\times47-13}{3\times47+21} = \frac{1}{2}$

$\therefore \quad \frac{T_{24}}{T'_{24}} = \frac{1}{2} \quad \Rightarrow \quad T_{24} : T'_{24} = 1 : 2$

➤ **Example 20.** Suppose $a_1, a_2, \ldots$ are in A.P. and S_k denotes the sum of the first k terms of this A.P. If $S_n/S_m = n^4/m^4$ for all m, n, $\in$ **N**, then prove that

$$\frac{a_{m+1}}{a_{n+1}} = \frac{(2m+1)^3}{(2n+1)^3}$$

➤ ***Solution*** Putting $a_1 = a$, we have

$$\frac{S_n}{S_m} = \frac{n[2a+(n-1)d]/2}{m[2a+(n-1)d]/2} = \frac{n^4}{m^4}$$

$$\Rightarrow \quad \frac{2a+(n-1)d}{2a+(m-1)d} = \frac{n^3}{m^3}$$

Replacing n by 2n + 1 and m by 2m + 1, we get

$$\frac{2a+(2n+1-1)d}{2a+(m+1-1)d} = \frac{(2n+1)^3}{(2m+1)^3}$$

$$\Rightarrow \quad \frac{a_{n+1}}{a_{m+1}} = \frac{(2n+1)^3}{(2m+1)^3} \Rightarrow \frac{a_{m+1}}{a_{n+1}} = \frac{(2m+1)^3}{(2n+1)^3}$$

Practice Problems [C]

1. Can the numbers 2, $\sqrt{6}$, 4.5 be terms of an arithmetic progression?
2. The lengths of the sides of a quadrilateral form an arithmetic progression. Can a circle be inscribed into the quadrilateral?
3. Divide 20 into four parts which are in A.P. such that the product of the first and fourth is to the product of the second and third is 2 : 3.
4. The sum of four integers in A.P. is 24, and their product is 945; find them.

www.pdfworld.net

www.pdfworld.net

5. If length of sides of a right angled triangle are in A.P., show that their ratio is 3 : 4 : 5.

6. Sum $\frac{6}{\sqrt{3}}, 3\sqrt{3}, \frac{12}{\sqrt{3}}$,.... to 50 terms.

7. Solve

$\log_{\sqrt{3}} x + \log_{\sqrt[4]{3}} x + \log_{\sqrt[6]{3}} x + ... + \log_{\sqrt[16]{3}} x = 36$

8. The sum of 15 terms of an A.P. is 600, and the common difference is 5; find the first term.

9. Find the arithmetic progression if

$a_1 = 1$ and $S_5 = \frac{1}{4}(S_{10} - S_5)$

10. The third term of an A.P. is 18, and the seventh term is 30; find the sum of 17 terms.

11. How many terms of the arithmetic progression 18, 16, 14,... should be taken so that their sum be equal to zero?

12. How many terms of the series $15 + 12 + 9 + ...$ must be taken to make 15 ? Explain the double answer.

13. The sum of n terms of a series is $2n^2 + 3n$. Is the series arithmetic? If so, find it .

14. If the sum of n terms of an A.P. is $2n + 3n^2$, find the rth term.

15. Find the sequence in which the sum of any number of terms, beginning with the first, is four times as large as the square of the number of terms.

16. Find a three digit number which is divisible by 45 and whose digits are terms of an arithmetic progression.

17. The sum of n terms of two arithmetic series are in the ratio of $7n + 1 : 4n + 27$; find the ratio of their 11^{th} terms.

3.4 PROPERTIES OF A.P.

An arithmetic progression (a_n) possesses the following characteristic property : any of its terms, beginning with the second, is an arithmetic mean of the predecessor and successor.

***Proof*:**

By the definition of the arithmetic progression,

$a_{n+1} = a_n + d$; $a_{n+2} = a_{n+1} + d$,

$\Rightarrow a_{n+1} - a_n = a_{n+2} - a_{n+1}$

Hence, $a_{n+1} = \frac{a_n + a_{n+2}}{2}$

The converse is also true : if a sequence is such that any of its terms, beginning with the second, is an arithmetic mean of the predecessor and successor, then this sequence is an arithmetic progression. Indeed, let for any three neighbouring terms of a certain sequence (a_n) the following relationship be fulfilled:

$$a_{n+1} = \frac{a_n + a_{n+2}}{2}$$

Then $2a_{n+1} = a_n + a_{n+2}$, or $a_{n+1} - a_n = a_{n+2} - a_{n+1}$, that is, the difference between any term of the sequence (a_n) and its predecessor is equal to one and the same number. Hence (a_n) is an arithmetic progression.

Thus, the established property is inherent in an arithmetic progression and only in it.

Important Properties of A.P.

(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP.

If $a_1, a_2, a_3, ...$ are in A.P. then

(a) $a_1 \pm k, a_2 \pm k, a_3 \pm k$ ..... are in A.P.

(b) a_1k, a_2k, a_3k ..... are in A.P.

(c) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}$ ..... are in A.P.

(d) a_p, a_{p+q}, a_{p+2q} ..... are in A.P.

(e) $a_1^r, a_2^r, a_3^r, ..., a_n^r$ are not in A.P. if $r \neq 0, 1$

(ii) The sum of the terms of an AP equidistant from the beginning and the end is constant and equal to the sum of the first and last terms. If

www.pdfworld.net

www.pdfworld.net

$a_1, a_2, a_3, ..., a_n$ are in A.P., then
$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ...$

(iii) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it. If $a_1, a_2, a_3, ..., a_n$ are in A.P., then

$$a_r = \frac{a_{r-k} + a_{r+k}}{2} \quad \forall\, k,\ 0 \le k \le n - r$$

(iv) If a_1, a_2, a_3 ..... and b_1, b_2, b_3,..... are two A.P.'s, then

(a) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3$ ..... are in A.P.

(b) a_1b_1, a_2b_2, a_3b_3,..... are not in A.P.

If c_1 and d_1 be the common difference of the two sequences respectively, then for the new sequence, we have

$t_n = \{a_1 + (n-1)c_1\}\{b_1 + (n-1)d_1\}$
$= a_1b_1 + (a_1d_1 + b_1c_1)(n-1) + c_1d_1(n-1)^2$
which is not a linear expression in n, and hence, is not an A.P.

(c) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$,..... are also not in A.P.

➤ **Example 1** The successive terms of an A.P. are a_1, a_2, a_3,..... Find S_{20} if $a_6 + a_9 + a_{12} + a_{15} = 20$.

➤ ***Solution*** Noting that a_6 and a_{15} are equidistant from the ends also a_9 and a_{12} and that the sum of 2 terms equidistant from the ends is the same, we have

$$a_6 + a_{15} = a_9 + a_{12} = \frac{20}{2} = 10$$

$\therefore \quad a_1 + a_{20} = 10$

$\therefore \quad S_{20} = \frac{20}{2}(a + l) = 10(a_1 + a_{20}) = 10 \times 10 = 100$

Practice Problems [D]

1. If a, b, c be in A.P., prove that

(i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

(ii) b + c, c + a, a + b are in A.P.

(iii) $a^2(b + c), b^2(c + a), c^2(a + b)$ are in A.P.

(iv) $\frac{ab + ac}{bc}, \frac{bc + ba}{ca}, \frac{ca + bc}{ab}$ are in A.P.

2. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P. show that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. provided $a + b + c \neq 0$

3. Prove that for any arithmetic progression the following equalities hold true :

(i) $a_1 - 2a_2 + a_3 = 0$

(ii) $a_1 - 3a_2 + 3a_3 - a_4 = 0$

(iii) $a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 = 0$

4. Prove that for the terms of an arithmetic progression the equality $a_p + a_m = a_{p+k} + a_{m-k}$ is valid.

5. Find the arithmetic progression if it is known that $a_1 + a_3 + a_5 = -12$, $a_1a_2a_3 = 80$.

6. The sum of three numbers which are consecutive terms of an arithmetic progression is 2, and the sum of the squares of those numbers is 14/9. Find the numbers.

7. The first term of an A.P. a_1, a_2, a_3,....., a_n is unity. For what value of the common difference is $a_1a_3 + a_2a_3$ minimum?

8. Suppose a_1, a_2, ..., a_n are in A.P. with first term $a_1 = 0$ and common difference $d \neq 0$. Show that

$$\frac{a_3}{a_2} + \frac{a_4}{a_3} + \frac{a_5}{a_4} + ... \frac{a_n}{a_{n-1}}$$
$$- d\left(\frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$$

3.5 GEOMETRIC PROGRESSION (G.P.)

A number sequence whose first term is different from zero and each term, beginning with the second, is equal to its predecessor multiplied by one and the same number is called a geometric progression.

If a sequence (a_n) is a geometric progression, then, by definition,

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = ... = \frac{a_n}{a_{n-1}} = \frac{a_{n+1}}{a_n} = ...,$$

www.pdfworld.net

www.pdfworld.net

that is, the ratio of any term to its predecessor is equal to one and the same number. This number is called the common ratio of a geometric progression and is denoted by r.

Each of the following series forms a geometric progression

3, 6, 12, 24,.....

$1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, ...$

$p, pq^2, pq^4, pq^6, ...$

The constant factor is also called the common ratio, and it is found by dividing any term by that which immediately preceedes it. In the first of the above examples the common ratio is 2, in the second it is $\frac{1}{3}$; in the third it is q^2.

Thus, a geometric progression (a_n) is defined by the following conditions:

(i) $a_1 = a$ where a is called the first term $(a \neq 0)$;

(ii) $a_{n+1} = a_n r$ for any $n \geq 1$

If a is the first term and r the common ratio, then G.P. can be written as

$a, ar, ar^2, ar^3, ar^4, ..., ar^{n-1}, ...$

If, for instance, a = 1 and r = 2, then we have a geometric progression 1, 2, 4, 8, ...

The conditions a = 4 and $r = -\frac{1}{2}$ define the geometric progression $4, -2, 1, -\frac{1}{2}, ...$

Note that all the definitions and formulas in the theory of progressions (both arithmetic and geometric) remain valid when the terms of the progressions are complex numbers, but in most problems involving progressions it is assumed (if not otherwise stated) that the terms of the progressions are real numbers).

Let (a_n) be a geometric progression, a its first term, and r the ratio of the progression. Let us derive the formula for the nth term of a geometric progression. By the definition of the geometric progression,

$a_2 = ar,$
$a_3 = a_2 r,$
$a_4 = a_3 r,$
.....
$a_{n-1} = a_{n-2} r,$
$a_n = a_{n-1} r,$

Multiplying these n – 1 equalities termwise, we obtain.

$(a_2 a_3 a_4 .... a_{n-1}) a_n = (a_2 a_3 .... a_{n-2} a_{n-1}) ar^{n-1}.$

After reduction we have $a_n = ar^{n-1}$.

If n be the number of terms, and if ℓ denotes the last, or the n^{th} term, we have $\ell = ar^{n-1}$

This formula enables us to find any term of a geometric progression if its first term and common ratio are known. Therefore it is called the formula of the general term of a geometric progression.

STUDY TIP Did you notice that the exponent on r is 1 less than the subscript of a denoting the term number?

Thus, the formula for the nth term is

$a_n = ar^{n-1}$

1 less than the subscript of a

Let us now derive the formula for the **sum of n terms of a geometric progression**.

We denote the sum of n terms of a geometric progression by S_n .
Let a be the first term, r the common ratio, n the number of terms, and s the sum required.

Then $S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1};$
If the common ratio of the progression r is equal to 1, then $S_n = na_1$.

But if $r \neq 1$, then we proceed as follows: we multiply the above equality termwise by r:

$rS_n = ar + ar^2 + ... + ar^{n-2} + ar^{n-1} + ar^n.$

Hence by subtraction,

$rS_n - S_n = ar^n - a;$

$\therefore \quad (r-1) S_n = a(r^n - 1);$

$\therefore \quad S_n = \frac{a(r^n - 1)}{r - 1}$ ...(1)

www.pdfworld.net

www.pdfworld.net

Changing the signs in numerator and denominator,

$$S_n = \frac{a(1-r^n)}{1-r} \qquad ...(2)$$

It will be convenient to remember both forms given above for S_n, using (2) in all cases except when r is positive and greater than 1.

Since $ar^{n-1} = \ell$, the formula (1) may be written

$$S_n = \frac{r\ell - a}{r-1};$$ a form which is sometimes useful.

(i) If a, b, c are in G.P. $\Rightarrow b^2 = ac$

(ii) The common difference of a G.P. $= \left(\frac{\ell}{a}\right)^{\frac{1}{n-1}}$

where a = first term of G.P., ℓ = last term of G.P. and n = number of terms of G.P.

The common ratio can be zero, positive or negative.

(iii) Three numbers in G.P. can be taken as $\frac{a}{r}$, a, ar;

five numbers in G.P. can be taken as $\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2. In general (2m + 1) numbers in G.P. can be written as (m ∈ N)

$$\frac{a}{r^m}, \frac{a}{r^{m-1}}, ..., \frac{a}{r}, a, ar, ..., ar^{m-1}, ar^m$$

(iv) Four numbers in G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3; six numbers in G.P. can be taken as

$$\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5.$$

In general : (2m) numbers in G.P. can be written as (m ∈ N)

$$\frac{a}{r^{2m-1}}, \frac{a}{r^{2m-3}}, ..., \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, .., ar^{2m-3}, ar^{2m-1}$$

➤ **Example 1.** Find a_5 and a_n for the following geometric sequence

4, 12, 36, 108,....

➤ ***Solution*** The first term, a_1 is 4. Find r by choosing any term except the first and dividing it by the preceeding term. For example, $r = \frac{36}{12} = 3$

Since $a_4 = 108$, $a_5 = 3.\ 108 = 324$. The fifth term also could be found by using the formula for $a_n.a_n = a_1 r^{n-1}$, and replacing n with 4, r with 3, and a_1 with 4.

$$a_5 = 4\ .\ (3)^{5-1} = 4\ .\ 3^4 = 324$$

by the formula, $a_n = 4.\ 3^{n-1}$.

➤ **Example 2.** If a, b, c, d be in G.P., show that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$$

➤ ***Solution*** Let r be the common ratio of the given G.P., then

$$b = ar,\ c = ar^2 \text{ and } d = ar^3$$

Now L.H.S. $= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$

$= (1 + r^2 + r^4)\ .\ a^2r^2\ (1 + r^2 + r^4)$

$= a^4r^2\ (1 + r^2 + r^4)^2 = \{a^2r(1 + r^2 + r^4)\}^2$

$= (a^2r + a^2r^3 + a^2r^5)^2$

$= (a.\ ar + ar.\ ar^2 + ar^2\ .\ ar^3)^2 = (ab + bc + cd)^2$

➤ **Example 3.** Find all sequences which are simultaneously an arithmetic and a geometric progression.

➤ ***Solution*** Let the numbers $a_1, a_2, ... a_n, ...$ form an arithmetic progression. Then

$$a_{n+1} = \frac{a_n + a_{n+2}}{2} \quad (n \geq 1).$$

Since (a_n) is a geometric progression, applying the formula of the general form $a_n = a_1 r^{n-1}$, where $a_1 \neq 0$, we obtain

$$a_1 r^n = \frac{a_1 r^{n-1} + a_1 r^{n+1}}{2}$$

or, on reducing by $ar^{n-1} \neq 0$

$$r = \frac{1 + r^2}{2},$$

i.e. $(r - 1)^2 = 0$. Hence, r = 1 and the given sequence is a sequence of equal numbers $a_1, a_1, ..a_1$.

Is the sequence 1, 1, 1, ...,1,.... an arithmetic or a geometric progression? This sequence can actually

www.pdfworld.net

www.pdfworld.net

be regarded as an arithmetic progression (with common difference 0) or as a geometric progression (with common ratio 1).

➤ **Example 4.** Show that 10, 11, 12 cannot be terms of a G.P.

➤ ***Solution*** If possible, let 10, 11 and 12 be the pth, qth and kth terms of a G.P. whose first term is a and common ratio is r.

Now $10 = ar^{p-1}$ ...(1)

$11 = ar^{q-1}$ ...(2)

$12 = ar^{k-1}$ ...(3)

$\therefore \quad \frac{11}{10} = r^{q-p}$ ...(4)

and $\frac{12}{11} = r^{k-q}$ ...(5)

From (4), $\left(\frac{11}{10}\right)^{k-q} = r^{(q-p)(k-q)}$ ...(6)

From (5), $\left(\frac{12}{11}\right)^{q-p} = r^{(k-q)(q-p)}$ ...(7)

From (6) and (7), $\left(\frac{11}{10}\right)^{k-q} = \left(\frac{12}{11}\right)^{q-p}$

$\therefore \quad (11)^{k-q+q-p} = (10)^{k-q}\,(12)^{q-p}$

$\Rightarrow \quad 11^{k-p} = 5^{k-q} \cdot 2^{k-q} \cdot 2^{2(q-p)} \cdot 3^{q-p}$

$\Rightarrow \quad 11^{k-p} = 5^{k-q} \cdot 2^{k+q-2p} \cdot 3^{q-p}$

This is possible only when $k - p = 0$, $k - q = 0$, $k + q - 2p = 0$ and $q - p = 0$ or $k = p = q$ which is not possible as p, q, k are distinct.

➤ **Example 5.** If the third and fourth terms of an arithmetic sequence are increased by 3 and 8 respectively, then the first four terms form a geometric sequence. Find

(i) the sum of the first four terms of A.P.

(ii) the second term of the G.P.

➤ ***Solution*** a, (a + d), (a + 2d), (a + 3d) in A.P.

a, a + d, (a + 2d + 3), (a + 3d + 8) are in G.P.

hence $a + d = ar$

also $r = \frac{a+d}{a} = \frac{a+2d+3}{a+d} = \frac{a+3d+8}{a+2d+3}$

$\therefore \quad \frac{d+3}{d} = \frac{d+5}{d+3} \Rightarrow d^2 + 6d + 9 = d^2 + 5d$

$\Rightarrow \quad d = -9$

$\therefore \quad \frac{a-9}{a} = \frac{a-15}{a-9} \Rightarrow a^2 - 18a + 81 = a^2 - 15a$

$\Rightarrow \quad 3a = 81 \quad \Rightarrow \quad a = 27$

Hence A.P. is 27, 18, 9, 0

G.P. is 27, 18, 12, 8

The sum of the first four terms of A.P. = 54

The second term of G.P. = 18

➤ **Example 6.** Three positive numbers form a G.P. If the second term is increased by 8, the resulting sequence is an A.P. In turn, if we increase the last term of this A.P. by 64, we get a G.P. Find the three numbers .

➤ ***Solution*** Let the numbers be a, ar, ar^2 where $r > 0$

Hence a, (ar + 8), ar^2 in A.P. ...(1)

Also a, (ar + 8), ar^2 + 64 in G.P. ...(2)

$\Rightarrow \quad (ar + 8)^2 = a(ar^2 + 64)$

$\Rightarrow \quad a = \frac{4}{4 - r}$ ...(3)

Also (1) $\Rightarrow \quad 2(ar + 8) = (a + ar^2)$

$\Rightarrow \quad (1 - r)^2 = \frac{16}{a}$ ...(4)

From (3) and (4) $r = 3$ or -5 (rejected) .

Hence $a = 4$

The numbers are : 4, 12, 36

➤ **Example 7.** Three numbers are successive terms of a geometric progression. If we subtract 4 from the third number, these numbers will become successive terms of an arithmetic progression. Now if we substract unity from the second and from the third term of the resulting arithmetic progression, the numbers obtained will again become successive terms of a geometric progression. Find the numbers.

➤ ***Solution*** Let the required numbers be a, b, c.

Now we have $b^2 = ac$

Also, $2b = a + c - 4$.

And finally, $(b - 1)^2 = a(c - 5)$

We subtract the third equation from the first. We obtain a linear equation $2b - 1 = 5a$ relating b and a. Expressing now the unknowns a and c from the system of linear equations

$2b - 1 = 5a$,

$2b = a + c - 4$

www.pdfworld.net

www.pdfworld.net

in terms of b, we get $a = \frac{2b-1}{5}$, $c = \frac{8b+21}{5}$

We exclude the unknowns a and c from the system substituting their expressions in terms of b into the first equation of the system. Then we get a quadratic equation for b : $9b^2 - 34b + 21 = 0$, whose roots are equal to 3 and 7/9. Substituting these values of b into the expression for a and c, we get the required numbers.

The numbers are 1, 3, 9 or 1/9, 7/9, 49/4.

➤ **Example 8.** Find $\sum_{i=1}^{6} 2.3^i$

➤ ***Solution*** This sum is the sum of the first six terms of a geometric sequence having $a_1 = 2.3^1 = 6$ and $r = 3$.

From the formula for S_n,

$$\sum_{i=1}^{6} 2.3^i = S_6 = \frac{6(1-3^6)}{1-3} = \frac{6(1-729)}{-2}$$

$$= \frac{6(-728)}{-2} = 2184.$$

➤ **Example 9.** The sum of the first four terms of a geometric progression is equal to 30 and the sum of the next four terms is equal to 480. Find the first term of the progression.

Assume that a is the first term and r is the ratio of the geometric progression.

$$a\frac{1-r^1}{1-r} = 30. \qquad ...(1)$$

The next four terms ar^4, ar^5, ar^6, ar^7, form a geometric progression with the initial term ar^4 and the ratio r. Therefore, we obtain

$$ar^4\frac{1-r^4}{1-r} = 480 \qquad ...(2)$$

From these equations $r^4 = 16$, which means that $r = 2$ or $r = -2$.

If $r = 2$, then from (1) we find $a = 2$, and if $r = -2$, we get $a = -6$.

➤ **Example 10.** Find a three digit number such that its digits are in G.P., and the digits of the number obtained from it by subtracting 400 form an A.P.

➤ ***Solution*** Let the digits at hundreds, tens and units places be a, ar and ar^2 and the required number be x, then

$$x = 100a + 10ar + ar^2$$

Let $y = x - 400$, then

$$y = 100(a-4) + 10ar + ar^2 \qquad ...(1)$$

In the number y, the digit at hundreds place is $a - 4$

Clearly $1 \le a - 4 \le 5$ $[\because 1 \le a \le 9$ and $a - 4 \ge 1]$

$\Rightarrow$ $5 \le a \le 9$ ...(2)

According to question $a - 4$, ar, ar^2 are A.P.

$\therefore$ $2ar = a - 4 + ar^2$ or $a(r-1)^2 = 4$

$\Rightarrow$ $r - 1 = \pm\frac{2}{\sqrt{a}}$ ...(3)

$\because$ ar and a are integers.

$\therefore$ r is a rational number.

$\therefore$ From (2), a must be a perfect square

But from (2), $5 \le a \le 9$ $\therefore$ $a = 9$

$\therefore$ From (3), $r = 1 \pm \frac{2}{3} = \frac{5}{3}, \frac{1}{3}$

But $r \ne \frac{5}{3}$ otherwise $ar = 15$ $\therefore r = \frac{1}{3}$

Hence $a = 9$, $ar = 3$, $ar^2 = 1$

Thus required number is 931.

➤ **Example 11.** Find the sum

$$\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + ... + \left(x^n + \frac{1}{x^n}\right)^2$$

➤ ***Solution*** The required sum can be rewritten in the following way

$$x^2 + x^4 + ... + x^{2n} + \frac{1}{x^2} + \frac{1}{x^4} + ... + \frac{1}{x^{2n}} + 2n.$$

Summing each of the geometric progressions separately and joining the partial sums thus obtained, we have

$$\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + ... + \left(x^n + \frac{1}{x^n}\right)^2$$

$$= \frac{(x^{2n+2}+1)(x^{2n}-1)}{(x^2-1)x^{2n}} + 2n.$$

www.pdfworld.net

www.pdfworld.net

➤ **Example 12.** Find the sum of n terms of the series
$2 + 22 + 222 + ...$

➤ ***Solution*** $S_n = 2 + 22 + 222 + ...$ to n terms

$$= \frac{2}{9}[9 + 99 + 999 + ... \text{ to n terms}]$$

$$= \frac{2}{9}[(10-1) + (10^2-1) + (10^3-1) + ... + (10^n-1)]$$

$$= \frac{2}{9}[(10 + 10^2 + 10^3 + ... + 10^n) - n]$$

$$= \frac{2}{9}\left[10 \cdot \frac{10^n - 1}{10 - 1} - n\right]$$

$$= \frac{2}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$$

➤ **Example 13.** Find the sum to n terms of the series

$$\frac{5}{2} + \frac{9}{4} + \frac{17}{8} + \frac{33}{16} + ...$$

➤ ***Solution*** We have

$$S_n = \left(2 + \frac{1}{2}\right) + \left(2 + \frac{1}{4}\right) + \left(2 + \frac{1}{8}\right) + \left(2 + \frac{1}{16}\right) + ...\text{n terms}$$

$$= 2n + \frac{1}{2}\left(\frac{1 - \frac{1}{2n}}{1 - \frac{1}{2}}\right) = 2n + 1 - \frac{1}{2n}.$$

➤ **Example 14.** Let S_n be the sum of the first n terms of a geometric progression. Prove that

$$S_n (S_{3n} - S_{2n}) = (S_{2n} - S_n)^2.$$

➤ ***Solution*** Let the geometric progression be

$$u_1, u_2 ...., u_n, u_{n+1}, ... , u_{2n}, u_{2n+1}, ..., u_{3n}$$

Hence $S_{3n} - S_{2n} = u_{2n+1} + ... + u_{3n}$,

$S_{2n} - S_n = u_{n+1} + ... + u_{2n}$.

But $u_k = u_1 q^{k-1}$, $u_s = u_1 q^{s-1}$.

Therefore $u_k = u_s \cdot q^{k-s}$, $u_{2n+k} = u_k q^{2n}$,

Consequently,

$S_{3n} - S_{2n} = u_{2n+1} + ... u_{3n} = q^{2n}(u_1 + u_2 + ... u_n) = q^{2n} S_n$.

$S_{2n} - S_n = u_{n+1} + ... u_{2n} = q^n (u_1 + u_2 + ... u_n) = q^n S_n$.

Therefore

$$S_n(S_{3n} - S_{2n}) = q^{2n} S_n^2, \quad (S_{2n} - S_n)^2 = q^{2n} S_n^2$$

➤ **Example 15.** If $x_1, x_2, ...x_n$ are n non-zero real numbers such that

$$(x_1^2 + x_2^2 + ... + x_{n-1}^2)\ (x_2^2 + x_3^2 + ... + x_n^2) \le (x_1 x_2 + x_2 x_3 + ... + x_{n-1} x_n)^2$$

then prove that $x_1, x_2, ... x_n$ are in G.P.

➤ ***Solution*** We shall make use of the **Lagrange's Identity**

$$(a_1^2 + a_2^2 + ... + a_m^2)\ (b_1^2 + b_2^2 + ... + b_m^2) - (a_1 b_1 + a_2 b_2 + ... + a_m b_m)^2 = (a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + ... + (a_{m-1} b_m - a_m b_{m-1})^2$$

Thus,

$$(x_1^2 + x_2^2 + ... + x_{n-1}^2)(x_2^2 + x_3^2 + ... + x_n^2) - (x_1 x_2 + x_2 x_3 + ... + x_{n-1} x_n)^2 \le 0$$

$$\Rightarrow (x_1 x_3 - x_2 x_2)^2 + (x_1 x_4 - x_3 x_3)^2 + ...+ (x_{n-2} x_n - x_{n-1} x_{n-1})^2 \le 0$$

As $x_1, x_2, .. , x_n$ are real, this is possible if and only if

$$x_1 x_3 - x_2^2 = x_2 x_4 - x_3^2 = ... = x_{n-2} x_n - x_{n-1}^2 = 0$$

$$\Rightarrow \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = ... = \frac{x_n}{x_{n-1}}.$$

Hence $x_1, x_2, ... x_n$ are in G.P.

➤ **Example 16.** In a certain test, there are n questions, In this test 2^{n-i} students give wrong answers to atleast i questions $(1 \le i \le n)$. If the total number of wrong answers given is 2047, find the value of n.

➤ ***Solution*** Number of students giving wrong answers to atleast i questions $= 2^{n-i}$

Numbers of students giving wrong answers to atleast $(i + 1)$ questions $= 2^{n-i-1}$.

∴ Number of students giving wrong answers to exactly i questions $= 2^{n-i} - 2^{n-i-1}$

Also the number of students giving wrong answers to exactly n questions $= 2^{n-n} = 2^0 = 1$

∴ Total number of wrong answers

$1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + ...$

$+ (n-1)(2^1 - 2^0) + n(2^0)$

$= 2^{n-1} + (-2^{n-2} + 2.2^{n-2}) + (-2.2^{n-3} + 3.2^{n-3}) + ... + -(n-1)2^0 + n.2^0$

www.pdfworld.net

www.pdfworld.net

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + ... + 2^0 = 2^n - 1$

According to the question

$2^n - 1 = 2047 \Rightarrow 2^n = 2048 = 2^{11} \Rightarrow n = 11$

Practice Problems [E]

1. If a, b, c, d are in GP then prove that
(i) $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$
(ii) a + b, b + c, c + d are in GP.

2. Given four numbers, the first three of which are three successive terms of a geometric progression and the last three are successive terms of an arithmetic progression. The sum of the extreme numbers is 32 and that of the middle numbers is 24. Find the numbers.

3. Find the ordinal number of the term of the geometric progression 1280, 640, .., which equals 20.

4. Prove that if three terms a, b, c form a geometric progression, then
$(a+b+c)(a-b+c) = a^2 + b^2 + c^2$.

5. Among the 11 terms of an arithmetic progression, the first, the fifth and the eleventh term are three successive terms of a certain geometric progression. Find the arithmetic progression if its first term is equal to 24.

6. Find the sum of the following series:

$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$ ..... to n terms.

7. The sequence $\{b_n\}$ is a G.P. with

$\frac{b_4}{b_6} = \frac{1}{4}$ and $b_2 + b_5 = 216$.

Show that $b_1 = 12$ or $\frac{108}{7}$

8. Find the least value of n such that
$1 + 2 + 2^2 + ... + 2^{n-1} \geq 300$.

9. The sum of n terms of a series is $a.\ 2^n - b$, find its nth term. Are the terms of this series in G.P. ?

10. Does there exist a geometric progression having 20, 21 and 22 as three of its terms?

11. Find four numbers forming a geometric progression in which the third term is greater than the first by 9, and the second term is greater than the fourth by 18.

12. Determine the number of the terms of a geometric progression (a_n) if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.

13. The sum of three numbers which are consecutive terms of an arithmetic progression is equal to 21. Reducing the second number by 1 and increasing the third number by 1, we obtain three consecutive terms of a geometric progression. Find these numbers.

3.6 INFINITE GEOMETRIC SERIES

Consider the infinite geometric series

$1 + x + x^2 + x^3 + ...$upto infinity

where x is the common ratio.

The series is convergent when $|x| < 1$; the series diverges to ∞ when $x \geq 1$; the series oscillates finitely when $x = -1$ and oscillates infinitely when $x < -1$.

Proof **:**

The sum of the first n terms

$$S_n = \frac{1 - x^n}{1 - x} ; x \neq 1.$$

(i) If $|x| < 1$, $x^n \to 0$ as $n \to \infty$.

$\therefore \quad S_n = \frac{1}{1-x}$ as $n \to \infty$.

Hence when $|x| < 1$, Σx^{n-1} is convergent and its sum is $\frac{1}{1-x}$.

(ii) If $x > 1$, $x^n \to \infty$ as $n \to \infty$.
$\therefore \quad S_n \to \infty$ as $n \to \infty$.

www.pdfworld.net

www.pdfworld.net

If $x = 1$, $S_n = n \to \infty$ as $n \to \infty$.

Therefore, Σx^{n-1} is divergent and diverges to ∞.

(iii) If $x = -1$,

$S_n = 0$ or 1 according as n is even or odd.

$\therefore$ $S_n \to 0$ or 1 according as n is even or odd.

Moreover $|S_n| \leq 1$.

Hence (S_n) oscillates finitely.

Therefore, Σx^{n-1} oscillates finitely.

(iv) If $x < 1$, $1 - x^n \to \infty$ or $-\infty$ according as n is odd or even.

Thus $S_n \to \infty$ or $-\infty$ according as n is odd or even.

Hence (S_n) oscillates infinitely.

Therefore, the series Σx^{n-1} oscillates infinitely.

Thus, **the sum of an infinite geometric series**

$S = a + ar + ar^2 + \ldots\ldots$ upto infinity, when $|r| < 1$ is

$$S = \frac{a}{1-r}$$

Consider the series $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

The sum to n terms $= \dfrac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$

$$= 2\left(1-\frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}.$$

From this result it appears that however many terms be taken the sum of the above series is always less than 2.

Also we see that, by making n sufficiently large, we can make the fraction $\frac{1}{2^{n-1}}$ as small as we please. Thus by taking a sufficient number of terms the sum can be made to differ by as little as we please from 2.

➤ **Example 1.** Find the sum of the infinite series :

$$\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1} = 3 + \frac{3}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \frac{3}{2^4} + \ldots\ldots$$

➤ ***Solution*** We can find the sum by using the formula for the sum of an infinite geometric series.

$$S_\infty = \sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1} = \frac{a}{1-r}a$$

(the first term) = 3 and r (the common ratio)

$$= \frac{1}{2} = \frac{3}{1-\frac{1}{2}} = 6$$

Thus, the sum of the given infinite geometric series is 6. Putting in an informal way, as we continue adding more and more terms, the sum is approximately 6.

We can visualize $3 + \frac{3}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \frac{3}{2^4} + \ldots\ldots = 6$ by writing the formula for the sum as a function of x, namely

$$f(x) = \frac{a(1-r^x)}{1-r} = \frac{a}{1-r} - \frac{ar^x}{1-r}$$

With $a = 3$ and $r = \frac{1}{2}$, the sum function is

$$f(x) = \frac{3}{1-\frac{1}{2}} - \frac{3\left(\frac{1}{2}\right)^x}{1-\frac{1}{2}} = 6 - 6\left(\frac{1}{2}\right)^x$$

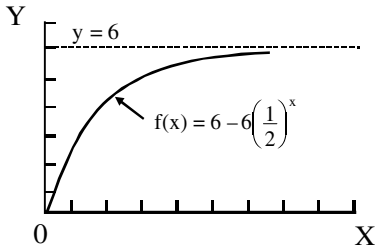


Figure shows the graph of f(x) and y = 6. Since the graph of S_n consists of points on the graph of f, we see that $S_n \to 6$ as $n \to \infty$.

There are some infinite geometric series to which we cannot assign a sum, such as

$2 + 4 + 8 + 16 + 32 + 64 + \ldots.$

As more and more terms are added, the sum is getting larger and larger without bound.

➤ **Example 2.** Find the sum of the infinitely decreasing geometric progression the sum of whose

www.pdfworld.net

www.pdfworld.net

first three terms is equal to 7 and the product of those terms is equal to 8.

Assume that a is the first term and r is the ratio of the geometric progression.

From the question, $\begin{cases} a(1+r+r^2) = 7 \\ a^3r^3 = 8 \end{cases}$

From the second equation of the system we find ar = 2 and substituting a = 2/1 into the first equation of the system, we obtain the quadratic equation $2r^2 - 5r + 2 = 0$ whose roots are r = 1/2 and r = 2. The sequence is infinitely decreasing, |r| < 1. Consequently, r = 1/2, and in that case a = 4. The sum of that progression can be found by the formula:

$$S = \frac{a}{1-r} = 8.$$

➤ **Example 3.** If the first 3 consecutive terms of a geometrical progression are the roots of the equation $2x^3 - 19x^2 + 57x - 54 = 0$ find the sum to infinite number of terms of G.P.

➤ ***Solution*** $2x^3 - 19x^2 + 57x - 54 = 0$

let the roots be $\frac{a}{r}$, a, ar

$$\therefore \quad a\left(1+r+\frac{1}{r}\right) = \frac{19}{2} \qquad ...(1)$$

$$a^2\left(1+r+\frac{1}{r}\right) = \frac{57}{2} \qquad ...(2)$$

$$a^3\left(1+r+\frac{1}{r}\right) = \frac{54}{2} = 27 \qquad ...(3)$$

$\therefore \quad a = 3$

from (1) $\quad 3(r^2+r+1) = \frac{19r}{2}$

$\Rightarrow \quad 6r^2 + 6r + 6 = 19r \quad \Rightarrow 6r^2 - 13r + 6 = 0$

$\Rightarrow \quad (2r-3)(3r-2) = 0 \Rightarrow r = \frac{2}{3}$ or $\frac{3}{2}$ (rejected)

$\therefore \quad$ Numbers are $\frac{a}{r}$, a, ar $\Rightarrow \frac{9}{2}$, 3, 2

$$S_\infty = \frac{9/2}{1-(2/3)} = \frac{9\times 3}{2} = \frac{27}{2}$$

➤ **Example 4.** If $x = 1 + a + a^2 + a^3 + ...\infty$ and $y = 1 + b + b^2 + b^3 + ... \infty$ show that

$$1 + ab + a^2b^2 + a^3b^3 + ... \infty = \frac{xy}{x+y-1}$$ where

$0 < a < 1$ and $0 < b < 1$

➤ ***Solution*** Given $x = 1 + a + a^2 + a^3 + ... \infty$

$$\therefore \quad x = \frac{1}{1-a}$$

$$\Rightarrow \quad x - ax = 1$$

$$\Rightarrow \quad a = \frac{x-1}{x} \qquad ...(1)$$

and $y = 1 + b + b^2 + b^3 + ... \infty$

Similarly $b = \frac{y-1}{y}$ ...(2)

since $0 < a < 1, 0 < b < 1$

$\therefore \quad 0 < ab < 1$

Now $1 + ab + a^2b^2 + a^3b^3 + ...\infty$

$$= \frac{1}{1-ab}$$

$$= \frac{1}{1-\left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)} \quad \{\text{from (1) \& (2)}\}$$

$$= \frac{xy}{xy - xy + x + y - 1} = \frac{xy}{x+y-1}$$

➤ **Example 5.** Find the sum of an infinitely decreasing G.P. whose first term is equal to (b + 2) and the common ratio is $2c^{-1}$, if b is the least value of the product of the roots of the equation $(m^2+1)x^2 - 3x + (m^2+1)^2 = 0$ and c is the greatest value of the sum of its roots.

➤ ***Solution*** b = least value of product of the roots of the given equation

$$= \text{least value of } \frac{(m^2+1)^2}{m^2+1} = \text{least value of}$$

$m^2 + 1$, which is 1

www.pdfworld.net

www.pdfworld.net

c = greatest value of sum of the roots

= greatest value of $\frac{3}{m^2+1}$, which is 3

$\therefore$ the first term of the G.P. = 3 and the common ratio is $\frac{2}{3}$

$\therefore \quad S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{2}{3}} = 9$

➤ **Example 6.** If S_n and S denote the sum to n terms and to infinity of the series $2 + 1\frac{1}{2} + 1\frac{1}{8} + ...$, then find the value of n such that $S - S_n < \frac{1}{1000}$

➤ ***Solution*** Here $a = 2, r = \frac{3}{4}$

$\therefore \quad S = \frac{a}{1-r} = \frac{2}{1-\frac{3}{4}} = 8$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{2\left[1-(3/4)^n\right]}{1-\frac{3}{4}} = 8 - 8\left(\frac{3}{4}\right)^n$$

$\therefore \quad S - S_n = 8\left(\frac{3}{4}\right)^n < \frac{1}{1000}$

if $(0.75)^n < 0.000125$

if $n \log_{10} 0.75 < \log_{10} 0.000125$

if $n(\bar{1}.8751) < \bar{4}.0969$

if $n > \frac{\bar{4}.0969}{\bar{1}.8751} = 31.25$

$\therefore \quad n = 32$

➤ **Example 7.** Let $P = \prod_{n=1}^{\infty}\left(10^{\frac{1}{2^{n-1}}}\right)$ then find $\log_{0.01}(P)$.

➤ ***Solution*** $P = 10^1 \cdot 10^{\frac{1}{2}} \cdot 10^{\frac{1}{2^2}} \cdot 10^{\frac{1}{2^3}} .....$

$$P = 10^{1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+.....+\infty} = 10^{\frac{1}{1-\frac{1}{2}}} = 10^2 = 100$$

$\therefore \quad \log_{0.01}(P) = -1$

➤ **Example 8.** Find the sum to n terms of the series,

$$1 + \left(1 + \frac{1}{2} + \frac{1}{2^2}\right) + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right) + ...$$

➤ ***Solution*** $S_n = 1 + \frac{1-\left(\frac{1}{2}\right)^3}{1-\frac{1}{2}} + \frac{1-\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}} .....$

n terms

$$= 1 + 2\left(1-\left(\frac{1}{2}\right)^3\right) + 2\left(1-\left(\frac{1}{2}\right)^5\right) + ...$$

n times

$$= 1 + (2 + 2 + ... (n-1) \text{ times}) - \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + ...(n-1)\text{times}\right]$$

$$= 1 + 2(n-1) - \frac{(1/2)^2}{1-\left(\frac{1}{2}\right)^2}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)$$

$$= 2n - 1 - \frac{1}{3} + \frac{2}{3.2^{2n}} = 2n - \frac{4}{3}\left(1 + \frac{1}{2^{2n}}\right)$$

Properties of G.P.

A geometric progression (a_n) possesses the following characteristic property: the square of any of its terms, beginning with the second, is equal to the product of its predecessor and successor:

$$a_{n+1}^2 = a_n a_{n+2} \ (n \geq 1).$$

Proof **:**

By the definition of the geometric progression,

$$\frac{a_{n+1}}{a_n} = \frac{a_{n+2}}{a_{n+1}} = r \quad \Rightarrow \quad a_{n+1}^2 = a_n a_{n+2}$$

The converse is also true ; if a certain sequence (a_n) is such that $a_{n+1}^2 = a_n a_{n+2}$ and $a_1 \neq 0$, then this sequence (a_n) relationship be fulfilled. Then

$$\frac{a_{n+1}}{a_n} = \frac{a_{n+2}}{a_{n+1}}$$

that is, the ratio of any term of the sequence (a_n) to its predecessor is equal to one and the same number. Hence, (a_n) is a geometric progression.

Thus, the established property is inherent in a geometric progression and only in it.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

In the case of a geometric progression with positive terms, the above relationship may be written in the form $a_{n+1} = \sqrt{a_n a_{n+2}}$.

A geometric progression with positive terms possesses the following characteristic property : any of its terms, beginning with the second, is a geometric mean of its predecessor and successor.

Important Properties of G .P.

(i) If $a_1, a_2, a_3, \ldots\ldots\ldots$ are in G.P. then

(a) $a_1k , a_2k, a_3k \ldots\ldots\ldots$ are in G.P.

(b) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k} \ldots\ldots\ldots$ are in G.P.

If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP .

(c) $a_1 \pm k , a_2 \pm k, a_3 \pm k \ldots\ldots$ are not in G.P., $k \neq 0$.

(d) $a_1^r, a_2^r, a_3^r, \ldots., a_n^r$ are in G.P.

(e) $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \ldots\ldots\ldots\ldots$ are in G.P.

(ii) If $a_1, a_2, a_3, \ldots\ldots$ and $b_1, b_2, b_3, \ldots\ldots$are in G.P. then $a_1b_1, a_2b_2, a_3b_3,\ldots\ldots\ldots\ldots$ are in G.P. and

$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3} \ldots\ldots\ldots$ are in G.P.

(iii) If $a_1, a_2, a_3, \ldots\ldots$ and $b_1, b_2, b_3, \ldots\ldots$are in G.P. then $a_1 \pm b_1 , a_2 \pm b_2 , a_3 \pm b_3 \ldots\ldots$ are not in G.P. But note that addition and substraction of two G.P. when common ratio of both are same, is G.P.

(iv) If $a_1, a_2, a_3, \ldots.$ are positive numbers in G.P. then $\log a_1, \log a_2, \log a_3, \ldots.$ are in A.P. & vice-versa.

(v) If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_n$ are in G.P. then $a_1n_1 = a_2a_{n-1} = a_3 a_{n-2} = \ldots\ldots\ldots$

(vi) If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots$ are in G.P.

then $a_2^2 = a_1a_3 ; a_3^2 = a_2a_4 ; a_4^2 = a_3 a_5 ; \ldots\ldots$

(vii) If $a_1, a_2, a_3,\ldots\ldots\ldots\ldots,$ are in G.P.

then $a_r = \sqrt{a_{r-k}a_{r+k}} \quad \forall\, k, \quad 0 \le k \le n - r$

➤ **Example 9.** Let r be the common ratio of the GP $a_1, a_2, a_3, \ldots., a_n$. Show that

$$\frac{1}{a_1^m + a_2^m} + \frac{1}{a_2^m + a_3^m} + \ldots\ldots + \frac{1}{a_{n-1}^m + a_n^m} = \frac{1 - r^{m(1-n)}}{a_1^m (r^m - r^{-m})}$$

➤ ***Solution*** LHS

$$= \frac{1}{a_1^m + (a_1r)^m} + \frac{1}{a_2^m + (a_2r)^m} + \ldots.. \frac{1}{a_{n-1}^m + (a_{n-1}r)^m}$$

$$= \frac{1}{1 + r^m}\left\{\frac{1}{a_1^m} + \frac{1}{a_2^m} + \ldots.. + \frac{1}{a_{n-1}^m}\right\}$$

As $a_1 , a_2, a_3,\ldots.$ form a GP, their reciprocals

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots.$$

are also in GP whose common ratio $= \frac{1}{r}$.

Again, their powers $\left(\frac{1}{a_1}\right)^m, \left(\frac{1}{a_2}\right)^m, \left(\frac{1}{a_3}\right)^m, \ldots.$ are also in GP whose common ratio will be $\left(\frac{1}{r}\right)^m$.

$$\therefore \quad \text{LHS} = \frac{1}{1 + r^m} \cdot \frac{\frac{1}{a_1^m}\left\{1 - \left(\frac{1}{r^m}\right)^{n-1}\right\}}{1 - \frac{1}{r^m}}$$

$$= \frac{1 - r^{-mn+m}}{a_1^m (1 + r^m)(1 - r^{-m})} = \frac{1 - r^{m(1-n)}}{a_1^m (r^m - r^{-m})}$$

➤ **Example 10.** Three positive distinct numbers x, y, z are three terms of geometric progression in an order and the numbers x + y, y + z, z + x are three terms of arithmetic progression in that order. Prove that

$x^x \cdot y^y \cdot z^z = x^y \cdot y^z \cdot z^x$.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Let x be first term of G.P. and y and z be the m^{th} and n^{th} of same G.P. respectively

$\Rightarrow \quad t_m = y = xr^{m-1}$

and $t_n = z = x\,r^{n-1}$, where r is a common ratio of G.P.

$$\Rightarrow \quad \frac{m-1}{n-1} = \frac{\log(y/x)}{\log(z/x)} \qquad ...(1)$$

Now, we have $y + z = x + y + (m-1)\,d$

and $z + x = x + y + (n-1)\,d$, where d is common difference of A.P.

$$\Rightarrow \quad \frac{m-1}{n-1} = \frac{z-x}{z-y}$$

From (1) and (2), $\dfrac{\log(y/x)}{\log(z/x)} = \dfrac{z-x}{z-y}$

$$\Rightarrow \quad \frac{y}{x} = \left(\frac{z}{x}\right)^{\frac{z-x}{z-y}} \quad \Rightarrow \quad y = z^{(z-x)/(z-y)} \cdot x^{(x-y)/(z-y)}$$

$$\Rightarrow \quad y^{(z-y)} = z^{(z-x)} \cdot x^{(x-y)}$$

$$\Rightarrow \quad x^x \cdot y^y \cdot z^z = x^y \cdot y^z \cdot z^x$$

Rational Representation of a Recurring Decimal

Let P denote the figures which do not recur, and suppose them p in number; let Q denote the recurring period consisting of q figures, let D denote the value of the recurring decimal; then

$D = 0.PQQQ.,,,;$

$\therefore \quad 10^n \times D = P.QQQ....;$

and $10^{p+q} \times D = PQ.QQQ....;$

therefore, by subtraction, $(10^{p+q} - 10^p)\,D = PQ - P$;

that is, $10^p\,(10^q - 1)\,D = PQ - P$;

$$\therefore \quad D = \frac{PQ - P}{(10^q - 1)10^p}$$

Now $10^q - 1$ is a number consisting of q nines; therefore the denominator consists of q nines followed by p zeros. Hence we have the following rule for reducing a recurring decimal to a fraction :

For the numerator subtract the integral number consisting of the non-recurring figures from the integral number consisting of the non-recurring and recurring figures; for the denominator take a number consisting of as many nines as there are recurring figures followed by as many zeros as there are non-recurring figures.

➤ **Example 11.** Using G.P. express

$0.\overline{3}$ and $1.2\overline{3}$ in $\dfrac{p}{q}$ form.

➤ ***Solution*** Let $x = 0.\overline{3} = 0.3333.....$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + ...$$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{100000} + ...$$

$$= \frac{\frac{3}{10}}{1 - \frac{1}{10}} + \frac{3}{9} = \frac{1}{3}.$$

Let $y = 1.2\overline{3} = 1.233333 = 1.2 + 0.03 + 0.003 + 0.0003 + ...$

$$= 1.2 + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + ...$$

$$= 1.2 + \frac{\frac{3}{10^2}}{1 - \frac{1}{10}} = 1.2 + \frac{1}{30} = \frac{37}{30}.$$

Practice Problems [F]

1. Sum $\dfrac{1}{5} + \dfrac{1}{7} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + ... \text{ to } \infty$
2. Find the value of $9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+....\text{to}\,\infty}$.
3. The sum of an infinite G.P. whose common ratio is numerically less than 1 is 32 and the sum of the first two terms is 24. Find the terms of G.P.
4. The sum of an infinitely decreasing geometric progression is 16 and the sum of the squares of its terms is $153\dfrac{3}{5}$. Find the fourth term and the common ratio of the progression.
5. If S be the sum, P the product & R the sum of the reciprocals of a G.P., find the value of $P^2\left(\dfrac{R}{S}\right)^n$.

www.pdfworld.net

www.pdfworld.net

6. The first term of an infinitely decreasing geometric progression is unity and its sum is S. Find the sum of the squares of the terms of the progression.

7. Find the sum : $\sum_{n=3}^{\infty}\left[\left(\frac{1}{4}\right)^n+\left(\frac{3}{4}\right)^n\right]$

8. Between what numbers must x lie in order that the series $\frac{2x}{x+3}+\left(\frac{2x}{x+3}\right)^2+\left(\frac{2x}{x+3}\right)^3+.....$ to ∞ may have a finite sum ?

9. Prove that in an infinite GP, whose common ratio r has its absolute value less than 1, the ratio of any term to the sum of all the succeeding terms is $\frac{1-r}{r}$.

10. Sum $\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{1}{2-\sqrt{2}}+\frac{1}{2}+.....$

11. Solve the equations :

(i) $1 + a + a^2 + ... + a^x = (1 + a)\ (1+ a^2)\ (1 + a^4)(1 + a^8)$

(ii) $2x + 1+x^2 - x^3 + x^4 - x^5 + ... = 13/6$ if $|x| < 1$

(iii) $(3(1 - 1/2 + 1/4 - 1/8 + ...))^{\log_a x}$
$= (20(1 - 1/4 + 1/16 - 1/64 + ...))^{\log_x a}$

(iv) $2^{x-1} + 2^{x-4} + 2^{x-2} = 6.5 + 3.25 + 1.625 + ...$

12. The continued product of three numbers in G.P. is 216 and the sum of the products of them in pairs is 156; find the numbers.

3.7 HARMONIC PROGRESSION (HP)

A sequence is said to HP if the reciprocals of its terms are in AP .

If the sequence $a_1, a_2, a_3, ... , a_n$ is an HP, then

$1/a_1, 1/a_2, ... , 1/a_n$ is an AP.

The general term of an H.P. is $a_n = \frac{1}{a+(n-1)d}$

In case we have to assume a certain number of terms which are in H.P., then suggested choice is as follows:

(i) for three terms use $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

(ii) for four terms use $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+2d}$

1. There is no general formula for finding out the sum of n terms of H.P.

2. No term of H.P. can be zero.

3. For HP whose first term is a and second term is b, the n^{th} term is $t_n = \frac{ab}{b+(n-1)(a-b)}$.

4. If a, b, c are in HP $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

5. Questions in H.P. are generally solved by inverting the terms, and making use of the properties of the corresponding AP.

Recognization of A.P, G.P., H.P.

Let a, b, c be three successive terms of a sequence.

(i) If $\frac{a-b}{b-c} = \frac{a}{a}$, then a, b, c are in A.P.

(ii) If $\frac{a-b}{b-c} = \frac{a}{b}$, then a,b, c are in G.P.

(iii) If $\frac{a-b}{b-c} = \frac{a}{c}$, then a, b, c are in H.P.

OR

(i) a, b,c are in A.P. if and only if $b = \frac{a+c}{2}$

(ii) a, b, c are in G.P. if and only if $b^2 = ac$

(iii) a, b, c are in H.P. if and only if $b = \frac{2ac}{a+c}$

In general we can use the following :
If $t_n - t_{n-1}$ = constant for all $n \geq 2$ then the sequence is in AP.

www.pdfworld.net

www.pdfworld.net

If $\frac{t_n}{t_{n-1}}$ = constant, for all $n \geq 2$ then the sequence is in GP.

If $\frac{1}{t_n} - \frac{1}{t_{n-1}}$ = constant, for all $n \geq 2$ then the sequence is in HP.

➤ **Example 1.** If a, b, c are in H.P., show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in H.P.

➤ ***Solution*** Given, a, b, c are in H.P.

$\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\Rightarrow$ $\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow$ $1+\frac{b+c}{a}, 1+\frac{c+a}{b}, 1+\frac{a+b}{c}$ are in A.P.

$\Rightarrow$ $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

➤ **Example 2.** If $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$ prove that a, b, c are in HP unless b = a + c.

➤ ***Solution***

$\Rightarrow$ $\frac{a+c}{ac}+\frac{c-a+a-b}{(a-b)(c-b}=0$

$\Rightarrow$ $\frac{a+c}{ac}+\frac{(a+c)-2b}{ac-b(a+c)+b^2}=0$

Let $a+c=\lambda$ $\therefore$ $\frac{\lambda}{ac}+\frac{\lambda-2b}{ac-b\lambda+b^2}=0$

$\Rightarrow$ $\frac{ac\lambda-b\lambda^2+b^2\lambda+ac\lambda-2abc}{ac(ac-b\lambda+b^2)}=0$

$\Rightarrow$ $ac\lambda - b\lambda^2 + b^2\lambda + ac\lambda - 2abc = 0$

$\Rightarrow$ $2ac(\lambda - b) - b\lambda(\lambda - b) = 0$

$\Rightarrow$ $(2ac - b\lambda)(\lambda - b) = 0$

$\Rightarrow$ $\lambda = b$ or $\lambda = \frac{2ac}{b}$

$\Rightarrow$ $a + c = b$ or $a + c = \frac{2ac}{b}$ $(\because a + c = \lambda)$

$\Rightarrow$ $a + c = b$ or $a + b = \frac{2ac}{a+c}$

$\Rightarrow$ a, b, c are in H.P. or $a + c = b$.

➤ **Example 3.** If $a_1, a_2, \ldots, a_n$ are in harmonic progression, show that

$a_1a_2 + a_2a_3 + \ldots\ldots a_{n-1}a_n = (n-1)a_1a_n$.

➤ ***Solution*** Here $\frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots, \frac{1}{a_n}$ are in A.P.

Let the common different be d.

$\therefore$ $\frac{1}{a_2}, \frac{1}{a_1} = d$ or $a_1a_2 = \frac{a_1 - a_2}{d}$,

$\frac{1}{a_3} - \frac{1}{a_2} = d$ or $a_2a_3 = \frac{a_2 - a_3}{d}$

....

$\frac{1}{a_n} - \frac{1}{a_{n-1}} = d$ or $a_{n-1}a_n = \frac{a_{n-1} - a_n}{d}$

By addition,

$\frac{1}{a_n} - \frac{1}{a_1} = (n-1)d$

and $a_1a_2 + a_2a_3 + \ldots\ldots + a_{n-1}a_n = \frac{a_1 - a_n}{d}$.

Now $(n-1)d = \frac{a_1 - a_n}{a_1a_n}$ or $\frac{a_1 - a_n}{d} = (n-1)a_1a_n$.

$\therefore$ $a_1a_2 + a_2a_3 + \ldots\ldots + a_{n-1}a_n = (n-1)a_1a_n$.

➤ **Example 4.** If a^2, b^2, c^2 are in A.P., show that b + c, c + a, a + b are in H.P.

➤ ***Solution*** By adding ab + ac + bc to each term, we see that

$a^2 + ab + ac + bc, b^2 + ba + bc + ac, c^2 + ca + cb + ab$ are in A.P.,

that is $(a + b)(a + c), (b + c)(b + a), (c + a)(c + b)$ are in A.P.,

$\therefore$ dividing each term by $(a + b)(b + c)(c + a)$,

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.,

that is, b + c, c + a, a + b are in H.P.

➤ **Example 5.** The value of x y z is 55 or $\frac{343}{55}$ according as the series a, x, y, z, b is an A.P. or H.P.

www.pdfworld.net

www.pdfworld.net

Find the values of a & b given that they are positive integers.

➤ ***Solution*** Let a, x, y, z, b are in A.P. $\Rightarrow b = a + 4d$

$\Rightarrow \quad d = \dfrac{b-a}{4}$

$\Rightarrow \quad x = a + d = \dfrac{3a+b}{4} \quad y = a + 2d = \dfrac{a+b}{2}$

& $\quad z = \dfrac{3a+b}{4}$

Similarly when a, x, y, z, b are in H.P. then

$$d = \frac{a-b}{4ab}$$

Hence $x = \dfrac{4ab}{a+3b}$; $y = \dfrac{2ab}{a+b}$; $z = \dfrac{4ab}{2a+b}$

In the first case

$$\frac{3a+b}{4} \cdot \frac{a+b}{2} \cdot \frac{3a+b}{4} = 55 \qquad \ldots(1)$$

In the second case

$$\frac{4ab}{a+3b} \cdot \frac{2ab}{a+b} \cdot \frac{4ab}{2a+b} = \frac{343}{55} \qquad \ldots(2)$$

dividing

$a^3 b^3 = 7^3 \Rightarrow a = 7$; $b = 1$ or $a=1$; $b=7$

Practice Problems [G]

1. Find the 7th term of the series $\dfrac{1}{3} + \dfrac{8}{23} + \dfrac{4}{11} + \ldots$.
2. Find the 4th term of an H.P. whose 7th term is $\dfrac{1}{20}$ and 13th term is $\dfrac{1}{38}$.
3. Find the largest positive term of the H.P. whose first two terms are 2/5 and 12/23.
4. If a, b, c, d be in G.P. and $a^x = b^y = c^z = d^w$, prove that x,y, z, w are in H.P.
5. If ℓ, m, n are three numbers in G.P., prove that the first term of an A.P. whose ℓ^{th}, m^{th}, and n^{th} terms are in H.P. is to the common difference as m + 1 to 1.
6. If a, b, c, be in A.P.; p, q, r be in H.P.; and ap, bq, cr be in G.P. show that $\dfrac{p}{r} + \dfrac{r}{p} = \dfrac{a}{c} + \dfrac{c}{a}$.
7. If a, b, c are in H.P. then prove that $a^3b^3 + b^3c^3 + c^3a^3 = (9ac - 6b^2)\, a^2c^2$.
8. The value of xyz is $\dfrac{15}{2}$ if a, x, y, z, b are in AP, while xyz is $\dfrac{18}{5}$ if a, x, y, z, b are in HP. If a and b are positive integers find them.

3.8 ARITHMETIC, GEOMETRIC & HARMONIC MEANS

Arithmetic Mean

When three quantities are in arithmetic progression, the middle one is said to be the arithmetic mean of the other two.

Thus, a is the arithmetic mean between a – d and a + d.

To find the arithmetic mean between two given quantities.

Let a and b be the two quantities and A the arithmetic mean.

Since a, A, b are in A.P. we must have

$b - A = A - a$,

each being equal to the common difference ;

$\therefore \quad A = \dfrac{a+b}{2}$

The arithmetic mean of a set of any n numbers

$a_1, a_2, \ldots, a_n$ is $A = \dfrac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$.

To insert a given number of arithmetic means between two given quantities.

Let a and b be the given quantities, n the number of means.

Let $A_1, A_2, A_3, \ldots, A_n$ be the A.M.'s between them.

Then $a, A_1, A_2, A_3, \ldots, A_n, b$ will be in A.P.

www.pdfworld.net

www.pdfworld.net

Including the extremes the number of terms will be n + 2; so that we have to find a sequence of n + 2 terms in A.P., of which a is the first, and b is the last. Let d be the common difference.

➤ **Example 1.** Find the first four terms for the sequences defined as follows :

then $b = a + (n + 2 - 1)\, d$

$\therefore \quad d = \dfrac{b-a}{n+1}$

$\therefore \quad A_1 = a + d,\ A_2 = a + 2d,\ \ldots\ldots\ldots\ldots,\ A_n = a + nd$

$\Rightarrow \quad A_1 = a + \left(\dfrac{b-a}{n+1}\right),\ A_2 = a + 2\left(\dfrac{b-a}{n+1}\right),\ \ldots\ldots\ldots,$

$A_n = a + n\left(\dfrac{b-a}{n+1}\right)$

Thus, the n A.M.'s between a and b are

$$a + \frac{b-a}{n+1},\quad a + \frac{2(b-a)}{n+1},\ \ldots.,\quad a + \frac{n(b-a)}{n+1}$$

For example, since 1, 3, 5, 7, 9, 11 are in A.P., therefore 3, 5, 7, 9 are the four A.M.'s between 1 and 11.

The sum of all the n A.M.'s inserted between a and b is equal to n times the single A.M. between a and b

i.e. $\displaystyle\sum_{r=1}^{n} A_r = A_1 + A_2 + \ldots\ldots + A_n = nA$

where A is the single A.M. between a and b.

Let two numbers a and b, and $A_1, A_2, A_3, \ldots\ldots,$ A_n are A.M.'s between them

If d is the common difference of this A.P. then

$A_1 = a + d$, A_n = second term from end = b – d

$\therefore \quad A_1 + A_2 + A_3 + \ldots. + A_n = \dfrac{n}{2}$ (first term + last term)

$= \dfrac{n}{2}(A_1 + A_n) = \dfrac{n}{2}(a + d + b - d)$

$= n\left(\dfrac{a+b}{2}\right) = n$ (A.M. of a and b)

➤ **Example 2.** Insert three arithmetic means between the numbers 3 and 19.

➤ ***Solution*** We have to find three numbers a_2, a_3, and a_4 such that the sequence 3, a_2, a_3, a_4, 19 is an arithmetic progression.

Let the difference of this progression be equal to d. By the formula of the general term of an arithmetic progression, 19 = 3 + 4d or d = 4.

Thus we have obtained the progression 3, 7, 11, 15, 19. The required numbers are 7, 11 and 15.

➤ **Example 3.** Insert 20 A.M.s between 2 and 86.

➤ ***Solution*** Here 2 is the first term and 86 is the 22nd term of A.P.

so $86 = 2 + (21)\, d \quad \Rightarrow \quad d = 4$

so the sequence is 2, 6, 10, 14,....., 82, 86

$\therefore$ The required means are 6, 10, 14,.....82

➤ **Example 4.** Find four numbers in A.P. whose sum is 6 and the product of whose extremes is 10 times the product of the means.

➤ ***Solution*** Let the four numbers be

a – 3d, a – d , a + d, a + 3d.

Sum of these numbers = 4a = 6 $\quad \therefore a = \dfrac{3}{2}$

Product of extremes = 10 times product of means

$\Rightarrow \quad (a - 3d)(a + 3d) = 10\,(a - d)\,(a + d)$

$\therefore \quad 10(a^2 - d^2) = a^2 - 9d^2$ or $d^2 = 9a^2 = 9 \times \dfrac{9}{4}$

Hence $d = \pm\dfrac{9}{2}$

When $d = \dfrac{9}{2}$, we get the four numbers as –12, –3, 6, 15

$d = \dfrac{-9}{2}$ gives the same numbers in reverse order.

➤ **Example 5.** Between two numbers whose sum is $\dfrac{13}{6}$, an even number of A.M. s is inserted, the sum of these means exceeds their number by unity. Find the number of means.

➤ ***Solution*** Let a and b be two numbers and 2n A.M. s are inserted between a and b then

$$\frac{2n}{2}(a + b) = 2n + 1.$$

www.pdfworld.net

www.pdfworld.net

$$n\left(\frac{13}{6}\right) = 2n + 1. \left[\text{given } a + b = \frac{13}{6}\right] \Rightarrow n = 6$$

$\therefore$ The number of means = 12.

➤ **Example 6.** Two consecutive numbers from 1, 2, 3,...., n are removed. The arithmetic mean of the remaining numbers is $\frac{105}{4}$. Find n and those removed numbers.

➤ ***Solution*** Let p and (p + 1) be the removed numbers from 1, 2,...,n then sum of the remaining numbers $= \frac{n(n+1)}{2} - (2p + 1)$

From given condition

$$\frac{105}{4} = \frac{\frac{n(n+1)}{2} - (2p+1)}{n-2}$$

$\Rightarrow \quad 2n^2 - 103n - 8p + 206 = 0$

Since n and p are integers so n must be even let n = 2r

we get $p = \frac{4r^2 + 103(1-r)}{4}$

Since p is an integer then (1 – r) must be divisible by 4.

Let r = 1 + 4t, we get

n = 2 + 8t and $p = 16t^2 - 95t + 1$,

Now $1 \le p < n$

$\Rightarrow \quad 1 \le 16t^2 - 95t + 1 < 8t + 2 \quad \Rightarrow \quad t = 6$

$\Rightarrow$ n = 50 and p = 7

Hence removed numbers are 7 and 8.

➤ **Example 7.** Prove that the arithmetic mean of the squares of 'n' quantities in A.P. exceeds the squares of their arithmetic mean by a quantity which depends only upon 'n' and upon their common difference .

➤ ***Solution*** $a + b, \; a + 2b, \; \ldots\ldots, \; a + nb$

Hence $\frac{(a+b)^2 + (a+2b)^2 + \ldots\ldots + (a+nb)^2}{n}$

$$= \frac{1}{n}\left[na^2 + nab(n+1) + b^2 \frac{n(n+1)(2n+1)}{6}\right]$$

$$= a^2 + ab(n+1) + \frac{b^2}{6}(n+1)(2n+1)$$

$$= a^2 + ab(n+1) + \frac{b^2}{6}(2n^2 + 3n + 1)$$

Again $\left\{\frac{(a+b) + (a+2b) + \ldots\ldots + (a+nb)}{n}\right\}^2$

$$= \left(a + \frac{n+1}{2} b\right)^2$$

$$= a^2 + ab(n+1) + \frac{b^2}{4}(n^2 + 2n + 1)$$

Hence A.M. of squares $-$ square of A.M.

$$= \left(\frac{n^2 - 1}{12}\right) b^2$$

Geometric Mean

When three positive quantities are in geometric progression, the middle one is said to be the geometric mean of the other two. Thus, if a, b, c are in GP, b is the G.M. between a and c (a , b, c > 0).

To find the geometric mean between two given quantities.

Let a and b be the two positive quantities and G the geometric mean.

Since a, G, b are in G.P., we must have

$G^2 = ab$

$\therefore \quad G = \sqrt{ab}$

The geometric mean of a set of any n positive numbers $a_1, a_2, \ldots, a_n$ is

$$G = (a_1 a_2 \ldots a_n)^{1/n}$$

To insert a given number of geometric means between two given positive quantities.

Let a and b be the given positive quantities, n the number of means.

Let $G_1, G_2, G_3, \ldots, G_n$ be the n G.M.s between a and b.

Then a, $G_1, G_2, G_3, \ldots, G_n$, b will be in G.P.

In all there will be n + 2 terms; so that we have to find a series of n + 2 terms in G.P., of which a is the first and b the last.

www.pdfworld.net

www.pdfworld.net

Let r be the common ratio;
then b = the $(n+2)^{th}$ term $= ar^{n+2-1} = ar^{n+1}$

$$r^{n+1} = \frac{b}{a}$$

$$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

$$\therefore \quad G_1 = ar,\ G_2 = ar^2, \ldots, G_n = ar^n$$

$$\Rightarrow \quad G_1 = a\left(\frac{b}{a}\right)^{1/n+1},\ G_2 = a\left(\frac{b}{a}\right)^{2/n+1}, \ldots,$$

$$G_n = a\left(\frac{b}{a}\right)^{n/n+1}$$

For example :

Since 1, 2, 4, 8, 16 are in G.P., therefore 2, 4, 8 are the three G.M.'s between 1 and 16.

The product of all the n G.M.s between a and b is equal to the nth power of the single GM between a and b.

i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a and b.

Let two numbers a and b, and $G_1, G_2, G_3, \ldots, G_n$ are A.M.'s between them

Then $a, G_1, G_2, G_3, \ldots G_n, b$ are in G.P. If r is the common ratio then

$$G_1 = ar,\ G_2 = ar^2,\ G_3 = ar^3, \ldots,$$

$$G_n = \text{2nd term from last} = \frac{b}{r}$$

$$G_{n-1} = \text{3rd term from last} = \frac{b}{r^2}$$

$$\therefore \quad G_1G_2G_3 \ldots G_n = ar.ar^2 \ldots ar^{n-1} = a^n r^{1+2+\ldots\ldots(n-1)}$$

$$= a^n r^{n(n-1)/2}$$

$$= (a^2 r^{n-1})^{n/2}$$

$$= (\sqrt{ab})^n$$

$$= (\text{G.M. of a and b})^n$$

➤ **Example 8.** Insert three geometric means between the numbers 1 and 256.

➤ ***Solution*** We have to find three numbers a_2, a_3 and a_4 such that
the sequence 1, a_2, a_3, a_4, 256 becomes a geometric progression.
Let the ratio of this progression be equal to r.
By the formula of the general term of a geometric progression, $256 = 1.r^4$, or $4^4 = r^4$, or r = 4. We have obtained the progression 1, 4, 16, 64, 256. The required numbers are 4, 16 and 64.

➤ **Example 9.** The sum of four numbers in G.P. is 60 and the arithmetic mean of the first and last numbers is 18. Find the numbers.

➤ ***Solution*** Let the four numbers be a, ar, ar^2, ar^3

$$a(1 + r + r^2 + r^3) = 60 \qquad \ldots(1)$$

$$\frac{a + ar^3}{2} = 18$$

$$\therefore \quad a(1+r)(1+r^2) = 60 \text{ and } a(1+r)(1-r+r^2) = 36$$

Division gives $\frac{1+r^2}{1-r+r^2} = \frac{5}{3}$

$$\therefore \quad 2r^2 - 5r + 2 = 0 \text{ giving } r = 2 \text{ or } \frac{1}{2}$$

Putting r = 2 in equation (1), we get a = 4
$\therefore$ The numbers are 4, 8, 16, 32

$r = \frac{1}{2}$ gives the same four numbers.

Harmonic Mean

$$b = 2ac/[a+c].$$

When three quantities are in harmonic progression, the middle one is said to be the harmonic mean of the other two. If a, b, c are in HP, b is the H.M. between a and c.

To find the harmonic mean between two given quantities.

Let a and b be the two quantities and H the harmonic mean.

Since a, H, b are in H.P. we must have

$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

$$\therefore \quad H = \frac{2ab}{a+b}$$

The harmonic mean of a set of any n numbers $a_1, a_2, \ldots, a_n$ is

$$H = \left[\frac{\left\{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right\}}{n}\right]^{-1}$$

or $$\frac{1}{H} = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots..+\frac{1}{a_n}}$$

To insert a given number of harmonic means between two given quantities.

Let a and b be two given numbers and $H_1, H_2, H_3, \ldots..,$ H_n are H.M.'s between them.

Then $a, H_1, H_2, H_3, \ldots.., H_n, b$ will be in H.P.

If d is the common difference of the corresponding A.P.

$\therefore$ b = (n + 2)th term of H.P.

$$= \frac{1}{(n+2)\text{th term of corresponding A.P.}}$$

$$\therefore \quad b = \frac{1}{\frac{1}{a}+(n+2-1)d}$$

$$\Rightarrow \quad \frac{1}{b} = \frac{1}{a} + (n+1)\,d$$

$$\Rightarrow \quad d = \frac{(a-b)}{(n+1)ab}$$

$$\therefore \quad \frac{1}{H_1} = \frac{1}{a}+d, \quad \frac{1}{H_2} = \frac{1}{a}+2d, \ldots., \quad \frac{1}{H_n} = \frac{1}{a}+nd$$

$$\Rightarrow \quad \frac{1}{H_1} = \frac{1}{a}+\frac{(a-b)}{(n+1)ab}, \quad \frac{1}{H_2} = \frac{1}{a}+\frac{2(a-b)}{(n+1)ab}, \ldots..,$$

$$\frac{1}{H_n} = \frac{1}{a}+\frac{n(a-b)}{(n+1)ab}$$

The sum of reciprocals of n Harmonic means between two given numbers is n times the reciprocal of single H.M. between them.

Let two numbers a and b and $H_1, H_2, H_3, \ldots.., H_n$ are H.M.'s between them then $a, H_1, H_2, H_3, \ldots..H_n, b$ are in H.P. If d is the difference of the corresponding A.P.

$$\therefore \quad \frac{1}{H_1} = \frac{1}{a}+d, \frac{1}{H_2} = \frac{1}{a}+2d, \frac{1}{H_3} = \frac{1}{a}+3d, \ldots.$$

$$\frac{1}{H_n} = 2\text{nd term from last} = \frac{1}{b}-d$$

$$\frac{1}{H_{n-1}} = 3\text{rd term from last} = \frac{1}{b}-2d$$

$$\therefore \quad \frac{1}{H_1}+\frac{1}{H_2}+\frac{1}{H_3}+\ldots+\frac{1}{H_{n-2}}+\frac{1}{H_{n-1}}+\frac{1}{H_n}$$

$$= \frac{n}{2}\left(\frac{1}{H_1}+\frac{1}{H_n}\right) = \frac{n}{2}\left(\frac{1}{a}+d+\frac{1}{b}-d\right)$$

$$= \frac{n}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

➤ **Example 10.** Insert 4 H.M.s between $\frac{2}{3}$ and $\frac{2}{13}$.

➤ ***Solution*** Let d be the common difference of corresponding A.P.

so $$d = \frac{\frac{13}{2}-\frac{3}{2}}{5} = 1$$

$$\therefore \quad \frac{1}{H_1} = \frac{3}{2}+1 = \frac{5}{2} \quad \text{or} \quad H_1 = \frac{2}{5}$$

$$\frac{1}{H_2} = \frac{3}{2}+2 = \frac{7}{2} \quad \text{or} \quad H_2 = \frac{2}{7}$$

$$\frac{1}{H_3} = \frac{3}{2}+3 = \frac{9}{2} \quad \text{or} \quad H_3 = \frac{2}{9}$$

$$\frac{1}{H_4} = \frac{3}{2}+4 = \frac{11}{2} \quad \text{or} \quad H_4 = \frac{2}{11}.$$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

➤ **Example 11.** A, B and C are distinct positive integers, less than or equal to 10. The arithmetic mean of A and B is 9. The geometric mean of A and C is $6\sqrt{2}$. Find the harmonic mean of B and C.

➤ ***Solution*** $A + B = 18$ ...(1)

$AC = 72$ ...(2)

There are only two possibilities

$A = 10$ and $B = 8$ or $A = 8$ and $B = 10$

If $A = 10$ then from (2) C is not an integer.

Hence $A = 8$ and $B = 10$; $C = 9$

$\therefore$ H.M. between B and C $= \dfrac{2 \cdot 10 \cdot 9}{10 + 9} = \dfrac{180}{19}$

➤ **Example 12.** If a be the A.M. and H the H.M. between two quantities a and b, prove that

$$\frac{a-A}{a-H} \cdot \frac{b-A}{b-H} = \frac{A}{H}$$

➤ ***Solution*** Given, $A = \dfrac{a+b}{2}$ and $H = \dfrac{2ab}{a+b}$

$$\text{Now, } \frac{a-A}{a-H} \cdot \frac{b-a}{b-H} = \frac{a - \dfrac{a+b}{2}}{b - \dfrac{2ab}{a+b}}$$

$$= \frac{(a-b)(b-a)(a+b)^2}{4(a^2 + ab - 2ab)(ab + b^2 - 2ab)}$$

$$= \frac{(a-b)(b-a)(a+b)^2}{4.a(a-b).b(b-a)} = \frac{(a+b)^2}{4ab}$$

$$= \frac{\dfrac{a+b}{2}}{\dfrac{2ab}{a+b}} = \frac{A}{H}$$

➤ **Example 13.** The A.M. between two positive numbers exceeds the G.M. by 5, and the G.M. exceeds the H.M. by 4. Find the numbers.

Solution If the numbers are a and b,

$$\frac{a+b}{2} - \sqrt{ab} = 5 \text{ and } \sqrt{ab} - \frac{2ab}{a+b} = 4$$

Substituting for $\dfrac{a+b}{2}$ in the second equation, we get

$$\sqrt{ab} - \frac{ab}{(5 + \sqrt{ab})} = 4$$

$$\Rightarrow \quad 5\sqrt{ab} + ab - ab = 20 + 4\sqrt{ab}$$

$$\therefore \quad \sqrt{ab} = 20 \text{ or } ab = 400$$

$$\therefore \quad \frac{a+b}{2} = 5 + \sqrt{ab} = 25 \qquad \therefore \ a + b = 50$$

Since $a + b = 50$ and $ab = 400$

$a = 40$ and $b = 10$

➤ **Example 14.** The arithmetic mean of two numbers exceed their geometric mean by $\dfrac{3}{2}$ and the geometric mean exceed their harmonic mean by $\dfrac{6}{5}$, find the numbers.

➤ ***Solution*** Let the two numbers be a and b and A, G, H be the A.M. , G.M. and H.M. respectively between a and b.

Then, $A = \dfrac{a+b}{2}$; $G = \sqrt{ab}$ and $H = \dfrac{2ab}{a+b}$ ...(1)

Given, $A = G + \dfrac{3}{2}$ and $G = H + \dfrac{6}{5}$

or, $H = G - \dfrac{6}{5}$ ...(2)

$\because \quad AH = G^2 \qquad \therefore \left(G + \dfrac{3}{2}\right)\left(G - \dfrac{6}{5}\right) = G^2$

or $(2G + 3)(5G - 6) = 10G^2$ or $3G = 18$

$\therefore \quad G = 6$

From (2), $A = 6 + \dfrac{3}{2} = \dfrac{15}{2}$ or $a + b = 15$ ...(3)

Also $G = 6 \ \therefore \ \sqrt{ab} = 6$ or $ab = 36$ ...(4)

Now $(a - b)^2 = (a + b)^2 - 4ab = 15^2 - 4.36 = 81$

$\therefore \quad a - b = \pm 9$ ...(5)

from (3) and (5), $a = 12$, $b = 3$ or $a = 3$, $b = 12$

Hence the numbers are 3 and 12.

➤ **Example 15.** If p be the first of n A.M.'s between two numbers; q be the first of n H.M.'s between the same numbers, prove that the value of q cannot lie between p and $\left(\dfrac{n+1}{n-1}\right)^2$ p.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Let the two numbers be a and b. Since n A.M.'s have been inserted between a and b

$\therefore$ Common difference of A.P., $d = \dfrac{b-a}{n+1}$

Now $p = a + d = a + \dfrac{b-a}{n+1} = \left(\dfrac{an+b}{n+1}\right)$ ...(1)

Since n H.M.'s have been inserted between a and b. Then common difference of the corresponding A.P.

$$D = \frac{\frac{1}{b} - \frac{1}{a}}{(n+1)} = \frac{(a-b)}{ab(n+1)}$$

$$\therefore \quad \frac{1}{q} = \frac{1}{a} + D = \frac{1}{a} + \frac{(a-b)}{ab(n+1)}$$

Then $q = \dfrac{ab(n+1)}{(a+bn)}$ ...(2)

Alternative Method 1

From (1),

$an + b = pn + p$

$\Rightarrow \quad b = pn - an + p$ ...(3)

from (2), $ap + bnq = abn + ab$ ...(4)

Now from (3) and (4)

$$pn - an + p = \frac{aq}{an + a - nq} \quad ...(5)$$

$\Rightarrow \quad apn^2 - a^2n^2 + apn + apn - a^2n + ap - n^2pq + aqn^2 - npq = aq$

$\Rightarrow \quad n(n+1)\,a^2 - ap\,(n+1)^2 + npq\,(n+1) - aq(n^2-1) = 0$

dividing by $(n+1)$

Then $na^2 - a\,[p\,(n+1) + q(n-1)] + npq = 0$

Since 'a' is real, the discriminant

$B^2 - 4AC \geq 0$

$\Rightarrow \quad [p\,(n+1) + q(n-1)]^2 - 4n^2\,pq \geq 0$

$\Rightarrow \quad p^2\,(n+1)^2 + q^2\,(n-1)^2 + 2pq\,(n^2-1) - 4n^2pq \geq 0$

$\Rightarrow \quad p^2\,(n+1)^2 + q^2\,(n-1)^2 + 2pq\,(n^2 - 1 - 2n^2) \geq 0$

$\Rightarrow \quad p^2\,(n+1)^2 + q^2\,(n-1)^2 - 2pq\,(n^2+1) \geq 0$

$\Rightarrow \quad p^2\,(n+1)^2 + q^2\,(n-1)^2 - pq\,\{(n+1)^2 + (n-1)^2\} \geq 0$

$\Rightarrow \quad (n+1)^2\,p\,(p-q) + (n-1)^2\,q\,(q-p) \geq 0$

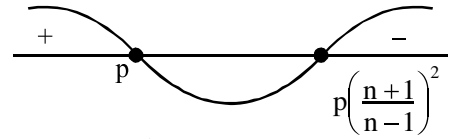
$\Rightarrow \quad (p-q)\,\{(n+1)^2\,p - (n-1)^2\,q\} \geq 0$

$$\Rightarrow \quad (p-q)\left\{\left(\frac{n+1}{n-1}\right)^2 p - q\right\} \geq 0$$

$$\Rightarrow \quad (q-p)\left\{q - p\left(\frac{n+1}{n-1}\right)^2\right\} \geq 0$$

By wavy curve method.

$$\therefore \quad q \leq p \text{ and } q \geq p\left(\frac{n+1}{n-1}\right)^2$$



Hence q cannot lie between p and $p\left(\dfrac{n+1}{n-1}\right)^2$

Alternative Method 2

Since $n + 1 > n - 1$

$$\Rightarrow \quad \left(\frac{n+1}{n-1}\right)^2 > 1$$

$$\Rightarrow \quad p\left(\frac{n+1}{n-1}\right)^2 > p \quad ...(6)$$

From (1) and (2) $\dfrac{p}{q} = \dfrac{n^2 + n\left(\frac{a}{b} + \frac{b}{a}\right) + 1}{(n+1)^2}$

$$\Rightarrow \quad \frac{p}{q} - 1 = \frac{n\left(\frac{a}{b} + \frac{b}{a} - 2\right)}{(n+1)^2}$$

$$= \frac{n}{(n+1)^2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 > 0$$

$\Rightarrow \quad p > q$ ...(7)

From (6) and (7)

$$q < p < p\left(\frac{n+1}{n-1}\right)^2$$

Hence q cannot lie between p and $p\left(\dfrac{n+1}{n-1}\right)^2$.

www.pdfworld.net

www.pdfworld.net

Practice Problems [H]

1. If a be the first term and *l* be the last term of an A.P. and if one arithmetic mean be inserted between every two successive terms of the A.P., prove that the sum of the original A.P. exceeds the sum of the arithmetic means by $\frac{1}{2}(a + l)$.
2. Prove that the sum of 10 arithmetic means between 8 and 30 is 190.
3. Find the arithmetic progression in which the arithmetic mean of the first n terms is equal to their number for any n.
4. If the arithmetic mean between a and b is twice as great as the geometric mean, show that $a : b = 2 + \sqrt{3} : 2 - \sqrt{3}$.
5. If odd number of G.M.'s are inserted between two given quantities a and b, show that the middle G.M. $= \sqrt{ab}$
6. Insert 6 geometric means between 14 and $-\frac{7}{64}$.
7. Prove that the product of 5 geometric means between 2 and 8 is 1024.
8. If 12 and $9\frac{3}{5}$ are the geometric and harmonic means, respectively, between two numbers, find them.
9. If the harmonic mean between two quantities is to their geometric means as 12 to 13, prove that the quantities are in the ratio of 4 to 9.
10. If b is the harmonic mean between a and c, prove that $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$.
11. Prove that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the GM or HM between positive numbers a and b according as $n = -\frac{1}{2}$ or -1.
12. If 9 AMs and again 9 HMs be inserted between 2 and 3 then prove that $A + \frac{6}{H} = 5$, A is any AM and H the corresponding HM.
13. If between any two quantities, there be inserted 2n – 1 arithmetic, geometric and harmonic means, show that nth means are in G.P.

3.9 ARITHMETICO-GEOMETRIC PROGRESSION

A sequence each term of which is formed by multiplying the corresponding terms of an A.P. and a G.P. is called Arithmetico-Geometric Progression.

For example

$1 + 3x + 5x^2 + 7x^3 + \ldots$ is an A.G.P.

Here 1, 3, 5, ... are in A.P. and 1, x, x^2, x^3..... are in G.P.

In general, let us consider

the A.P. $a, a+d, a+2d, \ldots\ldots, a+(n-1)d$

and the G.P. $1, r, r^2, \ldots\ldots\ldots\ldots, r^{n-1}$

then the A.G.P. is

$a, (a+d)r, (a+2d)r^2, \ldots\ldots, [a+(n-1)d]\, r^{n-1}$

The general term or the nth term of an A.G.P. can be obtained by multiplying the n^{th} term of an A.P. & n^{th} term of a G.P.

It is $t_n = [a + (n-1)d]\, r^{n-1}$

For instance, the n^{th} term of the series

$2x + 4x^2 + 6x^3 + 8x^4 + \ldots\ldots$

is $\{2 + (n-1)2\} . (x . x^{n-1}) = 2n.x^n$

Sum of n-terms of an Arithmetico-Geometric Series

Let $S_n = a + (a+d)r + a(a+2d)r^2 + \ldots. + [a + (n-1)d]\, r^{n-1}$ ...(1)

Multiplying both sides of (1) by common ratio r

$rS_n = 0 + ar + (a+d)r^2 + \ldots\ldots + [a + (n-1)d]\, r^n$ ...(2)

Subtracting (2) from (1), we get

$S_n(1-r) = a + dr + dr^2 + \ldots. + dr^{n-1}] - [a + (n-1)d]\, r^n$

$\Rightarrow S_n(1-r) = a + d[r + r^2 + \ldots. + r^{n-1}] - [a + (n-1)d]\, r^n$

$= a + \frac{dr(1-r^{n-1})}{(1-r)} - [a + (n-1)d]\, r^n$

$\Rightarrow S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)} - \frac{[a+(n-1)d]r^n}{(1-r)}$, $r \neq 1$ ...(3)

www.pdfworld.net

www.pdfworld.net

Sum to Infinity

If $|r| < 1$, i.e., $-1 < r < 1$ and $n \to \infty$ then $\underset{n\to\infty}{\text{Lim}}\ r^n = 0$

$\therefore \quad S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$.

➤ **Example 1.** If the sum to infinity of the series

$1 + 4x + 7x^2 + 10x^3 + \ldots.$ is $\dfrac{35}{16}$ then find x.

➤ ***Solution*** Let $S_\infty = 1 + 4x + 7x^2 + 10x^3 + \ldots \infty$ ...(1)

$\therefore \quad x\,S_\infty = 0 + x + 4x^2 + 7x^3 + \ldots. \infty$ ...(2)

Subtracting (2) from (1), we get

$(1-x)\,S_\infty = 1 + 3x + 3x^2 + 3x^3 + \ldots.. \infty$

$(1-x)S_\infty = 1 + \dfrac{3x}{1-x}$

$\Rightarrow \quad (1-x)\,\dfrac{35}{16} = \dfrac{1+2x}{1-x}$

$\Rightarrow \quad 35(1-x)^2 = 16 + 32x$

$\Rightarrow \quad 32x^2 - 102x + 19 = 0$

$\Rightarrow \quad (7x - 19)\,(5x - 1) = 0$

$x \neq \dfrac{19}{7}$ ($\because$ for infinite series $|x| < 1$)

$x = \dfrac{1}{5}$.

➤ **Example 2.** Find the sum to n terms of the series

$\dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \dfrac{4}{16} + \dfrac{5}{32} + \ldots\ldots\ldots$

Also find the sum (if it exists) if $n \to \infty$.

➤ ***Solution***

$S = \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \dfrac{4}{16} + \dfrac{5}{32} + \ldots.. + \dfrac{n}{2^n}$ ...(1)

$\dfrac{S}{2} = \dfrac{1}{4} + \dfrac{2}{8} + \dfrac{3}{16} + \ldots.. + \dfrac{n-1}{2^n} + \dfrac{n}{2^{n+1}}$ ...(2)

$\dfrac{S}{2} = \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \dfrac{1}{2^4} + \ldots\ldots\ldots + \dfrac{1}{2^n} - \dfrac{n}{2^{n+1}}$

$= \dfrac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} - \dfrac{n}{2^{n+1}} = 1 - \dfrac{1}{2^n} - \dfrac{n}{2^{n+1}}$

$S_n = 2\left[\dfrac{2^{n+1} - 2 - n}{2^{n+1}}\right] = \dfrac{2^{n+1} - 2 - n}{2^n}$

If $n \to \infty \qquad S_\infty = \underset{n\to\infty}{\text{Lim}}\left[2 - \dfrac{1}{2^{n-1}} - \dfrac{n}{2^n}\right] = 2$

➤ **Example 3.** If positive square root of,

$a^{\frac{1}{a}} \cdot (2a)^{\frac{1}{2a}} \cdot (4a)^{\frac{1}{4a}} \cdot (8a)^{\frac{1}{8a}} \ldots.. \infty$ is $\dfrac{8}{27}$,

find the value of 'a'.

➤ ***Solution***

$a^{\left(\frac{1}{a} + \frac{1}{2a} + \frac{1}{4a} + \ldots..\infty\right)} \cdot 2^{\frac{1}{2a} + \frac{1}{4a} + \frac{1}{8a} + \ldots.. \infty} = \dfrac{8}{27}$

now $\dfrac{1}{a}\left(1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \ldots..... \infty\right) = \dfrac{2}{a}$

and $\dfrac{1}{2a} + \dfrac{2}{4a} + \dfrac{3}{8a} + \ldots.. \infty = \dfrac{2}{a}$

(using sum of A.G.P.)

$\therefore \quad a^{\frac{1}{a}} \cdot 2^{\frac{1}{a}} = \dfrac{8}{27} = \left(\dfrac{1}{3}\right)^3 \cdot 2^3 \quad \Rightarrow \quad a = \dfrac{1}{3}$

➤ **Example 4.** Prove the identity :

$\left(x^{n-1} + \dfrac{1}{x^{n-1}}\right) + 2\left(x^{n-2} + \dfrac{1}{x^{n-2}}\right)$

$+ \ldots.. + (n-1)\left(x + \dfrac{1}{x}\right) + n = \dfrac{1}{x^{n-1}}\left(\dfrac{x^n - 1}{x - 1}\right)^2$.

➤ ***Solution*** The sum on the left may be rewritten as follows

$\left(\dfrac{1}{x^{n-1}} + \dfrac{2}{x^{n-2}} + \ldots.. + \dfrac{n-1}{x}\right) + [x^{n-1} + 2x^{n-2}$

$+ \ldots.. + (n-1)\,x] + n$

The first bracketed expression is equal to

$\dfrac{1}{x^n}[x + 2x^2 + \ldots.. + (n-1)\,x^{n-1}]$

$= \dfrac{x[(n-1)x^n - nx^{n-1} + 1]}{x^n(x-1)^2}$

The second bracketed expression is obtained from the first one by replacing x by 1/x.
Hence, we get the required result.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

Practice Problems [I]

1. Sum $1+3x+6x^2+10x^3+....$ to infinity, x being <1.
2. Find $\sum_{n=1}^{n} u_n$ if $u_n = \sum_{n=0}^{n} \frac{1}{2^n}$
3. Sum $a+\frac{1}{3}, 3a-\frac{1}{6}, 5a+\frac{1}{12}+....$ to 2p terms.
4. Determine the sums of the following series
   (i) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+.....+\frac{2n-1}{2^{n-1}}$;
   (ii) $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+.....+(-1)^{n-1}\frac{2n-1}{2^{n-1}}$.

3.10 SUMMATION OF SERIES

Series are often represented in compact form, called sigma notation, using the Greek letter Σ (sigma) as means of indicating the summation involved.
The sum of the first n terms of a series is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + ... + a_n$$

where i is the index of summation, 1 is the lower limit of summation and n is the upper limit of summation.

For example

(i) $\sum_{i=1}^{n=5} i$ stands for $1+2+3+4+5$

$\sum_{i=1}^{n=5} i$ is also written as $\sum_{n=1}^{5} n$ or $\sum_{1}^{5} n$

(ii) $\sum_{r=1}^{3}(3r+2)$ means $(3.1+2)+(3.2+2)+(3.3+2)$

(iii) $\sum_{1}^{4} n^2$ stands for $1^2+2^2+3^2+4^2$

(iv) $\sum_{1}^{3} \frac{r}{r+2}$ means $\frac{1}{1+2}+\frac{2}{2+2}+\frac{3}{3+2}$

Properties of Σ

(i) $\sum_{r=1}^{n} k = nk$, where k is a constant .

$\sum_{i=1}^{k} 2 = 2+2+..... \text{ to k terms} = 2k$

(ii) $\sum_{r=1}^{n} k\,a_r = k \sum_{r=1}^{n} a_r$,
(constant can be taken outside the Σ notation)

$\sum_{r=1}^{k} 2r = 2\sum_{r=1}^{k} r$

(iii) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$,
Σ notation can be separated in case of + and – signs.

$\sum_{r=1}^{n}(r^2 - r) = \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$

Sum of powers of natural numbers

(a) Sum of first n natural numbers:

$$\sum_{r=1}^{n} r = 1+2+3+....+n$$

(sum using formula of A.P.)

$$\Rightarrow \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

(b) Sum of squares of first n natural numbers:

$$\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+.....+n^2$$

$$\Rightarrow \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

***Proof*:**

We have $(x+1)^2 - x^3 = 3x^2 + 3x + 1$
Putting x = 1, 2, 3, 4,...., n then

$2^3 - 1^3 = 3.1^2 + 3.1 + 1$
$3^2 - 2^3 = 3.2^2 + 3.2 + 1$
$4^3 - 3^3 = 3.3^2 + 3.3 + 1$

www.pdfworld.net

www.pdfworld.net

$5^3 - 4^3 = 3.4^2 + 3.4 + 1$

.....

$(n+1)^3 - n^3 = 3.n^2 + 3.n + 1$

adding all of them, we get

$(n+1)^3 - 1^3 = 3(1^2 + 2^2 + 3^2 + ... + n^2)$

$+ 3(1 + 2 + 3 + ... + n) + (1 + 1 + 1 + ... + n \text{ times})$

$\Rightarrow \quad n^3 + 3n^2 + 3n = \Sigma n^2 + 3\Sigma n + n$

$\therefore \quad 3\Sigma n^2 = n^3 + 3n^2 + 3n - 3\,\frac{n(n+1)}{2} - n$

$\Rightarrow \quad \sum_{r=1}^{n} r^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

(c) Sum of cubes of first n natural numbers :

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ... + n^3$$

$$\Rightarrow \quad \sum_{r=1}^{n} r^3 = \frac{n^2\,(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$$

***Proof*:**

We have

$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$

Putting x = 1, 2, 3, ..., n then

$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$

$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$

$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$

.....

$(n+1)^4 - n^4 = 4.n^3 + 6.n^2 + 4.n + 1$

adding all we get

$(n+1)^4 - 1^4 = 4(1^3 + 2^3 + 3^2 + ... + n^3)$

$+ 4(1^2 + 2^2 + 3 + ... + n^2) + 4(1 + 2 + 3 + ... + n)$

$+ (1 + 1 + 1 + ... + n \text{ times})$

$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\Sigma n^3 + 5\Sigma n^2 + 4\Sigma n + n$

$= 4\,\Sigma n^3 + \frac{6.n(n+1)(2n+1)}{6} + \frac{4.n(n+1)}{2} + n$

$\Rightarrow \quad 4\Sigma\, n^2 = n^4 + 4n^3 + 6n^2$

$+ 4n - n\,(n+1)\,(2n+1) - 2n\,(n+1) - n$

$= n^2\,(n+1)^2$

$\therefore \quad \sum_{r=1}^{n} r^3 = \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

(d) Sum of fourth power of first n natural numbers :

$$\sum_{r=1}^{n} r^4 = 1^4 + 2^4 + 3^4 + ... + n^4$$

$\sum_{r=1}^{n} r^4$ can be found by using the identity

$k^5 - (k-1)^5 = 5k^4 - 10k^3 + 10k^2 - 5k + 1$

which gives $n^5 = \sum_{r=1}^{n} \{r^5 - (r-1)^5\}$

$= 5\sum_{r=1}^{n} r^4 - 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

After simplifying we get

$$\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)\,(2n+1)\,(3n^2 + 3n - 1)$$

If nth term of a sequence is given by

$t_n = an^3 + bn^2 + cn + d$

when a, b, c, d are constants,

Then sum of n terms of

$S_n = \Sigma\, t_n = a\,\Sigma\, n^3 + b\,\Sigma\, n^2 + c\,\Sigma\, n + \Sigma\, d$

This can be evaluated using the above results.

(e) Sum of products of numbers taking two at a time:

Given a set of n numbers $a_1, a_2, a_3, ... , a_n$, the sum of products of numbers taking two at a time is represented by $\sum_{1 \le i < j \le n} a_i a_j$

We have $(a_1 + a_2 + a_3 + ... + a_n)^2$

$= \sum_{i=1}^{n} a_i^2 + 2\sum_{1 \le i < j \le n} a_i a_j$

i.e., (sum of given numbers)^2 = sum of squares of numbers + 2 (sum of products of numbers taking two at a time)

This formula is useful to find the sum of products of numbers taking two at a time if the sum of the numbers and the sum of their squares can be obtained.

www.pdfworld.net

www.pdfworld.net

➤ **Example 1.** Find the sum of all the even positive integers less than 200 which are not divisible by 6 .

➤ ***Solution*** The required sum = (2 + 4) + (8 + 10) + (14 + 16) + ... + (194 + 196)

$= 1 \times 6 + 3 \times 6 + 5 \times 6 + ... + 65 \times 6$

$= 6(1 + 3 + 5 + ... + 65)$

$= 6 \times \frac{32}{2}(1 + 65) = 6 \times 33 \times 33 = 6534$

➤ **Example 2.** Sum to n terms the series whose nth term is $2^{n-1} + 8n^3 - 6n^2$.

➤ ***Solution*** Let the sum be denoted by S; then

$S = \Sigma 2^{n-1} + 8\Sigma n^3 - 6\Sigma n^2$

$= \frac{2^n - 1}{2-1} + \frac{8n^2(n+1)^2}{4} - \frac{6n(n+1)(2n+1)}{6}$

$= 2^n - 1 + n(n+1)[2n(n+1) - (2n+1)]$

$= 2^n - 1 + n(n+1)(2n^2 - 1).$

➤ **Example 3.** Find the sum to n terms of the series $1 + (1 + 2) + (1 + 2 + 3) + ...$

➤ ***Solution*** rth term of the given series = sum of numbers in the rth bracket

$t_r = (1 + 2 + 3 + 4 + ... + r)$

$= \frac{r(r+1)}{2} = \frac{1}{2}(r^2 + r)$

$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \frac{1}{2}\left(\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right)$

$= \frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$

$= \frac{1}{2} \cdot \frac{n(n+1)}{2}\left(\frac{2n+1}{3} + 1\right)$

$= \frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right) = \frac{n(n+1)(n+2)}{6}$

➤ **Example 4.** Find the sum $1^2+3^2+5^2 + ... + (2n-1)^2$.

➤ ***Solution*** We can write the sum in the form

$S_n = \{1^2 + 2^2 + 3^2 + 4^2 + ... + (2n-1)^2 + (2n)^2\} - \{2^2 + 4^2 + ... + (2n)^2\}$

$= \sum_{k=1}^{2n} k^2 - 4\sum_{k=1}^{n} k^2$

$= \frac{2n(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{6}$

$= \frac{n(4n^2 - 1)}{3}$

➤ **Example 5.** Find $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + ...$ to 10 terms.

➤ ***Solution*** Let

$S_n = (3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + ...$ to n terms ... (1)

Now nth term of the sequence 3, 5, 7....

$= 3 + (n-1)2 = 2n + 1$

$\therefore$ nth term of the series $3^3 + 5^3 + ...$ is $(2n+1)^3$

nth term of the sequence 2, 4, 6, ... is

$2 + (n-1)2 = 2n$

$\therefore$ nth term of the series $2^3 + 4^3 + ...$ is $(2n)^3 = 8n^3$

From (1), nth term of the given series $t_n = (2n+1)^3 - 8n^3$

$= 8n^3 + 3.4n^2 . 1 + 3. 2n . 1^2 + 1^3 - 8n^3$

$= 12n^2 + 6n + 1$

$\therefore \quad S_n = \Sigma 12n^2 + \Sigma 6n + \Sigma 1 = 12\Sigma n^2 + 6\Sigma n + n$

$= \frac{12n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$

$= 2n(n+1)2n+1) + 3n(n+1) + n$

$= n[2(2n^2 + 3n + 1) + 3n + 3 + 1]$

$= n(4n^2 + 9n + 6) = 4n^3 + 9n^2 + 6n$

Putting n = 10, we get

$S_{10} = 4. 10^3 + 9.10^2 + 6. 10 = 4960.$

➤ **Example 6.** Determine the sums of the series $1 - 2 + 3 - 4 + ... + (-1)^{n-1} n$;

➤ ***Solution*** First assume that n is even. Put n = 2m. Then $1 - 2 + 3 - 4 + ... + (-1)^{n-1}n$

$= 1 - 2 + 3 - 4 + ... + (2m-1) - 2m$

$= (1+3+...+2m-1) - (2+4+...+2m) = -m = -\frac{n}{2}.$

Now let n be odd and put n = 2m – 1. Then our sum takes the form

$[1-2+3-4+...-(2m-2)] + (2m-1) = -(m-1) + 2m - 1 = m = \frac{n+1}{2}$

Thus, if we put

$1 - 2 + 3 - 4 + ... + (-1)^{n-1}n = S,$

then $S = -\frac{n}{2}$ if n is even, $S = \frac{n+1}{2}$ if n is odd.

www.pdfworld.net

www.pdfworld.net

However, this result can be obtained in a simpler way. Indeed, if n is even, we have

$$S = [1-2] + [3-4] + [5-6] + \ldots.$$

$$+ [(2m-1) - 2m] = -1 \,.\, m = -m = -\frac{n}{2}$$

Hence we also get the result for odd n.

➤ **Example 7.** Find the sum of the series

$1^2 - 2^2 + 3^2 + \ldots. + (-1)^{n+1} n^2$.

➤ ***Solution*** We consider two cases, n odd and n even. Suppose n is odd so that n = 2m + 1 for some m. In this case the sum S_n is given by

$$S_n = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2m+1)^2$$

$$= \{1^2 + 2^2 + 3^2 + 4^2 + \ldots + (2m)^2 + (2m+1)^2\} - 2\{2^2 + 4^2 + \ldots + (2m)^2\}$$

$$= \sum_{k=1}^{2m+1} k^2 - 8\sum_{k=1}^{m} k^2$$

$$= \frac{(2m+1)(2m+2)(4m+3)}{6} - \frac{8m(m+1)(2m+1)}{6}$$

$$\therefore \quad S_n = \frac{(2m+1)(2m+2)}{2} = \frac{n(n+1)}{2}$$

If n is even. then n = 2m for some m. Hence the sum S_n is given by

$$S_n = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2m-1)^2 - (2m)^2$$

$$= \{1^2 + 2^2 + 3^2 + 4^2 + \ldots + (2m-1)^2 + (2m)^2\} - 2\{2^2 + 4^2 + \ldots + (2m)^2\}$$

$$= \sum_{k=1}^{2m} k^2 - 8\sum_{k=1}^{m} k^2$$

$$= \frac{2m(2m+1)(4m+1)}{6} - \frac{8m(m+1)(2m+1)}{6}$$

$$= -m(2m+1) = \frac{-n(n+1)}{2}$$

Thus for any n, $S_n = \dfrac{(-1)^{n+1} n(n+1)}{2}$

➤ **Example 8.** If $S_1, S_2, S_3, \ldots S_n, \ldots.$ are the sums of infinite geometric series whose first terms are 1, 2, 3, ... n, ... and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}, \ldots$ respectively, then find the value of $\sum_{r=1}^{2n-1} S_r^2$.

➤ ***Solution*** r^{th} series will have a = r and common ratio $= \dfrac{1}{r+1}$

$$\therefore \quad S_r = \frac{r}{1 - \frac{1}{r+1}} = \frac{r(r+1)}{r} = r+1 \quad \therefore \quad S_r^2 = (r+1)^2$$

$$\therefore \quad \sum_{r=1}^{2n-1} S_r^2 = \sum_{r=1}^{2n-1} (r+1)^2 = 2^2 + 3^2 + 4^2 + \ldots\ldots$$

$$+ (2n-1)^2 + (2n)^2$$

$$= 1^2 + 2^2 + 3^2 + \ldots\ldots + (2n)^2 - 1$$

= sum of the square of the first (2n) natural numbers

$$= \frac{(2n)(2n+1)(4n+1)}{6} - 1$$

$$= \frac{n(2n+1)(4n+1)}{3} - 1$$

➤ **Example 9.** Find the sum to n terms of the series,

$$0.7 + 7.7 + 0.77 + 77.7 + 0.777 + 777.7 + 0.7777 + \ldots\ldots$$ where n is even.

➤ ***Solution*** n = 2m

$$S = (0.7 + 0.77 + 0.777 + \ldots\ldots \text{ m term}) + (7.7 + 77.7 + 777.7 + \ldots\ldots \text{ m terms})$$

$$= \frac{7}{9}(0.9 + 0.99 + 0.999 + \ldots\ldots \text{ m terms})$$

$$+ \frac{7}{90}((10^2 - 1) + (10^3 - 1) + \ldots\ldots + (10^{m+1} - 1))$$

$$\frac{7}{9}\left\{\frac{n}{2} - \frac{1}{9}\left(1 - \frac{1}{10^{n/2}}\right)\right\}$$

$$+ \frac{7}{90}\left\{\frac{100}{9}\left(10^{n/2} - 1\right) - \frac{n}{2}\right\}$$

➤ **Example 10.** Find the number of shots arranged in a complete pyramid the base of which is an equilateral triangle, each side containing n shots.

➤ ***Solution*** Suppose that each side of the base contains n shots; then the number of shots in the lowest layer is

www.pdfworld.net

www.pdfworld.net

$$n + (n-1) + (n-2) + \ldots\ldots + 1$$

that is, $= \frac{n(n+1)}{2} = \frac{1}{2}(n^2 + n)$.

In this result write $n-1, n-2, \ldots$ for n , and we thus obtain the number of shots in the 2nd, 3rd,.... layers.

$$\therefore \quad S = \frac{1}{2}(\Sigma n^2 + \Sigma n) = \frac{n(n+1)(n+2)}{6}$$

➤ **Example 11.** There are n necklaces such that the first necklace contains 5 beads, the second contains 7 beads and, in general the ith necklace contains i beads more than the number of beads in (i– 1)th necklace. Find the total number of beads is all the n necklaces.

➤ ***Solution*** Let us write the sequence of the number of beads in the 1st, 2nd, 3rd, ..., nth necklaces.

$= 5, 7, 10, 14, 19, \ldots\ldots$

$= (4+1), (4+3), (4+6)(4+10), (4+15), \ldots\ldots,$

$\left[4 + \frac{n(n+1)}{2}\right]$

S_n = Total number of beads in the n necklaces

$$S_n = \left\{\underbrace{4+4+\ldots.+4}_{\text{n times}}\right\} + 1 + 3 + 6 + \ldots. + \frac{n(n+1)}{2}$$

$= 4n +$ sum of the first n triangular numbers.

$$= 4n + \frac{1}{2}(\Sigma n^2 + n) \quad = 4n + \frac{1}{2}(\Sigma n^2 + \Sigma n)$$

$$= 2n + \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{2}\frac{n(n+1)}{2}$$

$$= 4n + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{1}{12}[48n + 2n(n+1)(n+2)] = \frac{n}{6}[n^2 + 3n + 26].$$

Alternative :

Let us denote the number of beads in the kth necklace by a_k.

We are given that

$$a_k = a_{k-1} + k, \text{ for } k \geq 2. \qquad \ldots(i)$$

Also $a_1 = 5$, $a_2 = 7$ ...(ii)

We are required to find an and $a_1 + a_2 + a_3 + \ldots + a_n$.

Substituting k = 2, 3, ... , n successively in (i) throughout, we have

$a_2 = a_1 + 2,$

$a_3 = a_2 + 3,$

$\vdots$

$\vdots$

$a_n = a_{n-1} + n$

Adding column-wise and simplifying, we get

$$a_n = a_1 + (2 + 3 + \ldots. + n), \qquad \ldots(iii)$$

The terms $a_2, a_3 \ldots\ldots, a_{n-1}$ cancelling out on both sides.

From (iii), we have

$$a_n = 5 + \frac{1}{2}n(n+1) - 1, = \frac{1}{2}(n^2 + n + 8).$$

Since

$$\Sigma a_n = \frac{1}{2}\left\{\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) + 8n\right\},$$

$$= \frac{1}{6}n(n^2 + 3n + 26)$$

which is the total number of beads in all the n necklaces.

➤ **Example 12.** The squares of the natural numbers are grouped like (1^2); $(2^2, 3^2, 4^2)$; $(5^2, 6^2, 7^2, 8^2, 9^2)$; and so on. Find the sum of the elements in n^{th} group.

➤ ***Solution*** By observations, the last element of the n^{th} group = n^4

The number of elements in n^{th} group = $(2n-1)$

The first element of the n^{th} group = $(n^2 - 2n + 2)^2$

Hence sum of the numbers in n^{th} group are

$$S = (n^2 - 2n + 2)^2 + (n^2 - 2n + 3)^2 + \ldots.. + (n^2)^2$$

$$\Rightarrow \quad S = [1^2 + 2^2 + 3^2 + \ldots.. (n^2)^2] - [1^2 + 2^2 + 3^2 + ((n-1)^2)^2]$$

$$\Rightarrow \quad S = \frac{n^2(n^2+1)(2n^2+1)}{6} - \frac{(n-1)^2(n^2 - 2n + 2)(2n^2 - 4n + 3)}{6}$$

➤ **Example 13.** Find the value of the expression

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{k=1}^{i} j$

$$= \sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right]$$

$$\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}[2n+1+3] = \frac{n(n+1)(n+2)}{6}$$

➤ **Example 14.** The natural numbers are arranged in groups as given below;

1

2 3

4 5 6 7 8

9 10 11 12 13 14 15

.....

Prove that the sum of the numbers in the n^{th} group is $2^{n-2}\{2^n + 2^{n-1} - 1\}$.

➤ ***Solution*** Note that n^{th} group has 2^{n-1} terms. 1^{st} term in the n^{th} group is 2^{n-1} and the last term in the n^{th} group is $2^n - 1$.

$\therefore$ Sum of the terms in the n^{th} group

$$= \frac{N}{2}(A + L) \; ; \; N = 2^{n-1} \; ; \; A = 2^{n-1};$$

$$L = 2^n - 1 = 2^{n-2}\{2^n + 2^{n-1} - 1\}$$

Practice Problems [J]

1. Sum the series
   $2.3 + 3.6 + 4.11 + ... + (n + 1)(n^2 + 2)$
2. $11^2 - 12^2 + 13^2 - .... - 20^2 + 21^2$.
3. Find the sum $50^2 - 49^2 + 48^2 - 47^2 + ... + 2^2 - 1$.
4. $20^3 - 19^3 + 18^3 - ... + 2^3 - 1^3$.
5. If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that
   $9S_2^2 = S_3(1 + 8S_1)$
6. Show that
   $$\frac{1\times 2^2 + 2\times 3^2 + ... + n\times(n+1)^2}{1\times 2^2 + 2^2\times 3 + ... + n^2\times(n+1)} = \frac{3n+5}{3n+1}$$
7. Find the sum of n terms of the series whose n^{th} term is $4n(n^2 + 1) - (6n^2 + 1)$.
8. Find the sum of n terms of the series whose n^{th} term is $3(4^n + 2n^2) - 4n^3$.
9. Sum to n terms the series
   $1 . n^2 + 2(n - 1)^2 + 3(n - 2)^2 + ...$
10. Show that whether n is odd or even,
    $1^2 - 2^2 + 3^2 - 4^2 + ...$ to $(n - 1)$ terms
    $= \frac{1}{2}(-1)^n n(n-1)$.
11. Find (i) the sum of the squares, (ii) the sum of the products taken two together of the numbers 1, 4, 7, ... (3n – 2).
12. Find the number of shots which can be arranged in a pyramidal pile on a triangular base, each side of the base containing 10 shots.
13. If $S_1, S_2, S_3, ..., S_p$ are the sums of infinite geometric series, whose first terms are 1, 2, 3,....,p, and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \frac{1}{p+1}$ respectively, prove that
    $$S_1 + S_2 + S_3 + ... + S_p = \frac{p}{2}(p+3).$$

3.11 RECURRING SERIES

A series $a_0 + a_1 + a_2 + a_3 + ...$, in which from and after a certain term each term is equal to the sum of a fixed number of the preceeding terms multiplied respectively by certain constants is called a recurring series.

In the series $1 + 2x + 3x^2 + 4x^3 + 5x^4 + ...$, each term after the second is equal to the sum of the two

www.pdfworld.net

www.pdfworld.net

preceeding terms multiplied respectively by the constants 2x, and $-x^2$; these quantities being called constants because they are the same for all values of n. Thus $5x^4 = 2x \cdot 4x^3 + (-x^2) \cdot 3x^2$;
that is, $a_4 = 2xa_3 - x^2a_2$.

Consider the series

$$t_1 + t_2x + t_3x^2 + t_4x^3 + t_5x^4 + \ldots\ldots\ldots \quad \ldots(1)$$

In this type of series every term is a product of two numbers in which one form a G.P.

$x^0, x^1, x^2, x^3, x^4, \ldots\ldots$ form a G.P. in above series.

Case I : If $t_1, t_2, t_3, \ldots\ldots\ldots$ form a A.P.
(then the series (1) is called on arithmetico- geometric series, discussed before)

Case II : If $t_1, t_2, t_3, t_4, \ldots\ldots\ldots$ donot form an A.P. but $t_2 - t_1 = T_1$, $t_3 - t_2 = T_2$, $t_4 - t_3 = T_3$, ... from an A.P. Multiply equation (1) by x, the common ratio of geometric series and subtract this current equation from equation (1).

Case III: $T_1, T_2, T_3, \ldots$ donot form an A.P. But $T_1 - T_2$, $T_3 - T_2, T_4 - T_3, \ldots$ from an A.P.

For evaluating these type series you can multiply the last equation (means the equation at stage (A)) by the common ratio of G.P. & subtract this equation from (A). This process is repeated unitl you get the final equation in G.P.

➤ **Example 1.** Find the sum of

$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots$ to ∞.

➤ ***Solution*** Let $S = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots$

$(1 - x)S = 1 + 3x + 5x^2 + 7x^3 + \ldots\ldots$

$(1-x)^2 S = 1 + 2x + 2x^2 + 2x^3 + \ldots\ldots$

$= 1 + 2x \cdot \dfrac{1}{1-x}$ when $|x| < 1$,

$= \dfrac{1+x}{1-x} \qquad \therefore \quad S = \dfrac{1+x}{(1-x)^3}$

➤ **Example 2.** Evaluate $1^2 + 5^2 . 5 + 9^2 . 5^2 + 13^2 . 5^3 + \ldots\ldots$ to n terms.

➤ ***Solution*** This series is not an arithmetico-geometric series, because $1^2, 5^2, 9^2, 13^2, \ldots\ldots$ donot form an A.P. However their successive differences $5^2 - 1^2$,

$9^2 - 5^2, \ 13^2 - 9^2, \ldots$ form an A.P.

$S_n = 1 + 25 . 5 + 81 . 5^2 + 169 . 5^3 + \ldots.$

$\ldots + (4n + 1)^2 . 5^{n-1} \quad \ldots(1)$

$5 . S_n = \quad 1. 5 + 15 . 5^2 + 81. 5^3 + \ldots.$

$\ldots. + (4n - 3) . 5^{n-1} + (4n + 1)^2 . 5^n \quad \ldots(2)$

From (1) – (2)

$$S_n (1 - 5) = 1 + \underbrace{24.5 + 56.5^2 + 88.5^3 + \ldots(\ )}_{n-1 \text{ term}} - (4n + 1)^2 . 5^n \quad \ldots(3)$$

$$5(1 - 5) S_n = 1.5 + 24. 5^2 + 56 . 5^3 + \ldots.$$

$$\ldots + 8(4n - 1) 5^n - (4n + 1)^2 . 5^{n+1} \quad \ldots(4)$$

From (3) – (4)

$$S_n (1 - 5) - 5 (1 . 5) S_n = S_n (1 - 5)^2$$

$$= 1 + 23 . 5 + \underbrace{32.5^2 + 32.5^3 + \ldots.}_{(n-2) \text{ term}} - 8(4n - 1) 5^n + (4n + 1)^2 . 5^{n+1} \quad \ldots(5)$$

$$= 1 + 23. 5 + \frac{32.5^2(5^{n-2} - 1)}{5 - 1} - 8(4n - 1) 5^n + (4n + 1)^2 . 5^{n+1}$$

$$\Rightarrow \quad S_n = \frac{(4n+1)^2.5^{n+1} + 16.5^n - 40n.5^n - 84}{(1-5)^2}$$

3.12 METHOD OF SUBTRACTION

If the differences of the successive terms of a series are in A.P. or G.P., we can find nth term of the series by the following steps:

Step I : Denote the nth term and the sum of the series upto n terms of the series by T_n and S_n respectively.
Step II : Rewrite the given series with each term shifted by one place to the right.

Step III : Subtracting the above two forms of the series, find T_n.

Let $u_1, u_2, u_3, \ldots\ldots$ be a sequence, such that $u_2 - u_1, u_3 - u_2, \ldots\ldots\ldots$ is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

www.pdfworld.net

www.pdfworld.net

$S = u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n$ ...(i)

$S = u_1 + u_2 + \ldots\ldots\ldots + u_{n-1} + u_n$ ...(ii)

(i) – (ii)

$\Rightarrow \quad u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + \ldots\ldots + (u_n - u_{n-1})$

where the series $(u_2 - u_1) + (u_3 - u_2) + \ldots\ldots + (u_n - u_{n-1})$ is either in A.P. or in G.P. then we can find u_n and hence sum of this series as $S = \sum_{r=1}^{k} u_r$

Case 1 : If $(t_2 - t_1), (t_3 - t_2), \ldots\ldots, (t_{n-1} - t_n)$ are in A.P.

e.g. the series

$x, x + y, x + 2y + z, x + 3y + 3z, x + 4y + 6z, \ldots$;

the differences of consecutive terms are

$y, y + z, y + 2z, y + 3z, \ldots\ldots$ which are in A.P.

➤ **Example 1.** Evaluate $1 + 5 + 12 + 22 + 35 + \ldots\ldots\ldots$

➤ ***Solution*** $S_n = 1 + 5 + 12 + 22 + 35 + \ldots\ldots\ldots + t_n$ ...(1)

$S_n = 1 + 5 + 12 + 22 + \ldots\ldots\ldots + t_{n-1} + t_n$ ...(2)

From (1) – (2); $0 = 1 + 4 + 7 + 10 + 13 + \ldots\ldots$

$\ldots\ldots\ldots + t_n - t_{n-1} - t_n$

$\Rightarrow \quad t_n = 1 + \underbrace{4+7+10+13+\ldots\ldots}_{(n-1) \text{ terms are in A.P.}}$

$= 1 + \left(\frac{n-1}{2}\right)[2 \times 4 + (n-1-1)3] = \frac{3n^2 - n}{2}$

$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n} \left(\frac{3r^2 - r}{2}\right)$

$= \frac{1}{2}\left[\frac{3.n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$

$= \frac{n^3 + n^2}{2}$

Case 2 : If $t_2 - t_1, t_3 - t_2, \ldots\ldots, (t_{n-1} - t_n)$ are in G.P.

For example the series

$a, a + b, a + b + br, a + b + br + br^2, \ldots\ldots\ldots\ldots$;

the differences of consecutive terms are $b, br, br^2 \ldots\ldots\ldots\ldots$ which are in G.P.

➤ **Example 2.** Evaluate $2 + 6 + 22 + 86 + 342 + \ldots\ldots\ldots$ to n terms

➤ ***Solution*** $S_n = 2 + 6 + 22 + 86 + 342 + \ldots\ldots t_n$ ... (1)

$S_n = 2 + 6 + 22 + 86 + \ldots + t_{n-1} - t_n$ ... (2)

From (1) – (2) ; $0 = 2 + 4 + 16 + 64 + 256 + \ldots$

$\ldots\ldots\ldots + t_n - t_{n-1} - t_n$

$\Rightarrow \quad t_n = 2 + \underbrace{4+16+64+256+\ldots\ldots t_n - t_{n-1}}_{(n-1) \text{ terms are in A.P.}}$

$= 2 + \frac{4(4^{n-1} - 1)}{n - 1} = \frac{1}{3}(2 + 4^n)$

$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \frac{1}{3}\left[\sum_{i=1}^{n} 2 + \sum_{i=1}^{n} 4^r\right]$

$= \frac{(4^{n+1} + 3n^2 + 3n - 4)}{9}$

Case 3 :

It is not always necessary that the series of first order of differences i.e. $u_2 - u_1, u_3 - u_2 \ldots\ldots u_n - u_{n-1}$ is always either in A.P. or in G.P. In such case let

$u_1 = T_1, u_2 - u_1 = T_2, u_3 - u_2 = T_3 \ldots\ldots, u_n - u_{n-1} = T_n.$

So $u_n = T_1 + T_2 + \ldots\ldots + T_n$ .... (i)

$u_n = T_1 + T_2 + \ldots\ldots + T_{n-1} + T_n$ .... (ii)

(i) – (ii)

$\Rightarrow \quad T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots\ldots + (T_n + T_{n-1})$

Now, the series $(T_2 - T_1) + (T_3 - T_2) + \ldots\ldots + (T_n - T_{n-1})$ is series of second order of differences and when it is either in A.P. or in G.P., then $u_n = u_1 + \Sigma T_r$

Otherwise in the similar way we find series of higher order of differences and the n^{th} term of the series.

➤ **Example 3.** Evaluate

$2 + 5 + 12 + 31 + 86 + \ldots\ldots\ldots\ldots$ to n terms

➤ ***Solution***

$S = 2 + 5 + 12 + 31 + 86 + \ldots\ldots\ldots t_n$ ... (1)

$S = 2 + 5 + 12 + 31 + \ldots\ldots\ldots t_{n-1} + t_n$ ... (2)

From (1) – (2) ;

$2 + 3 + 7 + 19 + 55 + \ldots\ldots + t_n - t_{n-1} - t_n$

$\Rightarrow \quad t_n = 2 + 3 + 7 + 19 + 55 + \ldots\ldots + (t_n - t_{n-1})$ ...(3)

$t_n = 2 + 3 + 7 + 19 + \ldots\ldots\ldots$

$+ (t_{n-1} - t_{n-2}) - (t_n - t_{n-1})$ ...(4)

$0 = 2 + \underbrace{1+4+12+36+\ldots\ldots (\ldots\ldots) - T_n}_{(n-2) \text{ term}}$

$\Rightarrow \quad T_n = 2 + 1 + \underbrace{4+12+36+\ldots\ldots\ldots}_{n-2 \text{ term}}$

$\Rightarrow \quad T_n = 2 + 1 + 4(1 + 3 + 3^2 + \ldots\ldots\ldots) = 1 + 2 \cdot 3^{n-2}$

www.pdfworld.net

www.pdfworld.net

Now, $t_n = 2 + \sum_{r=1}^{n} T_r = 2+(n-1) + \frac{2.(3^{n-1}-1)}{3-1}$

$= n + 3^{n-1}$

$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n} \left(r + 3^{r-1}\right)$

$= \frac{n(n+1)}{2} + \frac{1(3^n-1)}{3-1} = \frac{1}{2}[n^2 + n + 3^n - 1]$

➤ **Example 4.** Find the sum of n terms

$3 + 7 + 13 + 21 + ...$

➤ ***Solution*** Let

$S = 3 + 7 + 13 + 21 + ... + T_n$ ...(i)

$S = 3 + 7 + 13 + ...+ T_{n-1} + T_n$ ...(ii)

(i) – (ii)

$\Rightarrow \quad T_n = 3 + 4 + 6 + 8 + ... + (T_n - T_{n-1})$

$= 3 + \frac{n-1}{2}[8 + (n + (n-2)2]$

$= 3 + (n-1)(n+2) = n^2 + n + 1$

Hence

$S = \Sigma(n^2 + n + 1) = \Sigma n^2 + \Sigma n + \Sigma 1$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$

$= \frac{n}{3}(n^2 + 3n + 5)$

➤ **Example 5.** Find the sum of n-terms

$1 + 4 + 10 + 22 + ...$

➤ ***Solution*** Let

$S = 1 + 4 + 10 + 22 + ... + T_n$ ... (i)

$S = 1 + 4 + 10 + ...+ T_{n-1} + T_n$ ... (ii)

(i) – (ii) $\Rightarrow T_n = 1 + (3 + 6 + 12 + ... + T_n - T_{n-1})$

$T_n = 1 + 3\left(\frac{2^{n-1}-1}{2-1}\right)$

$T_n = 3 \cdot 2^{n-1} - 2$

So $S = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2 = 3 \cdot \left(\frac{2^n - 1}{2-1}\right) - 2n$

$= 3.2^n - 2n - 3$

➤ **Example 6.** Find the nth term and the sum of n term of the series 2, 12, 36, 80, 150, 252.

➤ ***Solution*** Let

$S = 2 + 12 + 36 + 80 + 150 + 252 + ... + T_n$ ... (i)

$S = 2 + 12 + 36 + 80 + 150 + 252 + ... + T_{n-1} + T_n$ ... (ii)

(i) – (ii)

$\Rightarrow \quad T_n = 2 + 10 + 24 + 44 + 70 + 102 + ... + (T_n - T_{n-1})$ ... (iii)

$T_n = 2 + 10 + 24 + 44 + 70 + 102 + ...$

$+ (T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ ... (iv)

(iii) – (iv)

$\Rightarrow \quad T_n - T_{n-1} = 2 + 8 + 14 + 20 + 26 + ...$

$= \frac{n}{2}[4 + (n-1)6] = n[3n-1] = T_n - T_{n-1} = 3n^2 - n$

$\therefore$ The general term of given series is

$\Sigma T_n - T_{n-1} = \Sigma 3n^2 - n = n^3 + n^2.$

Hence sum of this series is

$S = \Sigma n^3 + \Sigma n^2 = \frac{n^2(n+1)^2}{4} + \frac{n(n+2)(2n+1)}{6}$

$= \frac{n(n+1)}{12}(3n^2 + 7n + 2)$

$\frac{1}{12} n(n+1)(n+2)(3n+1)$

➤ **Example 7.** Find the general term and sum of n terms of the series 9, 16, 29, 54, 103

➤ ***Solution*** Let

$S = 9 + 16 + 29 + 54 + 103 + ... + T_n$ ... (i)

$S = 9 + 16 + 29 + 54 + 103 + ... + T_{n-1} + T_n$ ... (ii)

(i) – (ii)

$\Rightarrow \quad T_n = 9 + 7 + 13 + 25 + 49 + ... + (T_n - T_{n-1})$ ... (iii)

$T_n = 9 + 7 + 13 + 25 + 49 + ... +$

$(T_{n-1} - T_{n-2}) + (T_n - T_{n-1})$ ...(iv)

(iii) – (iv)

$\Rightarrow \quad T_n - T_{n-1} = 9 + (-2) + \underbrace{6 + 12 + 24 + ...}_{(n-2)\text{terms}}$

$= 7 + 6[2^{n-2} - 1] = 6(2)^{n-2} + 1.$

$\therefore$ The general term is $T_n = 6(2)^{n-1} + n + 2$

Also sum $S = \Sigma T_n = 6\Sigma 2^{n-1} + \Sigma n + \Sigma 2$

www.pdfworld.net

www.pdfworld.net

$$= 6.\ \frac{(2^n - 1)}{2-1} + \frac{n(n+1)}{2} + 2n + 2n = 6(2^n - 1) + \frac{n(n+5)}{2}$$

➤ **Example 8.** The n^{th} term, a_n of a sequence of numbers is given by the formula $a_n = a_{n-1} + 2n$ for $n \geq 2$ and $a_1 = 1$. Find an equation expressing a_n as a polynomial in n. Also find the sum to n terms of the sequence.

➤ ***Solution*** $a_1 = 1$

$a_2 = a_1 + 4 = 1 + 4 = 5$

$a_3 = a_2 + 6 = 5 + 6 = 11$

$a_4 = a_3 + 8 = 11 + 8 = 19$ and so on hence

$S = 1 + 5 + 11 + 19 + ... + a_n$

$S = +1 + 5 + 11 + ... + a_{n-1} + a_n$

$(-)$ ———————————————————

$0 = 1 + 4 + 6 + 8 + ... + (a_n - a_{n-1}) - a_n$

$$a_n = 1 + 2\underbrace{(2 + 3 + 4 + ... + (a_n - a_{n-1}))}_{(n-1)\text{ terms}}$$

$$a_n = 1 + 2 \cdot \frac{n-1}{2}[4 + (n-2)] = 1 + (n-1)(n+2) = n^2 + n - 1.$$

$$\text{sum} = \sum a_n = \sum n^2 + \sum n - \sum 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - n$$

$$\frac{n}{6}[(n+1)(2n+1) + 3(n+1) - 6]$$

$$= \frac{n}{6}[2n^2 + 3n + 1 + 3n + 3 - 6]$$

$$= \frac{n}{6}[2n^2 + 6n - 2] = \frac{n(n^2 + 3n - 1)}{3}$$

Practice Problems [K]

Find the nth term and the sum of n terms of the series:

1. $2 + 7x + 25x^2 + 91x^3 + ...$
2. $1 + 2x + 6x^2 + 20x^3 + 66x^4 + 212x^5 + ...$
3. $1 + 3 + 7 + 15 + 31 + ...$
4. 2, 12, 36, 80, 150, 252,....
5. 9, 16, 29, 54, 103,....
6. 2, 5, 12, 31, 86,.....
7. Let $x = 1 + 3a + 6a^2 + 10a^3 + ..., |a| < 1;$
   $y = 1 + 4b + 10b^2 + 20b^3 + ..., |b| < 1.$
   Find $S = 1 + 3(ab) + 5(ab)^2 + ...$ in terms of x and y.

3.13 METHOD OF DIFFERENCES

If the rth term of a series can be expressed as the difference of two quantities one which is the same function r that the other is of r – 1, the sum of the series may be readily found.

For let the series be denoted by $t_1 + t_2 + t_3 + ... + t_n$, and its sum by S_n, and suppose that any term t_r can be put in the form $v_r - v_{r-1}$; then

$$S_n = (v_1 - v_0) + (v_2 - v_1) + ... + (v_{n-1} - v_{n-2}) + (v_n - v_{n-1}) = v_n - v_0.$$

In other words, if the nth term of a series can be written as $t_n = f(n) - f(n \pm 1)$

then $S_n = \sum_{n=1}^{n} t_n = \sum_{n=1}^{n} f(n) - \sum_{n=1}^{n} f(n \pm 1)$

$= \{f(1) + f(2) + ... + f(n)\} - \{f(2) + f(3) + ... + f(n+1)\}$ when '+' sign is taken

or $\{f(1) + f(2) + ... + f(n)\} - \{f(0) + f(1) + ... + f(n+1)\}$ when '–' sign is taken

$= f(1) - f(n+1)$ when '+' sign is taken

or $f(n) - f(0)$ when '–' sign is taken.

➤ **Example 1.** Find the sum to n-terms of the series $1.2 + 2.3 + 3.4 + ...$

➤ ***Solution*** Let T_r be the general term of the series

So $T_r = r(r+1)$

To express $t_r = f(r) - f(r-1)$ multiply and divide t_r by $[(r+2) - (r-1)]$

So $T_r = \frac{r}{3}(r+1)[(r+2) - (r-1)]$

$= \frac{1}{3}[r(r+1)(r+2) - (r-1)r(r+1)].$

Let $f(r) = \frac{1}{3}r(r+1)(r+2)$

so $T_r = [f(r) - f(r-1)].$

Now $S = \sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + ... + T_n$

www.pdfworld.net

www.pdfworld.net

$$T_1 = \frac{1}{3}[1.2.3 - 0]$$

$$T_2 = \frac{1}{3}[2.3.4 - 1.2.3]$$

$$T_3 = \frac{1}{3}[3.4.5 - 2.3.4]$$

$$\vdots$$

$$T_n = \frac{1}{3}[n(n+1)(n+2) - (n-1)n(n+1)]$$

$\therefore \quad S = \frac{1}{3} n(n+1)(n+2)$

Hence sum of series is $f(n) - f(0)$.

➤ **Example 2.** Sum the following series upto n-terms

$1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 + \ldots\ldots\ldots\ldots$

➤ ***Solution***

$S_n = 1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 + \ldots\ldots\ldots$

$$T_n = \frac{1}{5} n(n+1)(n+2)(n+3)[(n+4) - (n-1)]$$

$\therefore \quad$

$$T_1 = \frac{1}{5}[1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 - 0]$$

$$T_2 = \frac{1}{5}[2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 - 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5]$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$T_n = \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4) - (n-1)n(n+1)(n+2)(n+3)]$$

$\therefore \quad S_n = \sum T_n = \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4)]$

$$S_n = \frac{1}{5} n(n+1)(n+2)(n+3)(n+4).$$

➤ **Example 3.** Sum the series

$$\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \ldots\ldots + \frac{1}{n(n+3)}$$

➤ ***Solution*** $u_n = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$;

$\therefore \quad 3s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \ldots$

$$\frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$\therefore \quad s_n = \frac{1}{3}\left\{1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right\}$$

$$= \frac{11}{18} - \frac{1}{3}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right);$$

➤ **Example 4.** Sum to n terms of the series

$$\frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \frac{1}{(1+3x)(1+4x)} + \ldots$$

➤ ***Solution*** Let T_r be the general term of the series

$$T_r = \frac{1}{(1+rx)(1+(r+1)x)}$$

So $\quad T_r = \frac{1}{x}\left[\frac{[1+(r+1)x] - (1+rx)}{(1+rx)(1+(r+1)x)}\right]$

$$= \frac{1}{x}\left[\frac{1}{1+rx} - \frac{1}{1+(r+1)x}\right]$$

$T_r = f(r) - f(r+1)$

$\therefore \quad S = \Sigma T_r = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$$= \frac{1}{x}\left[\frac{1}{1+rx} - \frac{1}{1+(n+1)x}\right]$$

$$= \frac{n}{(1+x)[1+(n+1)x]}$$

➤ **Example 5.** Prove that

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \ldots.$$

to n terms $= \frac{6n}{n+1}$

➤ ***Solution*** Also find the sum to infinite terms of the series on the left hand side.

$$t_n = \frac{3 + (n-1)2}{1^2 + 2^2 + 3^2 + \ldots. + n^2}$$

$$= \frac{2n+1}{n(n+1)(2n+1)/6}$$

$$= \frac{6}{n(n+1)} = 6\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$$

$\therefore \quad t_1 = 6\left(1 - \frac{1}{2}\right), \; t_2 = 6\left(\frac{1}{2} - \frac{1}{3}\right),$

$t_3 = 6\left(\frac{1}{3} - \frac{1}{4}\right), \ldots\ldots\ldots\ldots\ldots, t_n = 6\left(\frac{1}{n} - \frac{1}{n+1}\right)$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

Adding, $S_n = 6\left(1-\frac{1}{n+1}\right) = \frac{6n}{n+1}$

$\therefore$ Sum of infinite terms

$= \lim_{n\to\infty} S_n = \lim_{n\to\infty} 6\left(1-\frac{1}{n+1}\right) = 6$

➤ **Example 6.** Find the sum to n terms of the series

$$\left[\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+.....+\frac{n}{1+n^2+n^4}\right]$$

➤ ***Solution*** $T_n = \frac{n}{1+n^2+n^4} = \frac{n}{(1+n^2)^2-n^2}$

$$= \frac{n}{(1+n^2-n)(1+n^2+n)}$$

$$= \frac{1}{2}\left(\frac{1}{1+n^2-n}-\frac{1}{1+n^2+n}\right)$$

$S_n = \sum_{r=1}^{n} t_r$

$$= \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{7}+.....+\frac{1}{1+n^2-n}-\frac{1}{1+n^2+n}\right)$$

$\Rightarrow \quad S_n = \frac{1}{2}\left(1-\frac{1}{1+n^2+n}\right) = \frac{n(n+1)}{2(n^2+n+1)}$

➤ **Example 7.** Show that

$$\frac{1^4}{1.3}+\frac{2^4}{3.5}+\frac{3^4}{5.7}+....+\frac{4^4}{(2n-1)(2n+1)}$$

$$= \frac{n(4n^2+6n+5)}{48}+\frac{n}{16(2n+1)}$$

➤ ***Solution*** The nth term of the given series is

$$T_n = \frac{n^4}{(2n-1)(2n+1)}$$

$$T_n = \frac{4n^2+1}{16}+\frac{1}{32}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$$

$$= \frac{1}{4}n^2+\frac{1}{16}+\frac{1}{32}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$$

$\therefore \quad S_n = \sum_{n=1}^{n} T_n$

$$= \frac{1}{4}\Sigma n^2+\frac{1}{16}\Sigma 1+\frac{1}{32}\left[\Sigma\frac{1}{2n-1}-\Sigma\frac{1}{2n+1}\right]$$

$$= \frac{1}{4}\frac{n(n+1(2n+1)}{6}+\frac{1}{16}n+\frac{1}{32}\left[\left(1+3+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+....+\frac{1}{2n-1}\right)-\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+....+\frac{1}{2n-1}+\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{4}\frac{n(n+1(2n+1)}{6}+\frac{1}{16}n+\frac{1}{32}\left[1-\frac{1}{2n+1}\right]$$

$$= \frac{n(n+1)(2n+1)}{24}+\frac{n}{16}+\frac{n}{16(2n+1)}$$

$$= \frac{n(4n^2+6n+5)}{48}+\frac{n}{16(2n+1)}$$

Partial Fractions

When numerator of the nth term t_n of the series is a constant and denominator is a polynomial in n which has linear factors.

1. Find the nth term t_n of the series and break it into partial fractions.
2. Then write down $t_1, t_2, t_3, ... t_n$ by putting n = 1, 2, 3,.... n and then add.
3. If sum to infinity is to be determined, first of all find S_n (sum to n terms of the series) and then find $\lim_{n\to\infty} S_n$

Method to find the partial fractions of t_n.

If $t_n = \frac{1}{(an+b)(cn+d)(pn+q)}$ then

$$t_n = \frac{A}{an+b}+\frac{B}{cn+d}+\frac{C}{pn+q}$$

where, A = value of $\frac{1}{(cn+d)(pn+q)}$ when

$an + b = 0$ i.e. $n = -\frac{b}{a}$

www.pdfworld.net

www.pdfworld.net

$$B = \text{value of } \frac{1}{(an+b)(pn+q)} \text{ when } n = -\frac{d}{c}$$

$$C = \text{value of } \frac{1}{(an+b)(cn+d)} \text{ when } n = -\frac{q}{p}$$

➤ **Example 8.** Find the sum of the following series to infinity : $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots\ldots\ldots$

➤ ***Solution*** Let t_n be the nth term and S_n the sum to n terms of the given series.

$$\therefore \quad t_n = \frac{1}{(\text{nth terms of seq. } 1, 2, 3, \ldots.)} \times \frac{1}{\begin{pmatrix}\text{nth term of the}\\ \text{seq. } 2, 3, 4, \ldots\end{pmatrix} \times \begin{pmatrix}\text{nth term of the}\\ \text{seq. } 3, 4, 5, \ldots.\end{pmatrix}}$$

$$= \frac{1}{n(n+1)(n+2)} \qquad \ldots(1)$$

Let $t_n = \frac{A}{n}+\frac{B}{n+1}+\frac{C}{n+2}$

then A = value of $\frac{1}{n(n+1)(n+2)}$ when $n = 0$

$$\therefore \quad A = \frac{1}{(0+1)(0+2)} = \frac{1}{2}$$

B=value of $\frac{1}{n(n+2)}$ when $n+1=0$ i.e. $n=-1$

$$\therefore \quad C = \frac{1}{-2(-2+1)} = \frac{1}{2}$$

From (1), $t_n = \frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2(n+2)}$

putting $n = 1\ 2, 3, \ldots., n$, we get

$$t_1 = \frac{1}{2.1}-\frac{1}{2}+\frac{1}{2.3}$$

$$t_2 = \frac{1}{2.2}-\frac{1}{3}+\frac{1}{2.4}$$

$$t_3 = \frac{1}{2.3}-\frac{1}{4}+\frac{1}{2.5}$$

.....

.....

$$t_{n-2} = \frac{1}{2(n-2)}-\frac{1}{n-1}+\frac{1}{2n}$$

$$t_{n-1} = \frac{1}{2(n-1)}-\frac{1}{n}+\frac{1}{2(n+1)}$$

$$t_n = \frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2(n+2)}$$

Adding we get,

$$S_n = \frac{1}{2.1}-\frac{1}{2}+\frac{1}{2.2}+\frac{1}{2(n+1)}-\frac{1}{n+1}+\frac{1}{2(n+2)}$$

$$= \frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}$$

$$\therefore \quad S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}\right]$$

$$= \frac{1}{4}$$

➤ **Example 9.** Sum to n terms

$$\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\ldots..$$

➤ ***Solution*** Here

$$u_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$$

$$= \frac{1}{4}.\frac{(2n+3)-(2n-1)}{(2n-1)(2n+1)(2n+3)}$$

$$\therefore \quad u_n = \frac{1}{4}.\frac{1}{(2n-1)(2n+1)} - \frac{1}{4}.\frac{1}{(2n+1)(2n+3)}$$

$$= v_{n-1} - v_n,$$

$$v_n = \frac{1}{4}.\frac{1}{(2n+1)(2n+3)};$$

$$\therefore \quad s_n = v_0 - v_n = \frac{1}{12}-\frac{1}{4}.\frac{1}{(2n+1)(2n+3)}$$

➤ **Example 10.** Sum the following series to infinity

$$\frac{1}{1\cdot4\cdot7}+\frac{1}{4\cdot7\cdot10}+\frac{1}{7\cdot10\cdot13}+\ldots\ldots\ldots$$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** $T_n = \dfrac{1}{[1+(n-1)3][1+3n][4+3n]}$

$$= \frac{1}{(3n-2)(3n+1)(3n+4)}$$

$$T_n = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$T_1 = \frac{1}{6}\left[\frac{1}{1\cdot 4} - \frac{1}{4\cdot 7}\right]$$

$$T_2 = \frac{1}{6}\left[\frac{1}{4\cdot 7} - \frac{1}{7\cdot 10}\right]$$

$$\vdots \quad \vdots \quad \vdots$$

$$T_n = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$\therefore \quad S_n = \sum T_n = \frac{1}{6}\left[\frac{1}{1\cdot 4} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$= \left[\frac{1}{24} - \frac{1}{6(3n+1)(3n+4)}\right]$$

as $n \to \infty$

$$\therefore \quad S_\infty = \frac{1}{24}.$$

➤ **Example 11.** Sum to n terms of the series

$$\frac{4}{1.2.3} + \frac{5}{2.3.4} + \frac{6}{3.4.5} + \ldots\ldots$$

➤ ***Solution*** Let $T_r = \dfrac{r+3}{r(r+1)(r+2)}$

$$= \frac{1}{(r+1)(r+2)} + \frac{3}{r(r+1)(r+2)}$$

$$= \left[\frac{1}{r+1} - \frac{1}{r+2}\right] + \frac{3}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$$

$$\therefore \quad S = \left[\frac{1}{2} - \frac{1}{n+2}\right] + \frac{3}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{5}{4} - \frac{1}{n+2}\left[1 + \frac{3}{2(n+1)}\right]$$

$$= \frac{5}{4} - \frac{1}{2(n+1)(n+2)}[2n+5]$$

➤ **Example 12.** Sum of the series

$$\frac{n}{1.2.3} + \frac{n-1}{2.3.4} + \frac{n-2}{3.4.5} + \ldots.$$

$$+ \frac{1}{n(n+1)(n+2)} \text{ is } \frac{n^2+3}{12(n+1)}$$

➤ ***Solution*** t_r, the rth term of the series is given by

$$t_r = \frac{n-r+1}{r(r+1)(r+2)}$$

$$= (n+1)\frac{1}{r(r+1)(r+2)} - \frac{1}{(r+1)(r+2)}$$

$$= (n+1)\left[\frac{1}{2r} - \frac{1}{r+1} + \frac{1}{2(r+2)}\right] - \left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

[resolving into partial fractions]

$$= (n+1)\left[\frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right) - \frac{1}{2}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)\right]$$

$$- \left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= \frac{n+1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right) - \left(\frac{n+1}{2} + 1\right)$$

$$\left(\frac{1}{r+1} - \frac{1}{r+2}\right)$$

$$\Rightarrow \quad \sum_{r=1}^{n} t_r = \left(\frac{n+1}{2}\right)\left(1 - \frac{1}{n+1}\right) -$$

$$\left(\frac{n+3}{2}\right)\left(\frac{1}{2} - \frac{1}{n+2}\right)$$

$$= \frac{n}{2} - \frac{n+3}{4} + \frac{n+3}{2(n+2)} = \frac{n(n+1)}{4(n+2)}$$

➤ **Example 13.** Find the sum of n terms of the series

$$\frac{1}{2.3}.2 + \frac{2}{3.4}.2^2 + \frac{3}{4.5}.2^3 + \ldots$$

➤ ***Solution*** Here $u_n = \dfrac{n}{(n+1)(n+2)}.2^n$

www.pdfworld.net

www.pdfworld.net

Let $\frac{n}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$,

or, $n = A(n+2) + B(n+1)$.

Equating the coeffs. of like powers of n,

$A + B = 1, \quad 2A + B = 0. \quad \therefore A = -1, B = 2.$

Now we may write $u_n = \frac{2^{n+1}}{n+2} - \frac{2^n}{n+1}$

Putting n = 1, 2, 3,...., n, we have

$u_1 = \frac{2^2}{3} - \frac{2}{2}$, $\quad u_2 = \frac{2^3}{4} - \frac{2^2}{3}$,

$u_3 = \frac{2^4}{5} - \frac{2^3}{4}$, ... $u_n = \frac{2^{n+1}}{n+2} - \frac{2^n}{n+1}$.

By addition, $S_n = \frac{2^{n+1}}{n+2} - 1$.

➤ **Example 14.** Sum to n terms of the series

$$\frac{1}{x+1} + \frac{2x}{(x+1)(x+2)} + \frac{3x^2}{(x+1)(x+2)(x+3)} + \ldots$$

➤ ***Solution*** Here

$$u_n = \frac{nx^{n-1}}{(x+1)(x+2)....(x+n)}$$

$$= \frac{(x+n)x^{n-1} - x^n}{(x+1)(x+2)....(x+n)}$$

$$= \frac{x^{n-1}}{(x+1)(x+2)....(x+n-1)} - \frac{x^n}{(x+1)(x+2)....(x+n)}$$

$u_1 = 1 - \frac{x}{x+1}$,

$u_2 = \frac{x}{x+1} - \frac{x^2}{(x+1)(x+2)}$,

$u_3 = \frac{x^2}{(x+1)(x+2)} - \frac{x^3}{(x+1)(x+2)(x+3)}$, ...

$u_n = \frac{x^{n-1}}{(x+1)(x+2)...(x+n-1)} - \frac{x^n}{(x+1)(x+2)....(x+n)}$

$\therefore \quad S_n = 1 - \frac{x^n}{(x+1)(x+2)....(x+n)}$.

➤ **Example 15.** Find the sum of n terms of the series

$$\frac{5}{1.2}\cdot\frac{1}{3} + \frac{7}{2.3}\cdot\frac{1}{3^2} + \frac{9}{3.4}\cdot\frac{1}{3^3} + \frac{11}{4.5}\cdot\frac{1}{3^4} + \ldots$$

➤ ***Solution*** Here $u_n = \frac{2n+3}{n(n+1)}\cdot\frac{1}{3^n}$

Assuming $\frac{2n+3}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

we find A = 3, B = –1.

Hence $u_n = \left(\frac{3}{n} - \frac{1}{n+1}\right)\frac{1}{3^n} = \frac{1}{n}\cdot\frac{1}{3^{n-1}} - \frac{1}{n+1}\cdot\frac{1}{3^n}$,

and therefore $S_n = 1 - \frac{1}{n+1}\cdot\frac{1}{3^n}$

➤ **Example 16.** Find the sum of n terms of the series

$2.2 + 6.4 + 12.8 + 20.16 + 30.32 + ...$

In the series 2, 6, 12, 20, 30,.... the nth term is $n^2 + n$; hence $t_n = (n^2 + n)2^n$.

Assume $(n^2 + n)\,2^n = (An^2 + Bn + C)\,2^n - \{A(n-1)^2 + B(n-1) + C\}2^{n-1}$;

dividing out by 2^{n-1} and equating coefficients of like powers of n, we have

$2 = A, 2 = 2A + B, 0 = C - A + B$;

whence A = 2, B = –2, C = 4.

$\therefore \quad t_n = (2n^2 - 2n + 4)\,2^n - \{2(n-1)^2 - 2(n-1) + 4\}\,2^{n-1}$;

and $S_n = (2n^2 - 2n + 4)\,2^n - 4 = (n^2 - n + 2)\,2^{n+1} - 4$.

➤ **Example 17.** Find the sum of n terms of the series

$$\frac{1}{3} + \frac{3}{3.7} + \frac{5}{3.7.11} + \frac{7}{3.7.11.15} + ...$$

The n^{th} term is $\frac{2n-1}{3.7.11....(4n-5)(4n-1)}$

➤ ***Solution*** Assume

$$\frac{2n-1}{3.7....(4n-5)(4n-1)}$$

$$= \frac{A(n+1)+B}{3.7...(4n-1)} - \frac{An+B}{3.7...(4n-5)}$$

$\therefore \quad 2n - 1 = An + (A + B) - (An + B)(4n - 1).$

www.pdfworld.net

www.pdfworld.net

On equating coefficients we have three equations involving the two unknowns A and B, and our assumption will be correct if values of A and B can be found to satisfy all three.

Equating coefficients of n^2, we obtain A = 0.

Equating the absolute terms, $-1 = 2B$; that is $B = -\frac{1}{2}$;

and it will be found that these values of A and B satisfy the third equation.

$$\therefore\ u_n = \frac{1}{2}.\frac{1}{3.7...(4n-5)} - \frac{1}{2}.\frac{1}{3.7...(4n-5)(4n-1)};$$

hence $S_n = \frac{1}{2} - \frac{1}{2}.\frac{1}{2.7.11....(4n-1)}$

➤ **Example 18.** Sum to n terms

$$\left[\frac{1}{1.3} + \frac{2}{1.3.5} + \frac{3}{1.3.5.7} + \frac{4}{1.3.5.7.9} + ...\right]$$

➤ ***Solution*** $t_n = \frac{n}{1.3.5.....(2n+1)}$

$$= \frac{1}{2}.\frac{(2n+1)-1}{1.3.5.....(2n+1)}$$

$$= \frac{1}{2}\left[\frac{1}{1.3.5.....(2n-1)} - \frac{1}{1.3.5.....(2n+1)}\right]$$

$$= \frac{1}{2}(T_{n-1} - T_n)$$

$$\therefore\ 2t_n = T_{n-1} - T_n \quad \left(\text{where } T_n = \frac{1}{1.3.5.....(2n+1)}\right)$$

$$2S_n = \sum_{n=2}^{n} 2t_n + 2t_1 = (T_1 - T_2) + (T_2 - T_3) + ... + T_{n-1} - T_n$$

$$\Rightarrow\ 2S_n = 2.\frac{1}{1.3} + \frac{1}{1.3} - \frac{1}{1.3.5.....(2n+1)}$$

$$\Rightarrow\ S_n = \frac{1}{2}\left[1 - \frac{1}{1.3.5.....(2n+1)}\right]$$

➤ **Example 19.** Find the sum of

$$\frac{3}{1.2}.\frac{1}{2} + \frac{4}{2.3}.\left(\frac{1}{2}\right)^2 + \frac{5}{3.4}.\left(\frac{1}{2}\right)^3 + ...$$

to n terms.

➤ ***Solution*** $t_n = \frac{3+(n-1)}{n(n+1)}\left(\frac{1}{2}\right)^n = \frac{n+2}{n(n+1)}\left(\frac{1}{2}\right)^n$

$$t_n = \left(\frac{2}{n} - \frac{1}{n+1}\right)\left(\frac{1}{2}\right)^n$$

$$\therefore\ S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\left(\frac{2}{r} - \frac{1}{r+1}\right)\left(\frac{1}{2}\right)^r$$

$$= \sum_{r=1}^{n}\left(\frac{2^{1-r}}{r} - \frac{1}{2^r(r+1)}\right)$$

$$= \sum_{r=1}^{n}\left(\frac{1}{r.2^{r-1}} - \frac{1}{(r+1).2^r}\right)$$

$$= \left(1 - \frac{1}{2.2^1}\right) + \left(\frac{1}{2.2^1} - \frac{1}{3.2^2}\right) + \left(\frac{1}{3.2^2} - \frac{1}{4.2^3}\right) + ...+ \left(\frac{1}{n.2^{n-1}} - \frac{1}{(n+1).2^n}\right)$$

$$= 1 - \frac{1}{2^n(n+1)}$$

➤ **Example 20.** Evaluate sum of n terms of the series

$$\frac{8}{5} + \frac{16}{65} + \frac{24}{325} + ...$$

➤ ***Solution*** Let $S_n = \frac{8}{5} + \frac{16}{65} + \frac{24}{325} + ...$ to n terms

$$= \sum_{r=1}^{n}\left(\frac{8r}{4r^4+1}\right)$$

www.pdfworld.net

www.pdfworld.net

$$= \sum_{r=1}^{n} \frac{8r}{(2r^2-2r+1)(2r^2+2r+1)}$$

$$= 2\sum_{r=1}^{n}\left(\frac{1}{(2r^2-2r+1)} - \frac{1}{(2r^2+2r+1)}\right)$$

$$= 2\left[\frac{1}{1} - \frac{1}{5} + \frac{1}{5} - \frac{1}{13} + \frac{1}{13} - \ldots + \frac{1}{(2n^2-2n+1)} - \frac{1}{(2n^2+2n+1)}\right]$$

$$= 2\left[1 - \frac{1}{2n^2+2n+1}\right] = \frac{4n^2+4n}{2n^2+2n+1}$$

Practice Problems [L]

Sum the following series to n terms and to infinity

1. $\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + \ldots$

2. $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + \ldots$

Sum the following series to n terms

3. $1.5.9 + 2.6\,10 + 3.7\,11 + \ldots$

4. $\frac{1}{2.4} + \frac{1.3}{2.4.6} + \frac{1.3.5}{2.4.6.8} + \ldots$

5. $1^2 + \frac{3^2}{2} + \frac{5^2}{2^2} + \frac{7^2}{2^3} + \ldots$

6. Find the sum of n terms of the series whose nth terms are :

   (i) $\frac{2n+3}{n(n+1)} \cdot \frac{1}{3^n}$, (ii) $n!.n$,

   (iii) $\frac{n+2}{n(n+1)(n+3)}$ (iv) $\frac{n^2(n^2-1)}{4n^2-1}$

   (v) $\frac{n}{1.3.5.....(2n+1)}$

7. Find the sum of the infinite series

   $\frac{1.2}{3} + \frac{2.3}{3^2} + \frac{3.4}{3^3} + \frac{4.5}{3^4} + \ldots$

8. Find the sum of n terms of the series

   $\frac{3}{1.2} \cdot \frac{1}{2} + \frac{4}{2.3} \cdot \frac{1}{2^2} + \frac{5}{3.4} \cdot \frac{1}{2^3} + \frac{6}{4.5} \cdot \frac{1}{2^4} + \ldots$

3.14 SUMMATION OF TERMS HAVING FACTORS IN A.P.

1. To find the sum of n terms of a series in which each term is the product of r factors in A.P. and also the first factors of all terms are in the same A.P.

   Let t_n (nth term) = (a + nb) {a + (n + 1)b} .... {a + (n + r – 1)b}, where a, b and r are constants.

   We may write

   $$u_n = \frac{u_n[\{a+(n+r)b\} - \{a+(n-1)b\}]}{(r+1)b}$$

   If we put $v_n = \frac{u_n\{a+(n+r)b\}}{(r+1)b}$,

   $$v_{n-1} = \frac{u_{n-1}\{a+(n+r-1)b\}}{(r+1)b}$$

   $$= \frac{u_n\{a+(n-1)b\}}{(r+1)b}.$$

   $\therefore \quad u_n = v_n - v_{n-1}.$

   $\therefore \quad S_n = \sum_{n=1}^{n} u_n = v_n - v_0.$

   Here v_0 is independent of n. For this we can write.

   Rule to write :

   $$S_n = v_n + C = \frac{u_n \,.\, \{a+(n+r)b\}}{(r+1)b} + C.$$

   C is determined by taking some particular values of n.

➤ **Example 1.** Show that the sum to n terms of the series whose rth term is 12(3r – 1) (3r + 2) (3r + 5) is (3n – 1)(3n + 2) (3n + 5) (3n + 8) + 80.

➤ ***Solution*** Here u_n = 12(3n – 1) (3n + 2) (3n + 5). The three factors 3n – 1, 3n + 2 and 3n + 5 are in

www.pdfworld.net

www.pdfworld.net

A.P. with c.d. 3. Moreover the first factors of the terms from the beginning are 2, 5, 8, ... These are also in A.P. with the same c.d. 3.

$$\therefore S_n = \frac{12(3n-1)(3n+2)(3n+5)(3n+8)}{(3+1).3} + C$$

To find out C, we put n = 1 and we have $S_1 = u_1$.

$$\therefore \quad 12 . 2 . 5 . 8 = \frac{12 . 2 . 5 . 8 . 11}{4.3} + C$$

$$\text{or,} \quad C = 12 . 2 . 5 . 8\left(1 - \frac{11}{12}\right) = 80.$$

Thus the sum of n terms

$$= (3n-1)(3n+2)(3n+5)(3n+8) + 80.$$

➤ **Example 2.** Find the sum of n terms of the series $1.5.9 + 2.6.10 + 3.7.11 + ...$

➤ ***Solution*** Here $u_n = n(n+4)(n+8)$.

The factors of u_n are in A.P. with the c.d. 4 but the first factors of all terms are not in the same A.P. Thus the sum of this series cannot be found out directly.

$$u_n = n\{(n+1)+3\}\{(n+2)+6\}$$
$$= n(n+1)(n+2) + 6n(n+1) + 3n(n+2) + 18n$$
$$= n(n+1)(n+2) + 6n(n+1) + 3n\{(n+1)+1\} + 18n$$
$$= n(n+1)(n+2) + 9n(n+1) + 21n.$$

$$\text{Now } S_n = \frac{n(n+1)(n+2)(n+3)}{4.1} + \frac{9n(n+1)(n+2)}{3.1} + \frac{21n(n+1)}{2.1} + C.$$

To determine C, we put n = 1.

$$1.5.9 = \frac{1.2.3.4}{4} + \frac{9.1.2.3}{3} + \frac{21.1.2}{2} + C$$

$$\text{or,} \quad 45 = 45 + C \qquad \text{or,} \qquad C = 0.$$

$$\therefore \quad S_n = n(n+1)\left[\frac{n^2+5n+6}{4} + 3n + 6 + \frac{21}{2}\right]$$

$$= \frac{n(n+1)}{4}[n^2 + 17n + 72]$$

$$= \frac{n(n+1)(n+8)(n+9)}{4}.$$

2. To find the sum of n terms of a series in which each term is the reciprocal of the product of r factors in A.P. and the first factors of the denominators of all terms are also in the same A.P.

$$\text{Let } u_n = \frac{1}{(a+nb)\{a+(n+1)b\} ....\{a+(n+r-1)b\}}.$$

$$\text{We may write } u_n = \frac{1}{(r-1)b} \cdot \frac{\{a+(n+r-1)b\} - (a+nb)}{(a+nb)\{(a+(n+1)b\}...\{a+(n+r-1)b\}}$$

$$= \frac{1}{(r-1)b}\left[\frac{a+(n-1)b}{\{a+(n-1)b\}(a+nb) .... \{a+(n+r-2)b\}} - \frac{a+nb}{(a+nb)\{a+(n+1)b\}....\{a+(n+r-1)b\}}\right]$$

$$= v_{n-1} - v_n,$$

where

$$v_n = \frac{a+nb}{(r-1)\,b(a+nb)\{a+(n+1)b\} ...\{a+(n+r-1)b\}}$$

$$\therefore \quad S_n = v_0 - v_n.$$

v_0 is independent of n, so we can write

$$S_n = C - \frac{(a+nb)u_n}{(r-1)b}.$$

The constant C is found out by taking some particular value of n.

Rule to write :

$$S_n = C - \frac{\left(\begin{matrix}\text{first factor of the}\\ \text{deno. of } u_n\end{matrix}\right) \times u_n}{\left(\begin{matrix}\text{number of factors in}\\ \text{the deno. of } u_n\end{matrix} - 1\right) \times \text{c.d.}}$$

➤ **Example 3.** Find the sum of the following series

$$\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + ... \text{ to infinity.}$$

➤ ***Solution*** First factor of the denominator of $u_n = 1 + (n-1)2 = 2n - 1$ and

www.pdfworld.net

www.pdfworld.net

$$u_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$$

The denominator consists of three factors which are in A.P. with the c.d. 2. Moreover the first factors of the denominators of all terms are also in the same A.P.

Thus $S_n = C - \frac{2n-1}{(2n-1)(2n+1)(2n+3)} \cdot \frac{1}{2.2}$

$$= C - \frac{1}{4(2n+1)(2n+3)}$$

Putting $n = 1$, $\frac{1}{1.3.5} = C - \frac{1}{4.3.5}$

or, $C = \frac{1}{3.5}\left(1+\frac{1}{4}\right) = \frac{1}{12}$

$\therefore \quad S_n = \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$

Now $\lim_{n\to\infty} S_n = \frac{1}{12}$

Hence the sum of the infinite series is $\frac{1}{12}$.

➤ **Example 4.** Find the sum to 'n terms of the series whose rth term is $\frac{3r+1}{(r+1)(r+2)(r+3)}$

➤ ***Solution*** The series is $\frac{4}{2.3.4} + \frac{7}{3.4.5} + \frac{10}{4.5.6} + \ldots$.

$$u_n = \frac{3n+1}{(n+1)(n+2)(n+3)}$$

$$= \frac{3n+3-2}{(n+1)(n+2)(n+3)}$$

$$= \frac{3}{(n+2)(n+3)} - \frac{2}{(n+1)(n+2)(n+3)}$$

$$S_n = C - \frac{3}{(n+3)(2-1).1} + \frac{2}{(n+2)(n+3)(3-1).1}$$

$$= C - \frac{3}{n+3} + \frac{1}{(n+2)(n+3)}$$

Putting $n = 1$, $\frac{4}{2.3.4} = C - \frac{3}{4} + \frac{1}{3.4} = C - \frac{2}{3}$, or, $C = \frac{1}{6} + \frac{2}{3} = \frac{5}{6}$

$\therefore \quad S_n = \frac{5}{6} - \frac{3n+5}{(n+2)(n+3)}$

➤ **Example 5.** Find the sum of the series

$\frac{1}{1.3.7} + \frac{1}{3.5.9} + \frac{1}{5.7.11} + \ldots\ldots$ to n terms

➤ ***Solution*** Here $u_n = \frac{1}{(2n-1)(2n+1)(2n+5)}$

$$= \frac{2n+3}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$= \frac{2n+5-2}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$= \frac{1}{(2n-1)(2n+1)(2n+3)} - \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

Now $S_n = C - \frac{1}{(2n+1)(2n+3).2.2} + \frac{2}{(2n+1)(2n+3)(2n+5).3.2}$

Putting

$$n = 1, \quad \frac{1}{1.3.7} = C - \frac{1}{3.5.4} + \frac{1}{3.5.7.3.}$$

$$= C - \frac{17}{3.3.4.5.7}$$

or $C = \frac{1}{1.3.7} + \frac{17}{3.3.4.5.7} = \frac{11}{180}$

$\therefore \quad S_n = \frac{11}{180} - \frac{6n+11}{12(2n+1)(2n+3)(2n+5)}$

If should be noticed that the above rule is only applicable to cases in which the factors of each terms form an arithmetical progression, and the first factors of the several terms are in the same arithmetical progression,

Thus the sum of the series

1. 3. 5 + 2. 4. 6 + 3.5.7 + ... to n terms, may be found as follows:

Here $u_n = (n+2)(n+4) = \{(n+1)+1\}\{(n+2)+2\}$

$= n(n+1)(n+2) + 2n(n+1) + n(n+2) + 2n$

$= n(n+1)(n+2) + 3n(n+1) + 3n.$

www.pdfworld.net

www.pdfworld.net

The rule can now be applied to each term; thus

$$S_n = \frac{1}{4} n(n+1)(n+2)(n+3) + n(n+1)(n+2) + \frac{3}{2} n(n+1) + 0$$

$$= \frac{1}{4} n(n+1)(n+4)(n+5),$$

the constant being zero.

To find the result of such summations the following result is useful:

Consider the sum of terms having r consecutive natural numbers as factors

$$S_n = 1 \cdot 2 \cdot 3 \cdots r + 2 \cdot 3 \cdot 4 \cdots (r+1) + 3 \cdot 4 \cdot 5 \cdots (r+2) + \text{.....upto n terms}$$

$$T_n = \frac{1}{r+1} n(n+1)(n+2) \cdots (n+r-1)\,[(n+r) - (n-1)]$$

$$\therefore \quad T_1 = \frac{1}{r+1}[1 \cdot 2 \cdot 3 \cdots (r+1) - 0]$$

$$T_2 = \frac{1}{r+1}[2 \cdot 3 \cdot 4 \cdots (r+2) - 1 \cdot 2 \cdot 3 \cdots (r+1)]$$

.....

$$T_n = \frac{1}{r+1}[n(n+1)(n+2) \cdots (n+r) - (n-1)n(n+1) \cdots (n+r-1)]$$

$$\therefore \quad S_n = \sum T_n = \frac{1}{r+1}[n(n+1)(n+2) \cdots (n+r)]$$

$$S_n = \frac{1}{r+1} n(n+1)(n+2) \cdots (n+r).$$

When the nth term of a series is a rational integral function of n it can be expressed in a form which will enable us readily to apply the method given above.

For suppose f(n) is a rational function of n of degree p, and assume

$f(n) = A + Bn + Cn(n+1) + Dn(n+1)(n+2) + \ldots.$,

where A, B, C, D,..... are undetermined constants p + 1 in number.

This identity being true for all values of n, we may equate the coefficients of like powers of n; we thus obtain p + 1 simple equations to determine the p + 1 constants.

➤ **Example 6.** Find the sum of n terms of the series whose general term is $n^4 + 6n^3 + 5n^2$.

➤ ***Solution*** Assume $n^4 + 6n^3 + 5n^2 = A + Bn + Cn(n+1) + Dn(n+1)(n+2) + En(n+1)(n+2)(n+3)$;

Equating coefficients of like powers of n, since this is an identity, gives A = 0, B = 0, C = –6, D = 0, E = 1.

Thus $n^4 + 6n^3 + 5n^2 = n(n+1)(n+2)(n+3) - 6n(n+1)$.

Hence

$$S_n = \frac{1}{5} n(n+1)(n+2)(n+3)(n+4) - 2n(n+1)(n+2)$$

$$= \frac{1}{5} n(n+1)(n+2)(n^2+7n+2).$$

➤ **Example 7.** Sum to terms the series

$6.9 + 12.21 + 20.37 + 30.57 + 42.81 + \ldots..$

➤ ***Solution*** We find that the nth term of the series

6, 12, 20, 30, 42,.... is $n^2 + 3n + 2$,

and the nth term of the series

9, 21, 37, 57, 81,.... is $2n^2 + 6n + 1$.

Hence $t_n = (n+1)(n+2)\{2n(n+3)+1\}$

$= 2n(n+1)(n+2)(n+3) + (n+1)(n+2)$;

$$\therefore \quad S_n = \frac{2}{5} n(n+1)(n+2)(n+3)(n+4) + \frac{1}{3}(n+1)(n+2)(n+3) - 2$$

Practice Problems [M]

1. Find the sum to n terms of
   $1.3 + 4.7 + 9.13 + 16.21 + 25.31 + \ldots...$
2. Find the sum to n terms of
   $1.5.9 + 2.6.10 + 3.7.11 + \ldots...$
3. Find the sum to n terms of
   $1.3.2^2 + 2.4.3^2 + 3.5.4^2 + \ldots...$
4. Find the sum of the following series
   $\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \frac{1}{4.7} \ldots$ to infinity.

www.pdfworld.net

www.pdfworld.net

5. Find the sum to n terms and to infinity :

$$\frac{1}{2.4.8}+\frac{1}{4.6.8}+\frac{1}{6.8.12}+.....$$

3.15 METHOD OF UNDETERMINED COEFFICIENTS

Let u_n denote some rational integral function of n, and let $u_1, u_2, u_3, u_4,.....$ denote the values of u_n when for n the values 1, 2, 3, 4,.... are written successively. We proceed to investigate a method of finding u_n when a certain number of the terms $u_1, u_2, u_3, u_4,....$ are given.

From the series $u_1, u_2, u_3, u_4, u_5,.....$ obtain a second series by subtracting each term from the term which immediately follows it

The series $u_2 - u_1, u_3 - u_2, u_4 - u_3, u_5 - u_4,.....$ thus found is called the series of the first order of differences, and may be conveniently denoted by

$$\Delta u_1, \Delta u_2, \Delta u_3, \Delta u_4,.....$$

By subtracting each term of this series from the term that immediately follows it, we have

$$\Delta u_2 - \Delta u_1, \Delta u_3 - \Delta u_2, \Delta u_4 - \Delta u_3,.....$$

which may be called the series of the second order of differences, and denoted by

$$\Delta_1 u_1, \Delta_2 u_2, \Delta_3 u_3,.....$$

If the first differences are constant, then assume the general term as $t_n = an + b$.

If the second differences are constant, then assume the general term as $t_n = an^2 + bn + c$.

If the third differences are constant, then assume the general term as $t_n = an^3 + bn^2 + cn + d$.

If the first differences are in G.P. with common ratio r, then assume the general term as $t_n = ar^n + b$.

If the second differences are in G.P. with common ratio r, then assume the general term as $t_n = ar^n + bn + c$.

If the third differences are in G.P. with common ratio r, then assume the general term as $t_n = ar^n + bn^2 + cn + d$.

The coefficients a, b, c,... are found by putting n = 1, 2, 3,.... and solving the resulting equations.

➤ **Example 1.** Find the nth term of the series
3, 7, 13, 21, 31,....

➤ ***Solution*** The successive order differences are
4, 6, 8, 10,.....
2, 2, 2, 2.....

Thus the second order differences are constant; hence we assume the general term $t_n = an^2 + bn + c$.

To determine the constants a, b, c make n equal to 1, 2, 3, successively, then

$a + b + c = 3$,
$4a + 2b + c = 7$,
$9a + 3b + c = 13$;

Solving these equations, we get a = 1, b = 1, c =1.
Thus $t_n = n^2 + n + 1$.

➤ **Example 2.** Find the nth term of the sequence
10, 23, 60, 169, 494,....

➤ ***Solution*** The successive order differences are
13, 37, 109, 335,.....
24, 72, 216,.....

Thus the second order differences are in geometrical progression whose common ratio is 3; hence we assume the general term $t_n = a.\ 3^{n-1} + bn + c$.

To determine the constants a, b, c make n equal to 1, 2, 3, successively,

then $a + b + c = 10$, $3a + 2b + c = 23$, $9a + 3b + c = 60$;
$a = 6$, $b = 1$, $c = 3$.

Thus $t_n = 6.3^{n-1} + n + 3 = 2\ .\ 3^n + n + 3$.

➤ **Example 3.** Sum to n terms the series
$1 + 4 + 11 + 26 + 57 + 120$ .....

➤ ***Solution***

1 4 11 26 57 120

First order differences

3 7 15 31 63

Second order differences

4 8 16 32

The second order of difference is a G.P. of common ratio 2.

In this case, t_n = A (common ratio)^n + (first degree polynomial in n since the second order differences are in G.P.)

$\therefore \quad t_n = A.2^n + Bn + C$

www.pdfworld.net

www.pdfworld.net

$$\left.\begin{aligned} t_1 &= 1 = 2A + B + C \\ t_2 &= 4 = 4A + 2B + C \\ t_3 &= 11 = 8A + 3B + C \end{aligned}\right\} \qquad \left.\begin{aligned} 3 &= 2A + B \\ 7 &= 4A + B \end{aligned}\right\}$$

$A = 2$

$B = -1$

$C = -2$

$\therefore \quad u_n = 2.2^n - n - 2$

$\therefore \quad \Sigma u_n = 2\,\Sigma\, 2^n - \Sigma n - 2\Sigma 1$

$$= 2\left\{\frac{2(2^n - 1)}{2 - 1}\right\} - \frac{n(n+1)}{2} - 2n$$

$$= 4(2^n - 1) - \frac{n}{2}(n + 5)$$

➤ **Example 4.** In the sequence 12, 40, 50, 168, 280, 432,..... find the general term t_n.

➤ ***Solution*** 12, 40, 90, 168, 280, 432, ...

28, 50, 78, 112, 152, ...,.....

(first differences)

22, 28, 34, 40 .....

(second differences)

6, 6, 6, 6, ...

(constant term)

then $t_n = a + bn + cn(n-1) + dn(n-1)(n-2)$ ...(1)

put in equation (1), n = 1, 2, 3, 4 and after solving these equations we get a = 0, b = 12, c = 8, d = 1.

$\therefore \quad t_n = 12n + 8n(n-1) + n(n-1)(n-2)$

$= n^3 + 5n^2 + 6n.$

Practice Problems [N]

1. Find the n^{th} term of the sequence 2, 7, 16, 29,.....
2. Find the sum to 'n' terms of the series, 3 + 15 + 63 + ...
3. Find the n^{th} term of the sequence 6, 11, 22, 45, 92.....
4. Find the n^{th} term and the sum of n terms of the series – 3 – 1 + 11 + 39 + 89 + 167 + ...
5. Find the n^{th} term and the sum of n terms of the series 1, 2, 3, 6, 17, 54, 171.....
6. The series of natural numbers is divided in to group (1), (2, 3, 4), (5, 6, 7, 8, 9) and so on. Show that the sum of the numbers in n^{th} group is $(n-1)^3 + n^3$.
7. The odd natural numbers have been divided into groups as (1, 3); (5, 7, 9, 11) ; (13, 15, 17, 19, 21, 23); ..... Show that the sum of numbers in the nth group is $4n^3$.

3.16 MISCELLANEOUS METHODS OF SUMMATION

Logarithmic Differentiation Method

There are certain series which can be evaluated using logarithmic differentiation.

➤ **Example 1.** Find the sum of

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \ldots + \frac{2^n x^{2^n - 1}}{1+x^{2^n}}$$

➤ ***Solution*** Let $f(x) = \log(1+x) + \log(1+x)^2 + \log(1+x^4) + \ldots \log\left(1 + x^{2^n}\right)$

Differentiating both sides w.r.t. x we have

$$f'(x) = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \ldots \frac{2^n . x^{2^n - 1}}{1+x^{2^n}}$$

Now $f(x) = \log\left\{(1+x)(1+x^2)(1+x^4)\ldots.(1+x^{2^n})\right\}$

$$= \log\left\{\frac{(1-x^2)}{(1-x)}(1+x^2)(1+x^4).....(1+x^{2^n})\right\}$$

$$= \log\left\{\frac{(1-x^4)}{(1-x)}(1+x^4)(1+x^4).....(1+x^{2^n})\right\}$$

$$= \log\left\{\frac{1}{1-x}(1-x^{2^{n+1}})\right\}$$

$$\Rightarrow \quad f'(x) = \frac{-2^{n+1}.\left(x^{2^{n+1}} - 1\right)}{\left(1 - x^{2^{n+1}}\right)} + \frac{1}{(1-x)}$$

$$= \frac{1}{(1-x)} - \frac{2^{2n+1}.(x^{2^{n+1}} - 1)}{(1 - x^{2^{n+1}})}$$

www.pdfworld.net

www.pdfworld.net

Alternative

If S be the sum then $S - \frac{1}{1-x} = \frac{1}{1+x} - \frac{1}{1-x}$

$$+\left(\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+.....+\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right)$$

$$= -\frac{2x}{1-x^2}+\frac{2x}{1+x^2}$$

$$+\left(\frac{4x^3}{1+x^4}+\frac{8x^4}{1+x^8}+...+\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right)$$

$$= -\frac{4x^3}{1-x^4}+\frac{4x^3}{1+x^4}+\left(\frac{8x^4}{1+x^8}+....+\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right)$$

.....

$$= -\frac{2^{n+1}.x^{2^{n+1}-1}}{1-x^{2^{n+1}}}$$

$$S = \frac{1}{1-x}+\frac{2^{n+1}.x^{2^{n+1}-1}}{x^{2^{n+1}}-1}$$

Integration Method

Some series can be evaluated using integration.

➤ **Example 2.** Evaluate

$$S = 1 - \frac{1}{4}+\frac{1}{6}-\frac{1}{9}+\frac{1}{11}-\frac{1}{14}+..... \text{ to infinity.}$$

➤ ***Solution*** The above series can be written in the form of integration as follows

$$S = \int_0^1 (1-x^3+x^5-x^8+x^{10}-x^{13}+.....)dx$$

$$S = \int_0^1 \left[(1-x^3)+x^5(1-x^3)+x^{10}(1-x^3)+.....\right]dx$$

$$= \int_0^1 \frac{(1-x^3)}{(1-x^5)}dx$$

$$= \int_0^1 \frac{(x^2+x+1)}{x^4+x^3+x^2+x+1}dx$$

$$= \int_0^1 \frac{adx}{(x^2+cx+1)}+\int_0^1 \frac{bdx}{(x^2+dx+1)}$$

From partial fractions we can calculate a, b, c, d,

$$a = \left(\frac{5+\sqrt{5}}{10}\right); b = \left(\frac{5-\sqrt{5}}{10}\right); c = \left(\frac{1-\sqrt{5}}{2}\right);$$

$$d = \left(\frac{1+\sqrt{5}}{2}\right)$$

After evaluating the integral we get

$$S = \frac{\pi}{50}\left[\sqrt{(10+2\sqrt{5})}+\sqrt{(10-2\sqrt{5})}\right]$$

➤ **Example 3.** If $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+..... \text{ upto } \infty = \frac{\pi^4}{90}$, then find the value of

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+ .....\text{upto } \infty$$

➤ ***Solution*** Let $S = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+.....\text{upto } \infty$

We have $\frac{\pi^4}{90} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}.....$

$$= S + \frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+.....$$

$$= S + \frac{1}{2^4}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+.....\right]$$

$$= S + \frac{1}{2^4}\times\frac{\pi^4}{90}$$

$$S = \frac{\pi^4}{90}\left(1-\frac{1}{16}\right) = \frac{\pi^4}{96}$$

➤ **Example 4.** Sum $\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+.....$ to n terms. Also find the sum to infinite terms when $|x| < 1$.

➤ ***Solution*** Here $t_n = \frac{x^{2^{n-1}}}{1-x^{2^n}}$

$$= \frac{(1+x^{2^{n-1}})-1}{(1-x^{2^{n-1}})(1+x^{2^{n-1}})}$$

$$= \frac{1}{1-x^{2^{n-1}}}-\frac{1}{1-x^{2^n}};$$

www.pdfworld.net

www.pdfworld.net

Putting n = 1, 2, 3, ... and adding, we get

$$\text{sum} = \frac{1}{1-x} - \frac{1}{1-x^2} + \frac{1}{1-x^2} - \frac{1}{1-x^4} + \frac{1}{1-x^4} - \frac{1}{1-x^8} + \ldots\ldots\ldots + \frac{1}{1-x^{2^{n-1}}} - \frac{1}{1-x^{2^n}}$$

$$= \frac{1}{1-x} - \frac{1}{1-x^{2^n}} = \frac{x - x^{2^n}}{(1-x)(1-x^{2^n})}$$

$$= \frac{x(1-x^{2^n-1})}{(1-x)(1-x^{2^n})}$$

$$\therefore \quad \text{Sum to infinite terms} = \lim_{n\to\infty} \frac{x(1-x^{2^n-1})}{(1-x)(1-x^{2^n})}$$

$$= \frac{x}{1-x}$$

since $x^{2^n} \to 0$ as $n \to \infty$ when $|x| < 1$.

➤ **Example 5.** Evaluate : $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n.3^m + m.3^n)}$

➤ ***Solution*** $S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n.3^m + m.3^n)}$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\left(\frac{3^m}{m}\right)\left(\frac{3^m}{m} + \frac{3^n}{n}\right)}$$

$$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{a_m(a_m + a_n)} \qquad \ldots(1)$$

$$\left(\text{Let } a_m = \frac{3^m}{m} \;\&\; a_n = \frac{3^n}{n}\right)$$

By interchanging m & n we get

$$S = \sum_{m=1}\sum_{n=1} \frac{1}{a_n(a_n + a_m)} \qquad \ldots(2)$$

adding (1) & (2) then

$$2S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{a_m a_n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{mn}{3^{m+n}} = \left(\sum_{n=1}^{\infty} \frac{n}{3^n}\right)^2$$

$$= \left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$

3.17 A.M. ≥ G.M. ≥ H.M. INEQUALITIES

Let a, b be two positive numbers. The positive numbers

$$A = \frac{a+b}{2},\ G = \sqrt{ab},$$

are called respectively the arithmetic mean and the geometric mean of a and b.

We know that $c^2 \geq 0$ for any real number c. This is an important inequality in itself. As a consequence of this property, we can derive many inequalities.

Since

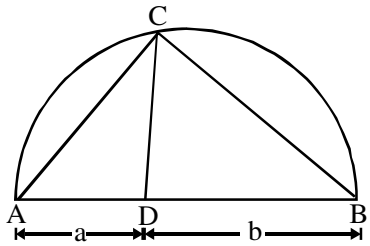
$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geq 0,$$

therefore, $\mathbf{A \geq G}$ ...(1)

i.e. $\frac{a+b}{2} \geq \sqrt{ab}$ ...(2)

The equality sign holds if and only if a = b. Hence we have the following result :

The arithmetic mean of two positive numbers is either greater than or equal to their geometric mean. The equality sign holds if the numbers are equal. This inequality has an interesting geometric interpretation. Let us consider a semicircle with diameter AB. If C is any point on the semicircle, then ABC forms a right angled triangle with the right angle at C. let CD be the perpendicular to AB dropped from C. Let AD = a and DB = b



The triangles DAC and DCB are similar, since $\angle ADC = \angle BDC$, $\angle DCA = \angle DBC$ and $\angle DAC = \angle DCB$. Hence $\frac{DA}{DC} = \frac{DC}{DB}$

www.pdfworld.net

www.pdfworld.net

Therefore $DC^2 = DA \cdot DB = ab$

and hence $DC = \sqrt{ab}$

If r is the radius of the semicircle, then $DC \le r$. But

$$r = \frac{a+b}{2}$$

thus we get $\sqrt{ab} \le \frac{a+b}{2}$

Thus we can interpret the inequality (2) as the statement that the perpendicular is the shortest distance from a point to a straight line. We also observe that DC = r if and only if D is the centre of the semicircle. And this is equivalent to a = b.

Relation between the Three Means

If a, b are positive real numbers, we have earlier defined their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as follows :

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b}.$$

(i) We first notice that A, G, H constitute a GP i.e. $G^2 = AH$.

(ii) If a, b are positive, then $A \ge G \ge H$. Also, equality occurs if and only if a = b.

Proof :

The first part can be easily verified and we have just proved that $A \ge G$.

$$\text{Now,} \quad G - H = \sqrt{ab} - \frac{2ab}{a+b} = \frac{\sqrt{ab}(\sqrt{a} - \sqrt{b})^2}{a+b}$$

shows that $G \ge H$.

If a = b, it is clear that A = G = H = a. Hence, in (ii) any equality sign holds if and only if the numbers are equal.

Also from the equation $G^2 = AH$, we see that G is intermediate in value between A and H; and it has been proved that $A \ge G$, therefore $G \ge H$; that is, the arithmetic, geometric, and harmonic means between any two positive quantities are in descending order of magnitude.

If A and G be respectively the arithmetic mean and the geometric mean of n positive numbers $a_1, a_2, \ldots, a_n$, then $A \ge G$. The equality sign holds if $a_1 = a_2 = \ldots = a_n$.

Proof : We shall first prove that the result holds for all those integers n that are powers of two.

The result holds when n = 2. For, if a_1, a_2 be two positive numbers, then

$$\frac{a_1 + a_2}{2} \ge (a_1 a_2)^{1/2},$$

and the equality sign holds if $a_1 = a_2$.

Let $a_1, a_2, \ldots, a_n$ where $n = 2^{k+1}$, be 2^{k+1} positive numbers.

Since the arithmetic mean of two positive numbers cannot be less than their geometric mean, therefore

$$\left.\begin{aligned} &\frac{a_1 + a_2}{2} \ge (a_1 a_2)^{1/2}, \\ &\frac{a_3 + a_4}{2} \ge (a_3 a_4)^{1/2}, \\ &\quad \ldots\ldots\ldots\ldots \\ &\quad \ldots\ldots\ldots\ldots \\ &\frac{a_{n-1} + a_n}{2} \ge (a_{n-1} a_n)^{1/2}, \end{aligned}\right\} \quad \ldots(1)$$

where $n = 2^{k+1}$

Adding the corresponding sides of all the above inequalities, we have

$$\frac{a_1 + a_2 + \ldots + a_n}{2} \ge \sqrt{(a_1 a_2)} + \ldots + \sqrt{(a_{n-1} a_n)} \quad \ldots(2)$$

Since the arithmetic mean of the 2^k $(= \frac{1}{2} n)$ numbers

$$\sqrt{(a_1 a_2)}, \ldots, \sqrt{(a_{n-1} a_n)},$$

cannot be less than their geometric mean, therefore

$$\frac{\sqrt{(a_1 a_2)} + \sqrt{(a_3 a_4)} + \sqrt{(a_{n-1} a_n)}}{2^k} \ge (\sqrt{[a_1 a_2 \ldots a_{n-1} a_n]})^{1/2^k} \quad \ldots(3)$$

www.pdfworld.net

www.pdfworld.net

From (2) and (3), we have

$$\frac{a_1 + a_2 + ... + a_n}{2} \geq 2^k (\sqrt{a_1 a_2 ... a_n})^{1/2^k},$$

or $$\frac{a_1 + a_2 + ... + a_n}{n} \geq (a_1 a_2 ... a_n)^{1/n} \quad ...(4)$$

The equality sign holds in (4) if it holds in (2) and (3), i.e. if it holds in each of the inequalities (1) and also in the inequality (3), i.e., if

$a_1 = a_2$, $a_3 = a_4$, ... , $a_{n-1} = a_n$,

and $\sqrt{(a_1 a_2)} = \sqrt{(a_3 a_4)} = ... = \sqrt{(a_{n-1} a_n)}$,

i.e., if $a_1 = a_2 = ... a_n$,

Therefore the result holds when $n = 2^{k+1}$

By the principle of induction, the result holds when n is any power of 2.

We shall now show that it holds for each positive integer n.

Let a_1, ... , a_n be n positive numbers, n being any positive integer whatever.

Determine a positive integer m such that $2^m > n$.

Let A, G denote respectively the arithmetic and geometric means of a_1, ... , a_n.

Applying (4) to 2^m numbers, $a_1, a_2, ... , a_n$, A, A, ..., A (the number of A's being $2^m - n$), we obtain

$$\frac{a_1 + a_2 + ... + a_n + A + A + ... + A}{2^m}$$

$$\geq (a_1 a_2 ..... a_n . A.A ... A)^{1/2^m},$$

or $$\frac{nA + (2^m - n)A}{2^m} \geq (G^n A^{2^m - n})^{1/2^m},$$

or $$A^{2^m} \geq G^n A^{2^m - n}$$

or $$A \geq G$$

The equality sign holds if $a_1 = a_2 = ... = a_n = A$

We have thus proved the result for all positive integers n.

Applying the inequality A.M. $\geq$ G.M. to the numbers $a_1^{-1}, a_2^{-1}, ... , a_n^{-1}$, we have

$$\frac{1}{2}(a_1^{-1} + a_2^{-1} + ...+ a_n^{-1}) \geq (a_1^{-1} a_2^{-1} .... a_n^{-1})^{1/n}$$

or $$(a_1 a_2 ... a_n)^{1/n} \geq \left[\frac{1}{n}\{a_1^{-1} + a_2^{-1} + ... + a_n^{-1}\}\right]^{-1}$$

The number on the right side is the harmonic mean of the numbers $a_1, a_2, ... , a_n$, denoted by H. The above relation can be written as $G \geq H$.

Together with the relation $A \geq G$, we get $A \geq G \geq H$.

Without loss of generality, we may assume that

$a_1 \leq a_2 \leq a_3 \leq ..... \leq a_n$.

Then A and G both lie between a_1 and a_n.

We have

$$a_1 = \frac{na_1}{n} \leq \frac{a_1 + a_n + ...+ a_n}{n} \leq \frac{na_n}{n} = a_n$$

i.e. $a_1 \leq A \leq a_n$

and similarly, we have

$$a_1 = ((a_1^n) \leq (a_1 a_2 ..... a_n)^{1/n} \leq ((a_n^n)^{1/n} = a_n.$$

i.e. $a_1 \leq G \leq a_n$

Similarly, it can be shown that $\frac{1}{H}$ will lie between $\frac{1}{a_1}$ and $\frac{1}{a_n}$.

We shall now define the arithmetic and geometric means of n positive numbers and extend the result $A \geq G$ for the general case.

Let $a_1, a_2, ... , a_n$ be n positive numbers. The positive numbers

$$A = \frac{a_1 + a_2 + ... + a_n}{n}$$

$$G = (a_1 a_2 ... a_n)^{1/n},$$

$$H = \left[\frac{\left\{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right\}}{n}\right]^{-1}$$

$$= \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$$

www.pdfworld.net

www.pdfworld.net

are called respectively the arithmetic mean, the geometric mean and the harmonic mean of

$a_1, a_2, \ldots, a_n$.

➤ **Example 1.** If a, b, c be positive, prove that $(b + c)(c + a)(c + b) \geq 8abc$.

➤ ***Solution*** Since the arithmetic mean of two positive numbers cannot be less than their geometric mean, therefore

$$\frac{b+c}{2} \geq \sqrt{bc},$$

$$\frac{c+a}{2} \geq \sqrt{ca},$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Multiplying the corresponding sides of the above inequalities, we obtain,

$$\frac{1}{8}(b + c)(c + a)(a + b) \geq abc,$$

i.e., $(b + c)(c + a)(a + b) \geq 8abc$.

➤ **Example 2.** If a, b, c are real numbers such that $0 < a < 1, 0 < b < 1, 0 < c < 1, a + b + c = 2$, prove that $\frac{a}{1-1} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$.

➤ ***Solution*** Let us put $a = y + z, b = z + x, c = x + y$, so that $x + y + z = 1$. Since $x = 1 - a, y = 1 - b, z = 1 - c$, and $0 < a < 1, 0 < b < 1, 0 < c < 1$, it follows that $x, y, z > 0$.

Now $\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c}$

$$= \frac{\{(y+z)(z+x)(x+y)\}}{(xyz)} \quad \ldots(1)$$

By the A.M.–G.M. inequality, we have

$$\frac{y+z}{2} \geq \sqrt{yz}, \frac{z+x}{2} \geq \sqrt{zx}, \frac{x+y}{2} \geq \sqrt{xy},$$

so that

$$\frac{\{(y+z)(z+x)(x+y)\}}{8} \geq (xyz) \quad \ldots(2)$$

From (1) and (2), we have the desired result.

➤ **Example 3.**

(i) If a, b, c, d be four distinct positive quantities in A.P., then show that

(a) $bc > ad$

(b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(ii) If a, b, c, d be four distinct positive quantities in G.P, then show that

(a) $a + d > b + c$

(b) $c^{-1} d^{-1} + a^{-1}b^{-1} > 2(b^{-1} d^{-1} + a^{-1}c^{-1} - a^{-1} d^{-1})$

(iii) If a, b, c, d be four distinct positive quantities in H.P., then show that

(a) $a + d > b + c$ (b) $ad > bc$

➤ ***Solution***

(i) ∵ a, b, c, d are in A.P.

(a) then A.M. > G.M. for first three terms

$\Rightarrow b > \sqrt{ac}$

$\therefore b^2 > ac$ ...(1)

and for last three terms

$c > \sqrt{bd}$

$\therefore c^2 > bd$ ...(2)

from (1) and (2)

then $b^2 c^2 > (ac)(bd)$

$\Rightarrow bc > ad$

(b) Again A.M. > H.M. for first three terms

$b > \frac{2ac}{a+c}$

$\Rightarrow ab + bc + 2ac$ ...(3)

and for last three terms

$c > \frac{2bc}{b+d}$

$\Rightarrow bc + cd > 2bd$ ...(4)

from (3) and (4)

$ab + cd + 2bc > 2ac + 2bd$

$\Rightarrow ab + cd > 2(ac + bd - bc)$

Dividing each term by abcd

then $c^{-1} d^{-1} + a^{-1} b^{-1} > 2(b^{-1} d^{-1} + a^{-1} c^{-1} - a^{-1} d^{-1})$

(ii) ∵ a, b, c, d are in G.P.

(a) Then A.M. > G.M. for first three terms

$\frac{a+c}{2} > b$

$\Rightarrow a > c > 2b$ ...(5)

www.pdfworld.net

www.pdfworld.net

and for last three terms $\frac{b+d}{2} > c$

$\Rightarrow \quad b + d > 2c$ ...(6)

from (5) and (6)

$a + c + b + d > 2b + 2c$

$\Rightarrow \quad a + d > b + c$

(b) Again G.M. > H.M. for first three terms

$$b > \frac{2ac}{a+c}$$

$\Rightarrow \quad ab + bc > 2ac$ ...(7)

for last three terms $c > \frac{2bd}{b+d}$

$\Rightarrow \quad bc + cd + 2bd$ ...(8)

from (7) and (8)

$ab + cd + 2bc > 2ac + 2bd$

Dividing each term by abcd

then $c^{-1} d^{-1} + a^{-1} b^{-1} > 2(b^{-1}d^{-1} + a^{-1} c^{-1} - a^{-1}d^{-1})$

(iii) $\because$ a, b, c, d are in H.P.

(a) Then A.M. > H.M. for first three terms,

$$\frac{a+c}{2} > b$$

$\Rightarrow \quad a + c > 2b$ ...(9)

and for last three terms, $\frac{b+d}{2} > c$

$\Rightarrow \quad b + d > 2c$ ...(10)

from (9) and (10)

$a + c + b + d > 2b + 2c$

$\Rightarrow \quad a + d > b + c$

(b) Again G.M. > H.M.

For first three terms, $\sqrt{ac} > b$

$\Rightarrow \quad ac > b^2$ ...(11)

add for last three terms

$\sqrt{bd} > c$

$\Rightarrow \quad bd > c^2$ ...(12)

from (11) and (12)

$(ac)(bd) > b^2c^2 \quad \Rightarrow \quad ad > bc.$

➤ **Example 4.** If a, b, c are in H.P. and they are distinct and positive then prove that $a^n + c^n > 2b^n$.

➤ ***Solution*** Let a^n and c^n be two numbers

then $\frac{a^n + c^n}{2} > (a^n c^n)^{1/2}$

$a^n + c^n > 2\,(ac)^{n/2}$ ...(1)

Also G.M. > H.M.

i.e. $\sqrt{ac} > b \Rightarrow (ac)^{n/2} > b^n$ ...(2)

Hence from (1) and (2) $a^n + c^n > 2b^n$

➤ **Example 5.** If a, b, c, d are four positive real numbers such that abcd = 1, prove that

$$(1 + a)\,(1 + b)\,(1 + c)\,(1 + d) \geq 16$$

➤ ***Solution*** Since AM ≥ GM, therefore,

$$\frac{1+c}{2} \geq \sqrt{(1.c)}, \quad \frac{1+d}{2} \geq \sqrt{(1.b)}.$$

Multiplying corresponding sides of the above inequalities, we have

$$(1 + a)\,(1 + b)\,(1 + c)\,(1 + d) \geq 16\sqrt{(abcd)}$$

$= 16$, since abcd = 1.

Hence $(1 + a)\,(1 + b)\,(1 + c)\,(1 + d) \geq 16.$

➤ **Example 6.** If a, b, c are any three real numbers, show that $a^4 + b^4 + c^2 \geq 2\sqrt{2}$ abc.

➤ ***Solution*** By applying the A.M.–G.M. inequality to the non-negative real numbers a^4, b^4 we have

$a^4 + b^4 \geq 2a^2b^2$,

so that $a^4 + b^4 + c^2 \geq 2a^2b^2 + c^2$

$$\geq 2\sqrt{(2a^2b^2.c^2)} \quad \text{...(1)}$$

From (1), we have

$a^4 + b^4 + c^2 \geq 2\sqrt{2}\ |abc| \geq 2\sqrt{2}$ abc.

➤ **Example 7.** If a, b, c are unequal positive numbers in HP then prove that $\frac{a+b}{2a-b} + \frac{c+b}{2c-b} > 4$

➤ ***Solution*** $\because$ a, b, c are in HP, $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

$$\text{LHS} = \frac{\frac{1}{b}+\frac{1}{a}}{\frac{2}{b}-\frac{1}{a}} + \frac{\frac{1}{b}+\frac{1}{c}}{\frac{2}{b}-\frac{1}{c}} = \frac{\frac{1}{b}+\frac{1}{a}}{\frac{1}{c}} + \frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{a}}, \text{ using (1)}$$

$$= \frac{c}{b} + \frac{c}{a} + \frac{a}{b} + \frac{a}{c} = \frac{a+c}{b} + \frac{a}{c} + \frac{c}{a}$$

$$\text{Now A} > \text{G} \Rightarrow \frac{\frac{a}{c}+\frac{c}{a}}{2} > \sqrt{\frac{a}{c}.\frac{c}{a}} \quad \text{or} \quad \frac{a}{c} + \frac{c}{a} > 2$$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

$$A > H \Rightarrow \frac{a+c}{2} > \frac{2ac}{a+c} = b, \text{ using (1)}$$

or $$\frac{a+c}{2} + \left(\frac{a}{c} + \frac{c}{a}\right) > 2 + 2 = 4$$

➤ **Example 8.** If x, y, z be positive numbers, show that $(x + y + z)^3 \geq 27\ xyz$.

➤ ***Solution*** Since A.M. (arithmetic mean) ≥ G.M. (geometric mean), therefore

$$\frac{x+y+z}{3} \geq (xyz)^{1/3}.$$

Cubing both sides and multiplying throughout by 27, we have $(x + y + z)^3 \geq 27\ xyz$.

➤ **Example 9.** Show that :

(i) $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} > 6$

(ii) $8xyz < (1 - x)(1 - y)(1 - z) < \frac{8}{27}$ when $x + y + z = 1$

(iii) $8(abcd + 1) > (a + 1)(b + 1)(c + 1)(d + 1)$ when each of a,b,c,d is greater than 1.

➤ ***Solution***

(i) Since $b + c > 2\sqrt{bc}$, $c + a > 2\sqrt{ca}$, $a + b > 2\sqrt{ab}$,

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} > 2\left(\frac{\sqrt{bc}}{a} + \frac{\sqrt{ca}}{b} + \frac{\sqrt{ab}}{c}\right)$$

$$> 2.3\left(\frac{\sqrt{bc}}{a} \cdot \frac{\sqrt{ca}}{b} \cdot \frac{\sqrt{ab}}{c}\right) \text{ [By A.M. > G.M.]}$$

> 6

(ii) By A.M. > G.M.,

$$\left(\frac{1-x+1-y+1-z}{3}\right)^3 > (1 - x)(1 - y)(1 - z),$$

or, $$\left\{\frac{3-(x+y+z)}{3}\right\}^3 > (1 - x)(1 - y)(1 - z),$$

or, $$\left(\frac{3-1}{3}\right)^3 > (1 - x)(1 - y)(1 - z),$$

or, $$\frac{8}{27} > (1 - x)(1 - y)(1 - z).$$

Since $x + y > 2\sqrt{xy}$. Similarly $1 - x > 2\sqrt{yz}$ and $1 - y > 2\sqrt{zx}$.

$\therefore\ (1 - x)(1 - y)(1 - z) > 8xyz.$

Thus $8xyz < (1 - x)(1 - y)(1 - z) < \frac{8}{27}$.

(iii) $2(ab + 1) - (a + 1)(b + 1) = ab - a - b + 1 = (a - 1)(b - 1) > 0.$

$\therefore\ (a + 1)(b + 1) < 2(ab + 1).$

Similarly $(c + 1)(d + 1) < 2(cd + 1).$

$\therefore\ (a + 1)(b + 1)(c + 1)(d + 1) < 4(ab + 1)(cd + 1) < 8(abcd + 1).$

Thus $8(abcd + 1) > (a + 1)(b + 1)(c + 1)(d+1).$

➤ **Example 10.** If α, β, γ are real numbers and $\beta + \gamma > \alpha$, $\gamma + \alpha > \beta$, $\alpha + \beta > \gamma$, show that $(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma) \leq \alpha\beta\gamma$.

➤ ***Solution*** By the A.M.–G.M. inequality,

$$\frac{\beta+\gamma-\alpha+\gamma+\alpha-\beta}{2} \geq (\beta + \gamma - \alpha)^{1/2}(\gamma + \alpha - \beta)^{1/2},$$

or $\gamma \geq (\beta + \gamma - \alpha)^{1/2}(\gamma + \alpha - \beta)^{1/2}.$

Similarly $\alpha \geq (\gamma + \alpha - \beta)^{1/2}(\alpha + \beta - \gamma)^{1/2}$ and $\beta \geq (\alpha + \beta - \gamma)^{1/2}(\beta + \gamma - \alpha)^{1/2}$.

Combining these results,

$$(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma) \leq \alpha\beta\gamma.$$

➤ **Example 11.** (i) If A be the area and 2s the sum of three sides of a triangle, show that $A \leq \frac{s^2}{3\sqrt{3}}$.

(ii) If a, b, c denote the sides of a triangle, show that $a^2(p–q)(p–r) + b^2(q–r)(q–p) + c^2(r–p)(r – q) \geq 0$, where p, q, r are real numbers in descending order.

➤ ***Solution*** (i) If a, b, c denote the sides of the triangle, then $A = \sqrt{s(s-a)(s-b)(s-c)}$.

www.pdfworld.net

www.pdfworld.net

By the A.M.–G.M. inequality

$$\frac{s-a+s-b+s-c}{3} \geq \{(s-a)(s-b)(s-c)\}^{1/2},$$

or $$\frac{3s-2s}{3} \geq \left(\frac{A^2}{s}\right)^{1/3}, \text{ or, } \frac{s}{3} \geq \left(\frac{A^2}{3}\right)^{1/3},$$

or $$A^2 \leq \frac{s^4}{3^3}, \text{ or, } A \leq \frac{s^2}{3\sqrt{3}}$$

Equality occurs when a = b = c.

(ii) Here p > q > r.

$\therefore$ (p – q) (p – r) > 0, (q – r) (q – p) < 0

and (r – p) (r – q) > 0.

$\therefore$ $a^2(p-q)(p-r) > 0$, $c^2(r-p)(r-q) > 0$

$\because$ $b^2 < (a+c)^2$, $b^2(q-r)(q-p) > (a+c)^2(q-r)(q-p)$.

The sign of inequality is changed due to the multiplication by a negative quantity.

$\therefore$ $a^2(p-q)(p-r) + b^2(q-r)(q-p) + c^2(r-p)(r-q)$

$> a^2(p-q)(q-r) + (a+c)^2(q-r)(q-p) + c^2(q-r)(p-r)$

$> a^2(p-q)^2 + 2ac(q-r)(q-p) + c^2(q-r)^2$

$> \{a(p-q) - c(q-r)\}^2 > 0$

If a = b = c and p, q, r are in A.P., then the given expression is equal to zero.

➤ **Example 12.** If n is positive integer, show that $2^{2n+1} > 1 + (2n+1)2^n$.

➤ ***Solution*** We have

$$\frac{1+2+2^2+.....+2^{2n}}{2n+1} > (1.2.2^2 \ldots 2^{2n})^{\frac{1}{2n+1}}$$

or, $$\frac{2^{2n+1}-1}{2n+1} > \left[2^{\frac{2n(2n+1)}{2}}\right]^{\frac{1}{2n+1}}, \text{ or, } \frac{2^{2n+1}-1}{2n+1} > 2^n$$

or $2^{2n+1} > 1 + (2n+1)2^n$.

➤ **Example 13.** Show that $n^n\left(\frac{n+1}{2}\right)^{2n} > (n!)^3$.

➤ ***Solution*** Consider the unequal positive numbers $1^3, 2^3, ...n^3$.

Since A.M. > G.M., therefore

$$\frac{1^3+2^3+.....+n^3}{n} > (1^3 . 2^3 ....n^3)^{1/n},$$

i.e., $$\frac{n(n+1)^2}{4} > \{(n!)^3\}^{1/n}.$$

Raising both sides to power n, we have

$$n^n\left(\frac{n+1}{2}\right)^{2n} > (n!)^3.$$

➤ **Example 14.** If a, b, c are positive real numbers such that a + b + c = 1, then find the minimum value of $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$.

➤ ***Solution*** $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} = \frac{a+b+c}{abc} = \frac{1}{abc}$

Also, $\frac{a+b+c}{3} \geq (abc)^{1/3} \Rightarrow abc \leq \frac{1}{27}$

$\Rightarrow$ $\frac{1}{abc} \geq 27$. Hence $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \geq 27$

➤ **Example 15.** If s be the sum of n positive unequal quantities a, b, c then prove the inequality;

$$\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+.....> \frac{n^2}{n-1} \ (n \geq 2).$$

➤ ***Solution*** AM > GM

$$[(s-a)+(s-b)+(s-c)+.....] > n[(s-a)+(s-b)+(s-c)+.....]^{1/n}$$

$$\left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+.....\right]$$

$$> n\left[\frac{1}{(s-a)(s-b)(s-c)}\right]^{1/n}$$

Multiplying $[(s-a)+(s-b)+.....]$

$$\left[\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+.....\right] > n^2$$

or $$\left[\frac{(s-a)}{s}+\frac{(s-b)}{s}+\frac{(s-c)}{s}+.....\right]$$

$$\left[\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+.....\right] > n^2$$

$$\left(\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+.....\right)\left(\frac{ns-s}{s}\right) > n^2$$

www.pdfworld.net

www.pdfworld.net

or $\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}+\ldots\ldots > \frac{n^2}{n-1}$

➤ **Example 16.** If a, b, c, d are all positive and the sum of any three is greater than twice the fourth, then show that, $abcd > (b+c+d-2a)(c+d+a-2b)(d+a+b-2c)(a+b+c-2d)$.

➤ ***Solution*** Use A.M. > G.M.

$a+b+c-2d = m_1$

$b+c+d-2a = m_2$

$c+d+a-2b = m_3$

$d+a+b-2c = m_4$

Now $m_1+m_2+m_3 = 3c$

$\Rightarrow \quad c = \frac{m_1+m_2+m_3}{3} > (m_1 m_2 m_3)^{1/3}$

$m_2+m_3+m_4 = 3d$

$\Rightarrow \quad d = \frac{m_2+m_3+m_4}{3} > (m_2 m_3 m_4)^{1/3}$

$m_3+m_4+m_1 = 3a$

$\Rightarrow \quad a = \frac{m_3+m_4+m_1}{3} > (m_3 m_4 m_1)^{1/3}$

$m_4+m_1+m_2 = 3b$

$\Rightarrow \quad b = \frac{m_4+m_1+m_2}{3} > (m_4 m_1 m_2)^{1/3}$

Hence $abcd > m_1 m_2 m_3 m_4$

This proves the inequality.

Practice Problems [O]

1. If $x \in R$, prove that $\frac{x^2}{1+x^4} \le \frac{1}{2}$.
2. If p, q, x, y are positive, show that $(px+qy)(pq+xy) \ge 4pqxy$.
3. If $a^2+b^2=1$, and $x^2+y^2=1$, show that $ax+by<1$.
4. Prove that $3ab^2 \le a^3+2b^3$.
5. Show that $n(n+1)^3 < 8(1^3+2^3+3^3+\ldots+n^3)$.
6. If a, b, c are positive numbers, prove that
   (i) $(a+b)(b+c)(c+a) \ge 8abc$
   (ii) $\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} \ge a+b+c$
7. If $a_1, a_2, a_3, \ldots\ldots, a_n$ are positive real numbers then prove that
   $\frac{(1+a_1+a_1^2)(1+a_2+a_2^2)\ldots\ldots(1+a_n+a_n^2)}{a_1 a_2 \ldots\ldots a_n} \ge 3^n$.
8. If a, b, c are positive, show that $a^3+b^3+c^3 \ge 3abc$.
9. If a, b, c, d be positive and inequal, show that $a^4+b^4+c^4+d^4 > 4abcd$.
10. Prove that $\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c} \ge 9$ when $2s = a+b+c$
11. Prove that
    $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 16$
12. If a, b, c be unequal and positive, show that
    (i) $(a+b+c)^2 > 3(bc+ca+ab)$
    (ii) $(bc+ca+ab)^2 > 3abc(a+b+c)$
13. If x, y, z be real show that $x^4+y^4+z^4 \ge y^2z^2+z^2x^2+x^2y^2$.
14. If a, b, c be positive, show that $(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) \ge 9a^2b^2c^2$.
15. If a, b, c be unequal and positive show that $(a+b+c)(bc+ca+ab) > 9abc$.

3.18 MISCELLANEOUS INEQUALITIES

1. Let a, b be positive real numbers and α, β be positive rational numbers such that $\alpha+\beta = 1$. Then $\alpha a+\beta b \ge a^\alpha . b^\beta$, and equality occurs if and only if a = b.

Proof :

We may assume that $\alpha = r/u$, $\beta = s/u$ where r, s, u $\in$ N. Then $\alpha+\beta = 1$ gives $u = r+s$. Applying the A.M.–G.M. inequality to the r + s numbers

$a_1 = a_2 = \ldots a_r = a$, $a_{r+1} = \ldots = a_{r+s} = b$,

we see that $\frac{ra+sb}{r+s} \ge (a^r b^s)^{1/(r+s)}$

i.e. $\alpha a+\beta b \ge a^\alpha . b^\beta$, with equality if and only if a = b.

2. Theorem on Weighted Means

If $(a_1, a_2, \ldots, a_n)$ be a set of n positive numbers and $(q_1, q_2, \ldots, q_n)$ be a set of n positive

www.pdfworld.net

www.pdfworld.net

rational numbers then $\left(\frac{\Sigma q_i a_i}{\Sigma q_i}\right)^{\Sigma q_i} > \prod a_i^{q_i}$

where i = 1, 2, ... , n and a's are not all equal.

Proof:

Case–I

Let $(q_1, q_2, ... , q_n)$ be the set of n positive integers. We may now consider q_i's positive quantities of which q_1 quantities are equal to a_1, q_2 quantities equal to a_2, ... , q_n quantities equal to a_n.
Applying the inequality A > G, we have

$$\frac{q_1a_1 + q_2a_2 + ... + q_na_n}{q_1 + a_2 + ... + q_n} > \left(a_1^{q_1} a_2^{q_2} .... a_n^{q_n}\right)^{\frac{1}{q_1 + q_2 + ... + q_n}}$$

or, $\left(\frac{\Sigma q_i a_i}{\Sigma q_i}\right)^{\Sigma q_i} > \prod a_i^{q_i}$

Case–II

If $q_1, q_2,....., q_n$ are not all integers, we can find out a positive number m such that $mq_1, mq_2,....., mq_n$ are all integers. By case I,

$$\left(\frac{mq_1a_1 + mq_2a_2 + ... + mq_na_n}{mq_1 + mq_2 + ... + mq_n}\right)^{mq_1+mq_2+....+mq_n} > a_1^{mq_1} a_2^{mq_2} .....a_n^{mq_n}$$

Extracting mth root and noting that m > 0, we have

$$\left(\frac{\Sigma q_i a_i}{\Sigma q_i}\right)^{\Sigma q_i} > \prod a_i^{q_i}$$

$\left(\frac{\Sigma q_i a_i}{\Sigma q_i}\right)^{\Sigma q_i} > \prod a_i^{q_i}$ when $a_1 = a_2 = ... = a_n$.

➤ **Example 1.** If $a_1, a_2, ... , a_n$ are positive and not all equal and S is their sum, show that

$$\left(\frac{S}{a_1} - 1\right)^{a_1} \left(\frac{S}{a_2} - 1\right)^{a_2} .... \left(\frac{S}{a_n} - 1\right)^{a_n} < (n - 1)^S$$

➤ ***Solution*** By the theorem of weighted means,

$$\left(\frac{S}{a_1} - 1\right)^{a_1} \left(\frac{S}{a_2} - 1\right)^{a_2} .... \left(\frac{S}{a_n} - 1\right)^{a_n}$$

$$< \left[\frac{a_1\left(\frac{S}{a_1} - 1\right) + a_2\left(\frac{S}{a_2} - 1\right) + ... + a_n\left(\frac{S}{a_n} - 1\right)}{a_1 + a_2 + ... + a_n}\right]^{a_1 + a_2 + ... + a_n}$$

$$< \left(\frac{nS - \Sigma a_i}{\Sigma a_i}\right)^{\Sigma a_i} < \left(\frac{nS - S}{S}\right)^S < (n - 1)^S$$

➤ **Example 2.** If a is any positive number except 1 and x, y, z are distinct rationals which may be positive or negative, prove that

$$a^x(y - z) + a^y(z - x) + a^z(x - y) > 0$$

➤ ***Solution*** It may be assumed that x > y > z.
Now y – z and x – y are positive.
By the theorem of weighted means,

$$\left\{\frac{(y - z)a^x + (x - y)a^z}{(y - z) + (x - y)}\right\}^{y-z+x-y} > a^{x(y-z)} \cdot a^{z(x-y)},$$

or $\left\{\frac{(y - z)a^x + (x - y)a^z}{(x - z)}\right\}^{x-z} > a^{y(x-z)},$

or $(y - z)a^x + (x - y)a^z > (x - z)a^y,$

or $(y - z)a^z + (z - x)a^y + (x - y)a^z > 0$

3. mth Powers Theorem

If $(a_1, a_2, ... , a_n)$ be a set of positive numbers and all a's are not equal, then $\frac{\Sigma a_i^m}{n} \gtrless \left(\frac{\Sigma a_i}{n}\right)^m$ according as m does not or does lie between 0 and 1.

www.pdfworld.net

www.pdfworld.net

If m = 0 or 1, both sides will be equal.

➤ **Example 3.** Prove that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n} < n\sqrt{\frac{n+1}{2}}$$

when n > 1

➤ ***Solution*** By mth power theorem,

$$\frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n}}{n} < \left(\frac{1 + 2 + 3 + ... + n}{n}\right)^{1/2}$$

or $$\frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n}}{n} < \left(\frac{n+1}{2}\right)^{1/2}$$

or $$\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n} < n\sqrt{\frac{n+1}{2}}$$

4. Cauchy-Schwartz inequality

If a, b, c and x, y, z are any real numbers (positive, zero or negative), then

$$(ax + by + cz)^2 \le (a^2 + b^2 + c^2)(x^2 + y^2 + z^2),$$

with equality if a : b : c :: x : y : z.

Proof : We have $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2$,

$= (a^2y^2 + b^2x^2 - 2abxy) + (b^2z^2 + c^2y^2 - 2bcyx) + (c^2x^2 + a^2z^2 - 2cazx)$,

$= (ay - bx)^2 + (bz - cy)^2 + (cx - az)^2 \ge 0.$

The equality sign holds if

$$ay - bx = 0,\ bz - cy = 0,\ cx - ax = 0,$$

i.e., if $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$,

i.e., if a : b : c :: x : y : z.

The above inequality can be put in the following more general form :

If $a_1, ... , a_n$ and $b_1, ... , b_n$ are any real numbers (positive, zero or negative), then

$(a_1b_1 + ... a_nb_n)^2 \le (a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2)$,

with equality if $a_1 : a_2 : ..... : a_n :: b_1 : b_2 : ..... : b_n$.

Proof : We have the identity

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) - (a_1b_1 + a_2b_2 + ... + a_nb_n)^2$$
$$= (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + ... + (a_{n-1}b_n - a_nb_{n-1})^2.$$

This is called **Cauchy-Lagrange's identity**.

Hence $(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \ge (a_1b_1 + a_2b_2 + ... + a_nb_n)^2$

Equality occurs if and only if

$$a_1b_2 - a_2b_1 = ... = a_{n-1}b_n - a_nb_{n-1} = 0,$$

i.e. $\frac{a_1}{b_1} = \frac{a_2}{b_2} ... = \frac{a_n}{b_n}$

Alternative Proof :

Consider the polynomial equation

$$(a_1x - b_1)^2 + (a_2x - b_2)^2 + ... + (a_nx - b_n)^2 = 0 \quad ...(1)$$

If (1) has a real root, say $x = \alpha$,

i.e., if $(a_1\alpha - b_1)^2 + (a_2\alpha - b_2)^2 + ... + (a_n\alpha - b_n)^2 = 0$,

then each of the numbers

$$a_1\alpha - b_1,\ a_2\alpha - b_2,\ .....,\ a_n\alpha - b_n \text{ equals zero,}$$

i.e. $a_1\alpha - b_1 = a_2\alpha - b_2 = ... = a_n\alpha - b_n = 0,$

i.e. $\alpha = b_1/a_1 = b_2/a_2 = ... = b_n/a_n$

Thus we have seen that the equation (1), which is the same as the equation

$$x^2(a_1^2 + a_2^2 + ... + a_n^2) - 2x(a_1b_1 + a_2b_2 + ... + a_nb_n) + (b_1^2 + b_2^2 + ... + b_n^2) = 0, \quad ...(2)$$

cannot have two different real roots and consequently the discriminant of equation (2) must be less than or equal to zero.

Hence $(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \le (a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2)$.

The equality sign holds if the equation (1) has a real root, i.e., if $b_1/a_1 = b_2/a_2 = ... = b_n/a_n$.

➤ **Example 4.** If x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 27$, then show that $x^3 + y^3 + z^3 \ge 81$.

➤ ***Solution*** Applying Cauchy-Schwartz inequality to the two sets of numbers $x^{3/2}, y^{3/2}, z^{3/2}, ; x^{1/2}, y^{1/2},, z^{1/2}$, we have

$$(x^2 + y^2 + z^2)^2 \le (x^3 + y^3 + z^3)(x + y + z). \quad ...(1)$$

www.pdfworld.net

www.pdfworld.net

Again, applying Cauchy-Schwartz inequality to the two sets of numbers x, y, z; 1, 1, 1,

we have $(x + y + z)^2 \le 3(x^2 + y^2 + z^2)$ ...(2)

Squaring both sides of (1), we have

$$(x^2 + y^2 + z^2)^4 \le (x^3 + y^3 + z^3)^2 (x + y + z)^2.$$

On using (2), the above inequality yields,

$$(x^2 + y^2 + z^2)^4 \le 3(x^3 + y^3 + z^3)^2 (x^2 + y^2 + z^2)^2 \quad ...(3)$$

Since $x^2 + y^2 + z^2 = 27$. we have from (3)

$$(x^3 + y^3 + z^3)^2 \ge (81)^2.$$

Taking positive square roots, we have

$$x^3 + y^3 + z^3 \ge 81.$$

➤ **Example 5.** For real numbers x, y, z prove that

$$\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{6}\right)^2 \le \frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{6},$$

with equality only when x = y = z.

➤ ***Solution*** Equivalently, we have to show that

$$(3x + 2y + z)^2 \le 6(3x^2 + 2y^2 + z^2).$$

This follows by applying the Cauchy-Schwartz inequality to the sequences

x, x, x, y, y, z are 1, 1, 1, 1, 1, 1.

5. RMS-AM inequality

Assume that $a_1, \ldots, a_n$ are positive real numbers.

$\sqrt{\frac{(a_1^2 + \ldots + a_n^2)}{n}}$ is called root-mean square (RMS) of the given numbers and the inequality

$$\sqrt{\frac{(a_1^2 + \ldots + a_n^2)}{n}} \ge \frac{a_1 + \ldots + a_n}{n}$$

is called as RMS-AM inequality.

Proof : Consider the **Cauchy-Lagrange's identity** :

$$(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \ge (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

If $b_1 = b_2 = \ldots\ldots = b_n$ then

$$\left(a_1^2 + a_2^2 + \ldots\ldots + a_n^2\right) n \ge (a_1 + a_2 + \ldots\ldots + a_n)^2$$

Hence, $\sqrt{\frac{a_1^2 + a_2^2 + \ldots\ldots + a_n^2}{n}} \ge \frac{a_1 + a_2 + \ldots\ldots + a_n}{n}$

6. Tchebycheff's Inequalities

If $a_1, a_2, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots, b_n$ are any real numbers such that $a_1 \le a_2 \le \ldots. \le a_n$ and $a_1 \le a_2 \le \ldots. \le a_n$, then $n(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \ge (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

For $i \ne j$, $(a_i - a_j)$ and $(b_i - b_j)$ are both non-negative or non-positive.

$\therefore (a_i - a_j)(b_i - b_j) \ge 0 \Rightarrow a_ib_i + a_jb_j \ge a_ib_j + a_jb_i$

There are ${}^nC_2 = n(n-1)/2$ such inequalities. Adding the corresponding sides we get

$$(n-1)(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \ge (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)$$

$$\Rightarrow \quad n(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \ge (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$$

For equality to hold, atleast one in every pair of $(a_i - a_j)$ and $(b_i - b_j)$ must be zero.

This certainly happens if $a_1 = a_2 = \ldots = a_n$, or if $b_1 = b_2 = \ldots = b_n$, and these are the only possibilities.

➤ **Example 6.** If a_1, a_2, a_3 and b_1, b_2, b_3 are any real numbers and $a_1 \le a_2 \le a_3$ then prove that

(i) $3\Sigma a_ib_i \ge \Sigma a_i \Sigma b_i$ when $b_1 \le b_2 \le b_3$ and

(ii) $3\Sigma a_ib_i \ge \Sigma a_i . \Sigma b_i$ when $b_1 \ge b_2 \ge b_3$

➤ ***Solution*** (i) Here $(a_1 - a_2)(b_1 - b_2) \ge 0$,

or, $a_1b_1 + a_2b_2 \ge a_1b_2 + a_2b_1$.

Similarly $a_2b_2 + a_3b_3 \ge a_2b_3 + a_3b_2$

and $a_3b_3 + a_1b_1 \ge a_3b_1 + a_1b_3$.

Adding these and then adding $a_1b_1 + a_2b_2 + a_3b_3$ to both sides, we get

$3(a_1b_1 + a_2b_2 + a_3b_3) \ge (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$

(ii) Here $(a_1 - a_2)(b_1 - b_2) \le 0$,

or $a_1b_1 + a_2b_2 \le a_1b_2 + a_2b_1$.

Similarly $a_2b_2 + a_3b_3 \le a_2b_3 + a_3b_2$

and $a_3b_3 + a_1b_1 \le a_3b_1 + a_1b_3$.

Adding these and then adding $a_1b_1 + a_2b_2 + a_3b_3$ to both sides, we get

$3(a_1b_1 + a_2b_2 + a_3b_3) \le (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$

www.pdfworld.net

www.pdfworld.net

➤ **Example 7.** If a, b, c, d, e are positive real numbers, such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, find the range of e.

➤ ***Solution*** As we know,

$$\left(\frac{a+b+c+d}{4}\right)^2 \le \frac{a^2+b^2+c^2+d^2}{4} \quad ...(i)$$

(using Tchebycheff's Inequality)

where $a+b+c+d+e=8$ and $a^2+b^2+c^2+d^2+e^2=16$

$\therefore$ Equation (i) reduces to, $\left(\frac{8-e}{4}\right)^2 \le \frac{16-e^2}{4}$

$\Rightarrow$ $64 + e^2 - 16e \le 4(16 - e^2)$

$\Rightarrow$ $5e^2 - 16e \le 0$

$\Rightarrow$ $e(5e - 16) \le 0 \Rightarrow 0 \le e \le \frac{16}{5}$

Thus range of $e \in \left[0, \frac{16}{5}\right]$

Maximum –Minimum Problems

Let us suppose that x, y, z, ..., w are n positive variables and c is a constant.

(i) We have the relation

$$xyz \ldots w \le \left(\frac{x+y+z+\ldots+w}{n}\right)^n.$$

The equality occurs when $x = y = z = \ldots = w$.

If $x + y + z + \ldots + w = c$ then $xyz \ldots w \le \left(\frac{c}{n}\right)^n$.

Therefore, **the greatest value of xyz ... w** $= \left(\frac{c}{n}\right)^n$.

Obviously this greatest value occurs when

$x = y = \ldots = w = \frac{c}{n}$.

If $xyz \ldots w = c$, then $\left(\frac{x+y+z+\ldots+w}{n}\right)^n \ge c$, or, $x + y + z + \ldots + w \ge nc^{1/n}$.

Therefore, **the least value of $x + y + z + \ldots + w = nc^{1/n}$** and it takes place when $x = y = z = \ldots w = c^{1/n}$.

(ii) We have

$$\frac{x^m + y^m + z^m + \ldots + w^m}{n} \ge \text{or} \le \left(\frac{x+y+z+\ldots+w}{n}\right)^m$$

according as m does not or does lie between 0 and 1.

Here $m \ne 0$ or 1.

The equality occurs when $x = y = z = \ldots = w$.

If $x + y + z + \ldots + w = c$,

then $\frac{\Sigma x^m}{n} \ge \text{or} \le \left(\frac{c}{n}\right)^m$.

The least value of Σx^m is $n\left(\frac{c}{n}\right)^m$ when m does not lie between 0 and 1 and it occurs when

$x = y = z = \ldots = w = \frac{c}{n}$.

The greatest value of Σx^m is $n\left(\frac{c}{n}\right)^m$ when m lies between 0 and 1 and it takes place when

$x = y = z = \ldots = w = \frac{c}{n}$.

(iii) Let us find the greatest value of $x^a y^b z^c \ldots$ when $\alpha x + \beta y + \gamma z + \ldots = k$.

Here a, b, c, ... ; α, β, γ, ... and k are constants.

We have

$$\left(\frac{\alpha x}{a}\right)^a \left(\frac{\beta y}{b}\right)^b \left(\frac{\gamma z}{c}\right)^c \ldots$$

$$\le \left(\frac{a\frac{\alpha x}{a} + b\frac{\beta y}{b} + c\frac{\gamma z}{c} + \ldots}{a+b+c+\ldots}\right)^{a+b+c+\ldots}$$

$$\le \left(\frac{\alpha x + \beta y + \gamma z + \ldots}{a+b+c+\ldots}\right)^{a+b+c+\ldots}$$

www.pdfworld.net

www.pdfworld.net

$$\le \left(\frac{k}{a+b+c+....}\right)^{a+b+c+...} \quad \text{or } x^a y^b z^c \ ....$$

$$\le \frac{a^a b^b c^c \ ...}{\alpha^a \beta^b \gamma^c \ ...}\left(\frac{k}{a+b+c+....}\right)^{a+b+c+...}$$

Therefore, **the greatest value of $x^a y^b z^c$ .... is**

$$\frac{a^a b^b c^c \ ...}{\alpha^a \beta^b \gamma^c ...}\left(\frac{k}{a+b+c+....}\right)^{a+b+c+...}$$

It occurs when

$$\frac{\alpha x}{a} = \frac{\beta y}{b} = ... = \frac{\alpha x + \beta y + \gamma z + ...}{a+b+c+...}$$

$$= \frac{k}{a+b+c \ ...}$$

(iv) Let us find the least value of $\alpha x + \beta y + \gamma z +$ .... when $x^a y^b z^c$ .... = k and $\alpha, \beta, \gamma, ...$; a, b, c, ... and k are constants.

We have

$$\left(\frac{a\frac{\alpha x}{a} + b\frac{\beta y}{b} + ...}{a+b+....}\right)^{a+b+c+...} \ge \left(\frac{\alpha x}{a}\right)^a \left(\frac{\beta y}{b}\right)^b \left(\frac{\gamma z}{c}\right)^c ....$$

or, $(\alpha x + \beta y + \gamma z + ...)^{a+b+c+....}$

$$\ge (a+b+c+...)^{a+b+c+...}\left\{\left(\frac{\alpha}{a}\right)^a \left(\frac{\beta}{b}\right)^b ....\right\} \quad (x^a y^b z^c ...)$$

or $(\alpha x + \beta y + \gamma z + ...) \ge (a+b+c+...)$

$$\left\{k\left(\frac{\alpha}{a}\right)^a \left(\frac{\beta}{b}\right)^b \left(\frac{\gamma}{c}\right)^c ....\right\}^{\frac{1}{a+b+c+...}}$$

Therefore, **the least value of $\alpha x + \beta y + \gamma z +$ .... is**

$$\textbf{(a+b+c+....)}\left\{k\left(\frac{\alpha}{a}\right)^a \left(\frac{\beta}{b}\right)^b ....\right\}^{\frac{1}{a+b+c+...}}$$

It takes place when $\frac{\alpha x}{a} = \frac{\beta y}{b} = \frac{\gamma z}{c} = ...$

➤ **Example 8.** Find the greatest value of the function $y = \frac{x}{ax^2+b}$ $(a>0, b>0)$.

➤ ***Solution*** It is obviously sufficient to consider only positive values of x.

We have $\frac{ax^2+b}{2} \le \sqrt{ax^2 b} = x\sqrt{ab}$

consequently, for all

$$x>0, \ y = \frac{x}{ax^2+b} \le \frac{x}{2x\sqrt{ab}} = \frac{1}{2\sqrt{ab}}$$

➤ **Example 9.** In the equation $x^4 + px^3 + qx^2 + rx + 5 = 0$ has four positive real roots, then find the minimum value of pr.

➤ ***Solution*** Let $\alpha, \beta, \gamma, \delta$ be the four positive real roots of the given equation. Then

$\alpha+\beta+\gamma+\delta = -p.$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = q.$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r.$

$\alpha\beta\gamma\delta = 5.$

Using A.M. $\ge$ G.M.

$$\Rightarrow \quad \frac{\alpha+\beta+\gamma+\delta}{4} \cdot \frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{4} \ge \sqrt[4]{\alpha\beta\gamma\delta}\ \sqrt[4]{\alpha^3\beta^3\gamma^3\delta^3} = \alpha\beta\gamma\delta = 5$$

$$\Rightarrow \quad \left(-\frac{p}{4}\right)\left(-\frac{r}{4}\right) \ge 5$$

$\Rightarrow$ $pr \ge 8$

$\Rightarrow$ minimum value of $pr = 80$.

➤ **Example 10.** (i) Find the greatest value of abc for positive values of a, b, c subject the condition $bc + ca + ab = 12$.

(ii) If a and b are positive variables such that $a + b = c$ (constant), show that the greatest value of a^2b^3 is $\frac{108}{3125}c^5$.

(iii) Find the least value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ for positive values of x, y, z which satisfy the condition $x + y + z = c$.

www.pdfworld.net

www.pdfworld.net

(iv) Find the minimum value of $\frac{(a + x)(b + x)}{c + x}$.

➤ ***Solution*** (i) The greatest value of (bc).(ca).(ab) occurs when $bc = ca = ab = \frac{12}{3} = 4$.

Therefore, the greatest value of $a^2b^2c^2 = 4^3 = 8^2$. Consequently the greatest value of abc = 8.

(ii) a^2b^3 is greatest when $\frac{a}{2} = \frac{b}{3} = \frac{a+b}{5} = \frac{c}{5}$ ($\because$ a + b = c).

Therefore, the greatest value of

$$a^2b^3 = \left(\frac{2c}{5}\right)^2 \cdot \left(\frac{3c}{5}\right)^3 = \frac{108}{3125}c^5$$

(iii) Since x + y + z = c, the least value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ occurs when $x = y = z = \frac{c}{3}$.

Therefore, the required value

$$= \frac{3}{c} + \frac{3}{c} + \frac{3}{c} = \frac{9}{c}.$$

(iv) Let $c + x = y$.

Now

$$\frac{(a + x)(b + x)}{c + x} = \frac{(a - c + y)(b - c + y)}{y}$$

$$= \frac{(a - c)(b - c)}{y} + a + b - 2c + y$$

$$= \left(\sqrt{\frac{(a - c)(b - c)}{y}} - \sqrt{y}\right)^2$$

$$+ a + b - 2c + 2\sqrt{(a - c)(b - c)}$$

Since y is variable, the above expression is minimum when

$$\sqrt{\frac{(a - c)(b - c)}{y}} - \sqrt{y} = 0,$$

or, $y = \sqrt{(a - c)(b - c)}$ or

$x = \sqrt{(a - c)(b - c)} - c$

and the minimum value is

$a + b - 2c + 2\sqrt{(a - c)(b - c)}$.

➤ **Example 11.** Find the greatest value of $(a+x)^3 (a - x)^4$ for any real value of x numerically less than 4.

➤ ***Solution*** The given expression is greatest when $\left(\frac{a+x}{3}\right)^3\left(\frac{a-x}{4}\right)^4$ is greatest, but the sum of the factors of this expression is $3\left(\frac{a+x}{3}\right) + 4\left(\frac{a-x}{4}\right)$, or 2a; hence $(a+x)^3 (a-x)^4$ is greatest when $\frac{a+x}{3} = \frac{a-x}{4}$, or $x = -\frac{a}{7}$. Thus the greatest value is $\frac{6^3 . 8^4}{7^7} a^7$.

➤ **Example 12.** If $x_1, x_2, \ldots, x_n$ are real numbers such that $(x_1 - 6)(x_2 - 6) \ldots (x_n - 6) = 8^n$ and $x_j > 6$ for j = 1, 2, ... , n, prove that the least value of $x_1, x_2 \ldots x_n$ is 14^n.

➤ ***Solution*** We have

$$\frac{8 . \frac{x_i - 6}{8} + 6.1}{8 + 6} \geq \left\{\left(\frac{x_i - 6}{8}\right)^8 . 1^6\right\}^{1/14}$$

or, $\frac{x_i}{14} \geq \left(\frac{x_i - 6}{8}\right)^{8/14}$

$\therefore$ $x_1 . x_2 \ldots x_n$

$$\geq 14^n \left\{\frac{x_1 - 6}{8} \cdot \frac{x_2 - 6}{8} \ldots \frac{x_n - 6}{8}\right\}^{8/14},$$

or, $x_1x_2 \ldots x^n \geq 14^n$.

$\therefore$ The least value of $x_1x_2 \ldots x_n = 14^n$

➤ **Example 13.** If 2a + b + 3c = 1 and a > 0, b > 0, c > 0, find the greatest value of $a^4b^2c^2$ and obtain the corresponding values of a, b, c.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Consider the positive numbers

$$\frac{2a}{4},\frac{2a}{4},\frac{2a}{4},\frac{2a}{4},\frac{b}{2},\frac{b}{2},\frac{3c}{2},\frac{3c}{2}$$

[As there is a^4, take 4 equal parts of 2a; as there is b^2, take 2 equal parts of b; as there is c^2, take 2 equal parts of 3c.]

For the numbers,

$$A=\frac{\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{b}{2}+\frac{b}{2}+\frac{3c}{2}+\frac{3c}{2}}{4+2+2}$$

$$=\frac{2a+b+3c}{8}=\frac{1}{8} \qquad (\because\ 2a+b+3c=1)$$

$$G=\left(\frac{2a}{4}.\frac{2a}{4}.\frac{2a}{4}.\frac{2a}{4}.\frac{b}{2}.\frac{b}{2}.\frac{3c}{2}.\frac{3c}{2}\right)^{\frac{1}{8}}$$

$$=\left(\frac{1}{2^4}.\frac{1}{2^2}.\frac{1}{2^2}.3^2a^4b^2c^2\right)^{\frac{1}{8}}$$

$$\therefore \quad A\geq G\Rightarrow \frac{1}{8}\geq\left(\frac{3^2}{2^8}a^4b^2c^2\right)^{\frac{1}{8}}$$

or $$\frac{2^8}{3^2.8^8}\geq a^4b^2c^2$$

or $$\frac{1}{9.4^8}\geq a^4b^2c^2$$

So the greatest value of $a^4b^2c^2=\frac{1}{9.4^8}$

When the equality holds, the greatest value is attained. We know A = G when the numbers are all equal,

i.e., $$\frac{2a}{4}=\frac{b}{2}=\frac{3c}{2} \quad\Rightarrow\quad a=b=3c$$

$$\therefore \quad \frac{a}{3}=\frac{b}{3}=\frac{c}{1}=k$$

$$\therefore \quad a=3k,\ b=3k,\ c=k$$

$$\therefore \quad 2a+b+3c=1 \quad\Rightarrow\quad 6k+3k+3k=1$$

$$\therefore \quad k=\frac{1}{12}$$

$$\therefore \quad a=\frac{3}{12},\ b=\frac{3}{12},\ c=\frac{1}{12},$$

i.e., $$a=\frac{1}{4},\ b=\frac{1}{4},\ c=\frac{1}{12}$$

Practice Problems [P]

1. If a, b, c are positive, then prove that $(a^2b + b^2c + c^2a)\,(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

2. If $c_1, \ldots, c_n$ are positive real numbers, then prove that $\left(\sum_{i=1}^{n} c_i\right)\left(\sum_{i=1}^{n}\frac{1}{c_i}\right)\geq n^2$.

3. If a, b, c > 0, prove that

$$a^3+b^3+c^3\geq\frac{1}{2}[bc(b+c)+ca(c+a)+ab(a+b)].$$

4. Prove that for positive quantities a, b, c we have

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\geq\frac{3}{2}.$$

5. Prove that $\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b}$ $\geq a+b+c$

6. Prove that $(n+1)^{n-1}\,(n+2)^n > 3^n(n!)^2$.

7. Show that the least value of $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$ is 1 when $x^2 + y^2 + z^2 = 9$.

8. If x + y + z = 1, show that the least value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is 9; and that

$$(1-x)\,(1-y)\,(1-z) > 8xyz.$$

9. Prove that

$$\left(\frac{bc+ca+ab}{a+b+c}\right)^{a+b+c}\geq\{(bc)^a\,(ca)^b\,(ab)^c\}^{1/2}$$

10. Prove that the cube is the rectangular parallelopiped of maximum volume for given surface and of minimum surface for given volume.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

11. Show that the greatest value of xy^2z^3 is $\frac{27}{16}$ where x, y, z are positive real numbers subject to x+y+z = 3. Determine x, y, z in this case.

Target Problems for JEE Advanced

➤ **Problem 1.** If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms prove that

$$(m+n)\left(\frac{1}{m}-\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$$

➤ ***Solution*** Let the A.P. be a, a + d, a + 2d,....

Given $T_1+T_2+\ldots+T_m=T_{m+1}+T_{m+2}+\ldots+T_{m+n}$ ...(1)

Adding $T_1+T_2+\ldots+T_m$ on both sides in (i), Then

$2(T_1+T_2+\ldots+T_m)$

$=T_1+T_2+\ldots+T_m+T_{m+1}+\ldots+\ldots+T_{m+n}$

$\Rightarrow \quad 2S_m=S_{m+n}$

$\therefore \quad 2\cdot\frac{m}{2}\{2a+(m-1)d\}$

$=\frac{m+n}{2}\{2a+(m+n-1)d\}$

Let $2a+(m-1)d=x$

$\Rightarrow \quad mx=\frac{m+n}{2}\{x+nd\}$ ...(2)

$\Rightarrow \quad (m-n)x=(m+n)nd$

again $T_1+T_2+\ldots+T_m=T_{m+1}+T_{m+2}+\ldots$

$+T_{m+p}$ ...(3)

Similarly $(m-p)x=(m+p)pd$

dividing (2) by (3), we get

$$\frac{m-n}{m-p}=\frac{(m+n)n}{(m+p)p}$$

$\Rightarrow \quad (m-n)(m+p)p=(m-p)(m+n)n$

dividing both sides by mnp, we have (m + p)

$$\left(\frac{1}{n}-\frac{1}{m}\right)=(m+n)\left(\frac{1}{p}-\frac{1}{m}\right)$$

Hence $(m+n)\left(\frac{1}{p}-\frac{1}{m}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$

➤ **Problem 2.** Show that any positive integral power (except the first) of a positive integer m, is the sum of m consecutive odd positive integers. Find the first odd integer for m^r (r > 1).

➤ ***Solution*** Let us find k such that

$m^r=(2k+1)+(2k+3)+\ldots.+(2k+2m-1)$

$\therefore \quad m^r=\frac{m}{2}[2k+1+2k+2m-1]$

$\Rightarrow \quad m^{r-1}=2k+m$

Note that $m^{r-1}-m$ is an even integer for all r, $m\in\mathbf{N}$ and r > 1. Therefore, $k=(m^{r-1}-m)/2$ is an integer. Thus, the first term is given by $m^{r-1}-m+1$.

➤ **Problem 3.** If $S_1, S_2, S_3, \ldots\ldots, S_p$ are the sums of n terms of arithmetic series whose first terms are 1,2, 3, 4,..... and whose common differences are 1, 3, 5, 7,.....; find the value of $S_1+S_2+S_3+\ldots\ldots+S_p$.

➤ ***Solution*** We have

$$S_1=\frac{n}{2}\{2+(n-1)\}=\frac{n(n+1)}{2},$$

$$S_2=\frac{n}{2}\{4+(n-1)3\}=\frac{n(3n+1)}{2},$$

$$S_3=\frac{n}{2}\{6+(n-1)5\}=\frac{n(5n+1)}{2},$$

$$S_p=\frac{n}{2}\{2p+(n-1)(2p-1)\}=\frac{n}{2}\{(2p-1)n+1\};$$

$\therefore$ The required sum

$=\frac{n}{2}\{(n+1)+(2n+1)+\ldots\ldots.(\overline{2p-1}\,.\,n+1)\}$

$=\frac{n}{2}\{(n+3n+5n+\overline{2p-1}\,.\,n+p\}$

$=\frac{n}{2}\{n(1+3+5+\ldots.\overline{2p-1})+p\}$

$=\frac{n}{2}(np^2+p)$

$=\frac{np}{2}(np+1).$

➤ **Problem 4.** Let S_k be the sum of the first k terms of an arithmetic progression. What must this progression be for the ratio $\frac{S_{kx}}{S_x}$ to be independent of x?

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Let the nth term of the required progression be a_n, its common difference being equal to d. Then

$$S_x = \frac{a_1 + a_x}{2}.x\,,\quad S_{kx} = \frac{a_1 + a_{kx}}{2}kx$$

Hence $$\frac{S_{kx}}{S_x} = \frac{a_1 + a_{kx}}{a_1 + a_2}.k = \frac{2a_1 + d(kx-1)}{2a_1 + d(x-1)}k$$

$$= \frac{2a_1 - d + kxd}{2a_1 - d + dx}.k$$

For the last relation to have a value independent of x it is necessary and sufficient that $2a_1 - d = 0$
i.e. the common difference of the required progression must be equal to double the first term.

➤ **Problem 5.** Suppose that $a_1, a_2, \ldots, a_n, \ldots$ is an A.P. Let $S_k = a_{(k-1)n+1} + a_{(k-1)n+2} + \ldots + a_{kn}$
Prove that $S_1, S_2, \ldots$ are in A.P. having common difference equal to n^2 times the common difference of the A.P. $a_1, a_2, \ldots$

➤ ***Solution*** We have

$$S_k = a_{(k-1)n+1} + a_{(k-1)n+2} + \ldots + a_{(k-1)n+n}$$

Now we have

$$a_{(k-1)n+1} = a_1 + (k-1)nd\ ;$$
$$a_{(k-1)n+2} = a_1 + (k-1)nd + d$$
$$a_{(k-1)n+3} = a_1 + (k-1)nd + 2d$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{(k-1)n=n} = a_1 + (k-1)d + \frac{(n-1)nd}{2}$$

and $$S_{k+1} = na + n^2 kd + \frac{(n-1)nd}{2}$$

[putting k + 1 in place of k]

Hence, we have $S_{k+1} - S_k = n^2 d$
which is a constant independent of k. This proves that the sequence (S_k) is an A.P. having common difference $n^2 d$.

➤ **Problem 6.** Given a three digit number whose digits are three successive terms of a G.P. If we subtract 792 from it, we get a number written by the same digits in the reverse order. Now if we subtract four from the hundred's digit of the initial number and leave the other digits unchanged, we get a number whose digits are successive terms of an A.P. Find the number.

➤ ***Solution*** Let the number be $100x + 10y + z$

$$y^2 = xz \qquad \ldots(1)$$
$$100x + 10y + z - 792 = 100z + 10y + z$$

or $$100(x - z) + (z - x) = 792$$

$$\Rightarrow\quad x - z = \frac{792}{99} = 8$$

Also $2y = x - 4 + z$

$$\Rightarrow\quad y = \frac{x + z - 4}{2} \qquad \ldots(2)$$

$$(x + z - 4)^2 = 4xz$$
$$(x + z)^2 - 8(x + z) + 16 = 4xz$$
$$(x - z)^2 - 8(x + z) + 16 = 0$$
$$x + z = 10$$

Hence the required number is 931.

➤ **Problem 7.** One of the roots of the equation

$$2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$$

is of the form $\frac{m + \sqrt{n}}{r}$, where m is non zero integer and n and r are relatively prime natural numbers. Find the value of $m + n + r$.

➤ ***Solution*** $2000x^6 + \underbrace{100x^5 + 10x^3 + x}_{\text{a G.P.}} - 2 = 0$

$$2000x^6 + \frac{x\left((10x^2)^3 - 1\right)}{10x^2 - 1} - 2 = 0$$

$$\frac{x(1000x^6 - 1)}{10x^2 - 1} = -2(1000x^6 - 1)$$

$\therefore\quad 1000x^6 - 1 = 0$

or $$\frac{x}{10x^2 - 1} = -2 \qquad \Rightarrow \qquad x = -(10x^2 - 1)$$

$$x^2 = \frac{1}{10}$$ which is not possible $20x^2 + x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{1 + 160}}{40} = \frac{-1 + \sqrt{161}}{40}$$

or $$\frac{-1 - \sqrt{161}}{40}$$

$\therefore\quad m = -1;\ n = 161;\ r = 40 \qquad m + n + r = 200.$

www.pdfworld.net

www.pdfworld.net

➤ **Problem 8.** In a G.P. if S_1, S_2 & S_3 denote respectively the sums of the first n terms, first 2 n terms and first 3 n terms, then prove that

$$S_1^2 + S_2^2 = S_1 (S_2 + S_3).$$

➤ ***Solution*** $S_1 = \dfrac{a\left(1 - r^n\right)}{1 - r}$; $S_2 = \dfrac{a\left(1 - r^{2n}\right)}{1 - r}$;

$$S_3 = \frac{a\left(1 - r^{3n}\right)}{1 - r}$$

To prove that $S_1 (S_1 - S_3) = S_2 (S_1 - S_2)$

$$\text{L H S } \frac{a\left(1 - r^n\right)}{1 - r}$$

$$\left[\frac{a\left(1 - r^n\right) - a\left(1 - r^{2n}\right)}{1 - r}\right]$$

$$= \frac{a^2\left(1 - r^n\right)}{(1 - r)^2}\left(r^{3n} - r^n\right)$$

$$= \frac{a^2 r^n \left(1 - r^n\right)\left(r^{2n} - 1\right)}{(1 - r)^2}$$

Same expression is obtained for R H S.

➤ **Problem 9.** How many increasing 3-term geometric progressions can be obtained from the sequence

1, 2, 2^2, 2^3,....,2^n ? (e.g., {2^2, 2^5 ,2^8} is a 3-term geometric progression for n ≥ 8.)

➤ ***Solution*** Let us start counting 3–term G.P.'s with common ratios 2, 2^2, 2^3,...

The 3–term G.P.'s with common ratio 2 are

$$1, 2, 2^2; 2, 2^2, 2^3; \ldots\ldots; 2^{n-2}, 2^{n-1}, 2^n.$$

They are (n – 1) in number. The 3–term GP's with common ratio 2^2 are

$$1, 2^2, 2^4; 2, 2^3, 2^5; \ldots\ldots; 2^{n-4}, 2^{n-2}, 2^n$$

They are (n – 3) in number. Similarly we see that the 3–term GP's with common ratio 2^3 are (n – 5) in number and so on. Thus the number of 3–term GP's which can be formed from the sequence 1, 2, 2^2, 2^3,....,2^n is equal to

$$S = (n-1) + (n-3) + (n-5) + \ldots\ldots\ldots.$$

Here the last term is 2 or 1 according as n is odd or even. If n is odd, then

$$S = (n-1) + (n-3) + (n-5) + \ldots. + 2$$

$$= 2(1 + 2 + 3 + \ldots + \frac{n-1}{2}) = \frac{n^2 - 1}{4}.$$

If n is even, then

$$S = (n-1) + (n-3) + \ldots. + 1 = \frac{n^2}{4}.$$

Hence the required number is $(n^2 - 1)/4$ or $n^2/4$ according as n is odd or even.

➤ **Problem 10.** The first two terms of an arithmetic and a geometric progression with positive terms are equal. Prove that all other terms of the arithmetic progression are not greater other terms of the arithmetic progression are not greater than the corresponding terms of the geometric progression.

➤ ***Solution*** Let the first common term of the progression be a, and the second b. Then the nth term of the arithmetic progression will be equal to

$$a + (b - a)(n - 1)$$

and the corresponding terms of the geometric progression has the form $a(b/a)^{n-1}$

And so, we have to prove that

$$a + (b - a)(n - 1) \le a\,(b/a)^{n-1}.$$

In other words, that $a + (b - a)(n - 1) - a\,(b/a)^{n-1} \le 0$.

or $$a\left\{\left(\frac{b}{a} - 1\right)(n-1) - \left[\left(\frac{b}{a}\right)^{n-1} - 1\right]\right\} \le 0.$$

Let us rewrite the left member of this inequality as follows $a\left(\dfrac{b}{a} - 1\right)$

$$\left\{(n-1) - \left[\left(\frac{b}{a}\right)^{n-2} + \left(\frac{b}{a}\right)^{n-3} + \ldots. + \left(\frac{b}{a}\right) + 1\right]\right\}.$$

Considering separately the three cases :

$$\frac{b}{a} > 1,\ \frac{b}{a} < 1,\ \frac{b}{a} = 1,$$

we easily prove the validity of our inequality.

➤ **Problem 11.** Two geometric series have the same first term 1. The common ratio of the first series is greater than that of the second series and both are positive. If S_n and s_n denote sum of n terms of the two series respectively then prove that $\dfrac{S_n}{S_{n+1}} < \dfrac{s_n}{s_{n+1}}$.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Clearly S_n, S_{n+1}, s_n and s_{n+1} are all positive.

Let a, b be their common ratios respectively. So $a > b > 0$.

$$\begin{aligned} & s_n \cdot S_{n+1} - s_{n+1} \cdot S_n \\ &= (1 + b + b^2 + ... + b^{n-1})(1 + a + a^2 + ... + a^n) \\ &\quad - (1 + b + b^2 + ... + b^n)(1 + a + a^2 + ... + a^{n-1}) \\ &= 1 + b + b^2 + ... + b^{n-1})(1 + a + a^2 + ... + a^{n-1}) \\ &\qquad + a^n(1 + b + b^2 + ... + b^{n-1}) \\ &\quad - (1 + b + b^2 + ... + b^{n-1}) \times (1 + a + a^2 + ... \\ &\qquad + a^{n-1}) - b^n(1 + a + a^2 + ... + a^{n-1}) \\ &= a^n(1+b+b^2 + ... + b^{n-1}) \\ &\qquad - b^n(1+ a + a^2 + ... + a^{n-1}) \\ &= (a^n - b^n) + ab(a^{n-1} - b^{n-1}) + a^2b^2(a^{n-2} - b^{n-2}) \\ &\qquad + ... + a^{n-1}b^{n-1}(a - b), \end{aligned}$$

each term of which are positive because $a > b > 0$ and so $a^n > b^n$ for all $n \in N$.

$\therefore \quad s_n \cdot S_{n+1} - s_{n+1} \cdot S_n > 0$ or $s_{n+1} \cdot S_n < s_n \cdot S_{n+1}$

$$\therefore \quad \frac{\Sigma_n}{\Sigma_{n+1}} < \frac{\sigma_n}{\sigma_{n+1}}$$

➤ **Problem 12.** Construct a geometric progression $1, q, q^2, ..., q^n.....$with $|q| < 1$ whose every term differs from the sum of all subsequent terms by a given constant factor k. For what values of k is the problem solvable?

➤ ***Solution*** We have $|q| < 1$ and consequently,

$$q^n = k(q^{n+1} + q^{n+2} + ...) = kq^{n+1}\frac{1}{1-q} \qquad ...(1)$$

Hence, $1 - q = kq$ and thus, if the problem has a solution, we have

$$q = \frac{1}{k+1} \qquad ...(2)$$

It is, however, easily seen that if, conversely, equality (2) implies that $|q| < 1$, then the equality (2) implies equality(1), and the corresponding progression satisfies the condition of the problem. Thus, the problem is solvable for any k satisfying the inequality $\left|\frac{1}{k+1}\right| < 1$. The latter holds for $k > 0$ and $k < -2$.

➤ **Problem 13.** Find the n^{th} term of the series $1 + 2 + 5 + 12 + 25 + 46 + ...$

➤ ***Solution*** Let the sum of the series by S_n and nth term of the series be T_n

Then $S_n = 1 + 2 + 5 + 12 + 46 + ...+T_{n-1} + T_n$ ...(1)

$\therefore \quad S_n = 1 + 2 + 5 + 12 + 25 +.... + T_{n-1} + T_n$ ..(2)

Subtracting (2) from (1), we get

$0 = 1 + 1 + 3 + 7 + 13 + 21 + ... + (T_n - T_{n-1}) - T_n$

$\therefore \quad T_n = 1 + 3 + 7 + 13 + 2 + ...+t_{n-1} + t_n$ ...(3)

(Here nth term of T_n is t_n)

$\therefore \quad T_n = 1 + 1+ 3 + 7 + 13 + ... t_{n-1} + t_n$ ...(4)

Subtracting (4) from (3), we get

$0 = 1 + 0 + 2 + 4 + 6 + 8 + ... +(t_n - t_{n-1}) - t_n$

$\therefore \quad t_n = 1 + 2 + 4 + 6 + 8 +....(n - 1)$ terms

$= 1 + (2 + 4 + 6 + 8 (n - 2)$ terms)

$$= 1 + \frac{(n-2)}{2}\{2.2 + (n - 2 - 1) 2\}$$

$$= 1 + (n - 2)(2 + n - 3)$$

$\therefore \quad t_n = n^2 - 3n + 3$

then $T_n = \Sigma t_n = \Sigma n^2 - 3\Sigma n + 3\Sigma 1$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + \frac{3n}{1}$$

$$= \frac{n}{6}\{2n^2 + 3n + 1 - 9n - 9 + 18\}$$

$$\therefore \quad T_n = \frac{n}{6}(2n^2 - 6n + 10) = \frac{n}{3}(n^2 - 3n + 5)$$

Alternative

The nth terms of the series can be written directly from the following procedure

1, 2, 5, 12, 25, 46,... (given series)

1, 3, 7, 13, 21,... (first consecutive differences)

2, 4, 6, 8,.... (second consecutive differences)

2, 2, 2,.... (constant terms)

then $T_n = a(n - 1)(n - 2)(n - 3) + b(n - 1)(n - 2) + c(n - 1) + d$

Putting n = 1, 2, 3, 4 then we get

$T_1 = 6a + 6b + 3c + d = 12$

$$\therefore \quad a = \frac{1}{3},\ b = 1,\ c = 1,\ d = 1$$

www.pdfworld.net

www.pdfworld.net

Hence $T_n = \frac{n}{3}(n^2 - 3n + 5)$

➤ **Problem 14.** Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$.

Show that

(A) $S_n = n - \left(\frac{1}{2} + \frac{2}{3} + ... + \frac{n-1}{n}\right)$;

(B) $nS_n = n + \left(\frac{n-1}{1} + \frac{n-2}{2} + ... + \frac{2}{n-2} + \frac{1}{n-1}\right)$.

➤ ***Solution***

(A) It is obvious that

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$$
$$= n + \left[(1-1) + \left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - 1\right) + ... + \left(\frac{1}{n} - 1\right)\right]$$
$$= n - \left(\frac{1}{2} + \frac{2}{3} + ... + \frac{n-1}{n}\right).$$

(B) $S_n = \sum_{k=1}^{k=n} \frac{1}{k}$, $nS_n = \sum_{k=1}^{k=n} \frac{n-k+k}{k} \sum_{k=1}^{k=n} \left(\frac{n-k}{k} + 1\right)$.

Hence, $S_n = n + \left(\frac{n-1}{1} + \frac{n-2}{2} + ... + \frac{1}{n-1}\right)$.

➤ **Problem 15.** If $f(t) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{t}$, prove that $\sum_{k=1}^{n} (2k + 1) f(k) = (n + 1)^2 . f(n) - \frac{n(n+1)}{2}$

➤ ***Solution*** L. H.S.

$$= 3 f(1) + 5 f(2) + 7 f(3) + ... + (2n + 1) f(n)$$
$$= 3 + 5\left(1 + \frac{1}{2}\right) + 7\left(1 + \frac{1}{2} + \frac{1}{3}\right) + ... + (2n+1)\left[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right]$$
$$= \frac{1}{1}[3 + 5 + 7 + ... + (2n + 1)] + \frac{1}{2}[5 + 7 + ... + (2n + 1)] + \frac{1}{3}[7 + 9 + ... + (2n+1)] + ... + \frac{1}{n}(2n + 1)$$
$$= [(n + 1)^2 - 1] + \frac{1}{2}[(n + 1)^2 - 2^2] + \frac{1}{3}[(n + 1)^2 - 3^2] + ... + \frac{1}{n}[(n + 1)^2 - n^2]$$
$$= (n + 1)^2 \left[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right] - (1 + 2 + 3 + ... + n)$$
$$= (n + 1)^2 f(n) - \frac{n(n+1)}{2}$$

➤ **Problem16.** Let $H_{1n} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ and $H_{2n} = \frac{n+1}{2} - \left[\frac{1}{n(n-1)} + \frac{2}{(n-1)(n-2)} + ... + \frac{n-2}{2.3}\right]$.

Show that $H_{1n} = H_{2n}$

➤ ***Solution*** We have

$$H_{2n} = \frac{n+1}{2} + \left\{\left(\frac{1}{n} - \frac{1}{n-1}\right) + \left(\frac{2}{n-1} - \frac{2}{n-2}\right) + \left(\frac{3}{n-2} - \frac{3}{n-3}\right) + ... + \left(\frac{n-3}{4} - \frac{n-3}{3}\right) + \left(\frac{n-2}{3} - \frac{n-2}{2}\right)\right\}$$
$$= \left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + ... + \frac{1}{3}\right] + \frac{n+1}{2} - \frac{n-2}{2}$$
$$= \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + ... + \frac{1}{3} + \frac{1}{2} + 1 = H_{1n}.$$

➤ **Problem 17.** $A_1, A_2, ..., A_n$ are n A.M's, and $H_1, H_2, ..., H_n$ are n H.M's inserted between a and b. Prove that $\frac{A_1 + A_n}{H_1 + H_n} < \frac{A^2}{G^2}$, where A is the arithmetic mean and G is the geometric mean of a and b.

➤ ***Solution*** Since $A_1, A_2, ..., A_n$ are n arithmetic means between a and b,

$$A_1 = \frac{na + b}{n+1}, A_n = \frac{a + nb}{n+1}$$
$$\Rightarrow \quad A_1 + A_n = a + b \qquad ...(1)$$

www.pdfworld.net

www.pdfworld.net

Also, $\frac{1}{H_1} = \frac{a+nb}{(n+1)ab}, \frac{1}{H_n} = \frac{an+b}{(n+1)ab}$ ...(2)

From (1) and (2)

$$\frac{A_1 + A_n}{H_1 + H_n} = \frac{a+b}{\frac{(n+1)ab}{ba+b} + \frac{(n+1)ab}{nb+a}}$$

$$= \frac{(na+b)(a+nb)(a+b)}{(n+1)ab(na+b+nb+a)}$$

$$= \frac{(na+b)(a+nb)}{(a+1)^2 ab}$$

$$< \left(\frac{na+b+a+nb}{2}\right)^2 \frac{1}{(n+1)^2 ab}$$

$$= \frac{(a+b)^2}{4ab} = \frac{A^2}{G^2}$$

$$\Rightarrow \quad \frac{A_1 + A_n}{H_1 + H_n} < \frac{A^2}{G^2}.$$

➤ **Problem 18.** If $a_1, a_2, \ldots, a_n$ are all positive and distinct, show that the A.M. of the numbers

$\sqrt{a_1a_2}, \sqrt{a_1a_3}, \sqrt{a_2a_3} \ldots\ldots$

is less than the A.M. of $a_1, a_2, \ldots, a_n$.

➤ ***Solution*** There are ${}^nC_2 \left(= \frac{n(n-1)}{2}\right)$ numbers of the form $\sqrt{a_1a_2}, \sqrt{a_1a_3}, \sqrt{a_2a_3} \ldots\ldots$

We have $\sqrt{a_1a_2} < \frac{a_1 + a_2}{2}$,

$\sqrt{a_1a_3} < \frac{a_1 + a_3}{2}$,

.....

By addition $\sum\sqrt{a_1a_2} < \frac{n-1}{2}\sum a_1$

($\because$ each of $a_1, a_2, \ldots$ occurs $n-1$ times.)

or, $\frac{\Sigma\sqrt{a_1a_2}}{\frac{n(n-1)}{2}} < \frac{\sum a_1}{n}$

➤ **Problem 19.** Let $a_1, a_2, \ldots, a_n$ be positive in A.P. If A and G are the arithmetic and the geometric means of a_1 and a_n respectively, prove that $G^n < a_1a_2 \ldots a_n < A^n$.

➤ ***Solution*** We have $A = \frac{a_1 + a_n}{2}$ and $G = \sqrt{a_1a_n}$. Since $a_1, a_2, \ldots, a_n$ are in A.P.,

$$a_1 + a_2 + \ldots + a_n = \frac{n}{2}(a_1 + a_n) = nA$$

We may assume that

$a_1 < a_2 < \ldots < a_n$ and $a_n = a_1 + (n-1)b$.

Now $a_1a_n - a_2a_{n-1} = a_1\{a_1 + (n-1)b\} - (a_1 + b)\{a_1 + (n-2)b\} = -(n-2)b^2$

$\therefore \quad a_1a_n < a_2a_{n-1}$.

Similarly $a_1a_n < a_3a_{n-2}$ and so on.

Therefore, $(a_1a_n)^{n/2} < a_1a_2 \ldots a_n$ when n is even.

If $n = 2m+1$ (odd), then $a_1a_{2m+1} < a^2_{m+1}$, or, $G < a_{m+1}$

Therefore, $(a_1a_n)^{n/2} < a_1a_2 \ldots a_n$ when n is odd.

By the theorem on means,

$$a_1a_2 \ldots a_n < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^n,$$

or $\quad a_1a_2 \ldots a_n < \left(\frac{a_1 + a_n}{2}\right)^n$.

$$\therefore \quad (a_1a_n)^{n/2} < a_1a_2 \ldots a_n < \left(\frac{a_1 + a_n}{2}\right)^n,$$

or $\quad G^n < a_1a_2 \ldots a_n < A^n$.

➤ **Problem 20.** Prove that

$$\frac{a^3 + b^3}{2} \geq \left(\frac{a+b}{2}\right)^3 \quad (a > 0, b > 0).$$

➤ ***Solution*** We have $a^2 - ab + b^2 \geq (a-b)^2 \geq 0$

Hence $a^2 - ab + b^2 \geq ab$,

$a^3 + b^3 \geq ab(a+b)$.

Consequently $3a^3 + 3b^3 \geq 3a^2b + 3ab^2$.

Add $a^3 + b^3$ to both members of the last inequality.

We have $4a^3 + 4b^3 \geq (a+b)^3$.

And so, indeed $\frac{a^3 + b^3}{2} \geq \left(\frac{a+b}{2}\right)^3$

➤ **Problem 21.** If a, b be positive numbers and p, q be rational numbers ($p > 1$) such that $\frac{1}{p} + \frac{1}{q} = 1$, then show that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Since $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, therefore $q > 1$. Let $p = \frac{u}{v}, q = \frac{u'}{v'}$, where u, v, u', v' are positive integers.

$$\frac{a^p}{p} + \frac{b^q}{q} = \frac{(vu'a^p + uv'b^q)}{uu'}, \geq (a^{uu'} \cdot b^{uu'})^{1/uu'} = ab,$$

by applying A.M.–G.M. inequality to the vu' numbers each equal to a^p and uv' numbers each equal to b^q.

➤ **Problem 22.** Let a, b and c be positive integers. Prove that $a^{\frac{a}{a+b+c}} \cdot b^{\frac{b}{a+b+c}}, c^{\frac{c}{a+b+c}} \geq \frac{1}{3}(a+b+c)$.

➤ ***Solution*** Consider a quantities equal to $\frac{1}{a}$, b quantities equal to $\frac{1}{b}$, and c quantities equal to $\frac{1}{c}$.

The arithmetic mean of these quantities will be

$$\frac{a.\frac{1}{a} + b.\frac{1}{b} + c.\frac{1}{c}}{a+b+c} = \frac{3}{a+b+c}$$

The geometric mean is equal to $\sqrt[a+b+c]{\frac{1}{a^a} \cdot \frac{1}{b^b} \cdot \frac{1}{c^c}}$

Consequently $\frac{3}{a+b+c} \geq \sqrt[a+b+c]{\frac{1}{a^a} \cdot \frac{1}{b^b} \cdot \frac{1}{c^c}}$

i.e. $\frac{a}{a^{a+b+c}} \frac{b}{b^{a+b+c}} \frac{c}{c^{a+b+c}} \geq \frac{1}{3}(a+b+c)$

➤ **Problem 23.** If a, b, c be any three positive real numbers, prove that

$$\frac{b+c}{b^2+c^2} + \frac{c+a}{c^2+a^2} + \frac{a+b}{a^2+b^2} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

➤ ***Solution*** $\frac{b+c}{b^2+c^2} - \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)$

$$= \frac{b+c}{b^2+c^2} - \frac{b+c}{2bc}$$

$$= (b+c)\left(\frac{1}{b^2+c^2} - \frac{1}{2bc}\right) \leq 0$$

$\because\ b^2 + c^2 \geq 2bc$, or, $\frac{1}{b^2+c^2} \leq \frac{1}{2bc}$.

$\therefore\ \frac{b+c}{b^2+c^2} \leq \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)$.

Similarly

$$\frac{c+a}{c^2+a^2} \leq \frac{1}{2}\left(\frac{1}{c} + \frac{1}{a}\right) \text{ and } \frac{a+b}{a^2+b^2} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$\therefore\ \frac{b+c}{b^2+c^2} + \frac{c+a}{c^2+a^2} + \frac{a+b}{a^2+b^2} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

➤ **Problem 24.** If p and q are positive real numbers such that

$$p + q = 1,$$

prove that $\left(p + \frac{1}{p}\right)^2 + \left(q + \frac{1}{q}\right)^2 \geq \frac{25}{2}$.

➤ ***Solution*** Let $S = \left(p + \frac{1}{p}\right)^2 + \left(q + \frac{1}{q}\right)^2$.

Then $S = \left(p^2 + q^2\right) + \left(\frac{1}{p^2} + \frac{1}{q^2}\right) + 4$,

$$= \left(p^2 + q^2\right) + \left(1 + \frac{1}{p^2q^2}\right) + 4 \qquad ...(1)$$

Now

$$p^2 + q^2 = \frac{1}{2}\left[(p+q)^2 + (p-q)^2\right] \geq \frac{1}{2}(p+q)^2$$

so that $p^2 + q^2 \geq \frac{1}{2}$, since $p + q = 1$. ...(2)

Also, since p, q are positive numbers, therefore

$$\frac{p+q}{2} \geq (pq)^{1/2}, \text{ so that } \frac{1}{pq} \geq 4. \qquad ...(3)$$

From (1), (2) and (3), we find that

$$S = \frac{1}{2}(1 + 4^2) + 4 = \frac{25}{2}.$$

Thus we find that $\left(p + \frac{1}{p}\right)^2 + \left(q + \frac{1}{q}\right)^2 \geq \frac{25}{2}$

➤ **Problem 25.** If the polynomial

$$f(x) = 4x^4 - ax^3 + bx^2 - cx + 5$$

www.pdfworld.net

www.pdfworld.net

where a,b,c $\in$ R has four positive real roots say r_1, r_2, r_3 and r_4, such that $\frac{r_1}{2} + \frac{r_2}{4} + \frac{r_3}{5} + \frac{r_4}{8} = 1$. Find the value of 'a'.

➤ ***Solution*** Consider 4 positive terms $\frac{r_1}{2}$, $\frac{r_2}{4}$, $\frac{r_3}{5}$, $\frac{r_4}{8}$

$$\text{A.M.} = \frac{1}{4}\left(\frac{r_1}{2}+\frac{r_2}{4}+\frac{r_3}{5}+\frac{r_4}{8}\right) = \frac{1}{4}\times 1 = \frac{1}{4}$$

$$\text{G.M.} = \left(\frac{r_1}{2}\cdot\frac{r_2}{4}\cdot\frac{r_3}{5}\cdot\frac{r_4}{8}\right)^{1/4} = \left(\frac{r_1\cdot r_2\cdot r_3\cdot r_4}{2\cdot 4\cdot 5\cdot 8}\right)^{1/4}$$

now, $r_1r_2r_3r_4 = \frac{5}{4}$

$$\therefore \quad \text{G.M.} = \left[\frac{5}{4(2\cdot 4\cdot 5\cdot 8)}\right]^{1/4} = \left(\frac{1}{2^8}\right)^{1/4} = \frac{1}{4}$$

hence A.M. = G.M.

$\Rightarrow$ All numbers are equal $\frac{r_1}{2} = \frac{r_2}{4} = \frac{r_3}{5} = \frac{r_4}{8} = k$

$r_1 = 2k;\ r_2 = 4k;\ r_3 = 5k;\ r_4 = 8k$

$\Rightarrow \quad \prod r_1 = (2\cdot 4\cdot 5\cdot 8)k^4$

$$\frac{5}{4} = (2\cdot 4\cdot 5\cdot 8)k^4$$

$\therefore \quad k = 1/4$

hence $r_1 = \frac{1}{2};\ r_2 = 1;\ r_3 = \frac{5}{4};\ r_4 = 2$

$\Rightarrow \quad \sum r_1 = \frac{19}{4}$ but $\quad r_1 + r_2 + r_3 + r_4 = \frac{a}{4}$

$\Rightarrow \quad \frac{19}{4} = \frac{a}{4} \quad \Rightarrow \quad a = 19$

➤ **Problem 26.** Let $u_1, u_2, u_3, \ldots\ldots, u_n$ be an A.P. of positive terms with common difference d. Prove that $\sum_{i=1}^{n} u_i \geq n\sqrt{u_1^2 + (n-1)\,d\,u_1}$.

➤ ***Solution*** L H S $= S_n = \frac{n}{2}[2u_1 + (n-1)\,d]$

T P T $\left[u_1 + \frac{(n-1)\,d}{2}\right]^2 \geq u_1^2 + (n-1)\,d\,u_1$

or $\frac{(n-1)^2\,d^2}{4} + u_1 d\,(n-1) \geq (n-1)\,d\,u_1$

or $(n-1)^2\,d^2 \geq 0 \qquad [(n-1)\,d]^2 \geq 0$

which is true, equally holds if $d = 0$

➤ **Problem 27.** If a, b, c are real and positive, prove that the inequality,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < \frac{a^8+b^8+c^8}{a^3b^3c^3}.$$

➤ ***Solution*** This is equivalent to showing that

$a^8 + b^8 + c^8 > a^2\,b^2\,c^2\,(ab + bc + ca)$

or $\frac{a^6}{b^2c^2} + \frac{b^6}{a^2c^2} + \frac{c^6}{a^2b^2} > ab + bc + ca$ ...(1)

Now $\left(\frac{a^3}{bc} - \frac{b^3}{ac}\right)^2 > 0$

$\Rightarrow \quad \frac{a^6}{b^2c^2} + \frac{b^6}{c^2a^2} > \frac{2a^2b^2}{c^2}$

Thus $\frac{a^6}{b^2c^2} + \frac{b^6}{a^2c^2} + \frac{c^6}{a^2b^2}$

$> \frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2}$

again $\left(\frac{ab}{c} - \frac{bc}{a}\right)^2 > 0 \Rightarrow \frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} > 2b^2$

Thus $\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} > a^2 + b^2 + c^2$

Finally it is well known that,

$a^2+b^2+c^2 > ab + bc + ca$

➤ **Problem 28.** If a, b, c are the sides of a triangle, show that $\frac{1}{2} < \frac{ab+bc+ca}{a^2+b^2+c^2} < 1$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** $a^2 + b^2 > 2ab$, $b^2 + c^2 > 2bc$, $c^2 + a^2 > 2ca$.

Adding $a^2 + b^2 + c^2 > ab + bc + ca$,

or, $\dfrac{ab + bc + ca}{a^2 + b^2 + c^2} < 1$.

Again $a < b + c$, or, $a^2 < a(b + c)$.

Similarly $b^2 < b(c + a)$ and $c^2 < c(a + b)$.

Adding, $\dfrac{1}{2} < \dfrac{ab + bc + ca}{a^2 + b^2 + c^2}$

➤ **Problem 29.** Prove that for any real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ satisfying the relations

$$\left.\begin{aligned} a_1^2 + a_2^2 + ... + a_n^2 = 1, \\ b_1^2 + b_2^2 + ... + b_n^2 = 1, \end{aligned}\right\}$$ the inequality

$|a_1b_1 + a_2b_2 + ... + a_nb_n| \le 1$ is valid.

➤ ***Solution*** As is known, $|a.b| \le \dfrac{a^2 + b^2}{2}$

Taking advantage of the fact that the absolute value of a sum does not exceed the sum of the absolute values of the summands we get

$$|a_1b_1 + a_2b_2 + ... + a_nb_n| \le |a_1b_1| + |a_2b_2| + ... + |a_nb_n|$$

$$\le \frac{a_1^2 + b_1^2}{2} + \frac{a_2^2 + b_2^2}{2} + ... + \frac{a_n^2 + b_n^2}{2}$$

$$= \frac{a_1^2 + a_2^2 + ... + a_n^2 + b_1^2 + b_2^2 + ... + b_n^2}{2}$$

$$\le \frac{1+1}{2} = 1$$

➤ **Problem 30.** Show that $1!\ 3!\ 5!\ ....\ (2n - 1)! > (n!)^n$.

➤ ***Solution*** We have $^{2n}C_r = \dfrac{(2n)!}{r!(2n - r)!}$

The greatest value of $^{2n}C_r$ is $^{2n}C_n = \dfrac{(2n)!}{n!n!}$

$\therefore \quad \dfrac{(2n)!}{r!(2n - r)!} < \dfrac{(2n)!}{n!n!}$, or $r!(2n - r)! > (n!)^2$

Putting $r = 1, 3, 5, (2n - 1)$,

$1!(2n - 1)! > (n!)^2$

$3!(2n - 3)! > (n!)^2$

.....

$(2n - 1)!\ 1! > (n!)^2$

Multiplying these results we get

$\{1!\ 3!\ 5!\ ....\ (2n - 1)!\}^2 > (n!)^{2n}$,

or, $1!\ 3!\ 5!\ .....\ (2n - 1)! > (n!)^n$.

➤ **Problem 31.** Show that for any natural n we have

$$\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} < \frac{n-1}{n}$$

➤ ***Solution*** We write the following obvious inequalities

$$\frac{1}{2^2} < \frac{1}{1.2} = \frac{1}{1} - \frac{1}{2},$$

$$\frac{1}{3^2} < \frac{1}{2.3} = \frac{1}{2} - \frac{1}{3}$$

.....

$$\frac{1}{n^2} < \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$$

Adding them we get

$$\frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} < 1 - \frac{1}{n} = \frac{n-1}{n}$$

➤ **Problem 32.** Establish the inequality,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2$$

.

➤ ***Solution*** $\sqrt{r} + \sqrt{r} < \sqrt{r} + \sqrt{r+1}$

or $2\sqrt{r} < \sqrt{r} + \sqrt{r+1}$

or $\dfrac{1}{2\sqrt{r}} > \dfrac{1}{\sqrt{r+1} + \sqrt{r}}$ or $\dfrac{1}{2\sqrt{r}} > \sqrt{r+1} - \sqrt{r}$

or $\dfrac{1}{\sqrt{r}} > 2\left[\sqrt{r+1} - \sqrt{r}\right]$

➤ **Problem 33.** If $a_1, a_2, ..., a_n$ are n distinct odd natural numbers not divisible by any prime greater than 5, then prove that $\dfrac{1}{a_1} + \dfrac{1}{a_2} + ... + \dfrac{1}{a_n} < 2$.

➤ ***Solution*** Since each a_i is an odd number not divisible by a prime greater than 5, a_i can be written as $a_i = 3^r 5^s$

where r, s are non-negative integers

www.pdfworld.net

www.pdfworld.net

Thus, for all $n \in N$. $\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}$

$$< \left(1+\frac{1}{3}+\frac{1}{3^2}+....\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+....\right)$$

$$= \left(\frac{1}{1-1/3}\right)\left(\frac{1}{1-1/5}\right) = \left(\frac{3}{2}\right)\left(\frac{5}{4}\right) = \frac{15}{8} < 2.$$

Things to Remember

1. If a is the first term, d the common difference, n be the number of terms, and if ℓ denotes the last, or nth term of an A.P., we have

$\ell = a + (n-1)\,d.$

$\therefore \quad S_n = \frac{n}{2}(a+\ell)$

and $S_n = \frac{n}{2}\{2a+(n-1)\,d\}$

2. Important Properties of A.P.

(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP .

If $a_1, a_2, a_3, ...$ are in A.P. then

(a) $a_1 \pm k, a_2 \pm k, a_3 \pm k$ ..... are in A.P.

(b) a_1k, a_2k, a_3k ..... are in A.P.

(c) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}$ ..... are in A.P.

(d) a_p, a_{p+q}, a_{p+2q} ..... are in A.P.

(e) $a_1^r, a_2^r, a_3^r, ..., a_n^r$ are not in A.P. if $r \neq 0, 1$

(ii) The sum of the terms of an AP equidistant from the beginning and the end is constant and equal to the sum of the first and last terms. If $a_1, a_2, a_3, ..., a_n$ are in A.P., then

$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ...$

(iii) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it. If $a_1, a_2, a_3, ..., a_n$ are in A.P., then

$a_r = \frac{a_{r-k} + a_{r+k}}{2} \quad \forall\, k, 0 \le k \le n-r$

(iv) If a_1, a_2, a_3 ..... and b_1, b_2, b_3, ... are two A.P.'s, then

(a) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3$ ..... are in A.P.

(b) a_1b_1, a_2b_2, a_3b_3, ... are not in A.P.

(c) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$, ... are also not in A.P.

3. If a is the first term, r the common ratio, n be the number of terms, and if ℓ denotes the last, or nth term of a G.P., we have

$\ell = ar^{n-1}$ and $S_n = \frac{a(r^n-1)}{r-1}$

The sum of an infinite geometric series $S = a + ar + ar^2 + ...$ upto infinity,

when $|r| < 1$ is $\frac{a}{1-r}$

4. Important Properties of G.P.

(i) If $a_1, a_2, a_3, ...$ are in G.P. then

(a) a_1k, a_2k, a_3k ..... are in G.P.

(b) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}$ ..... are in G.P.

If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP.

(c) $a_1 \pm k, a_2 \pm k, a_3 \pm k$ ..... are not in G.P., $k \neq 0$.

(d) $a_1^r, a_2^r, a_3^r, ..., a_n^r$ are in G.P.

(e) $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ ..... are in G.P.

(ii) If $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are in G.P.

www.pdfworld.net

www.pdfworld.net

then $a_1b_1, a_2b_2, a_3b_3, \ldots\ldots\ldots\ldots$ are in G.P.

and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3} \ldots\ldots\ldots\ldots$ are in G.P.

(iii) If $a_1, a_2, a_3, \ldots\ldots\ldots$ and $b_1, b_2, b_3, \ldots\ldots\ldots$ are in G.P. then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \ldots\ldots\ldots$ are not in G.P.
But note that addition and subtraction of two G.P. when common ratio of both are same, is G.P.

(iv) If $a_1, a_2, a_3, \ldots\ldots$ are positive numbers in G.P. then $\log a_1, \log a_2, \log a_3, \ldots$ are in A.P. & vice-versa.

(v) If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots a_n$ are in G.P. then $a_1n_1 = a_2a_{n-1} = a_3\, a_{n-2} = \ldots\ldots\ldots\ldots$

(vi) If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots$ are in G.P.
then $a_2^2 = a_1a_3;\ a_3^2 = a_2a_4;\ a_4^2 = a_3\, a_5; \ldots\ldots$

(vii) If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots$, are in G.P.
then $a_r = \sqrt{a_{r-k}a_{r+k}}\ \ \forall\, k,\ \ 0 \le k \le n-r$

5. Recognization of A.P, G.P., H.P.
Let a, b, c be three successive terms of a sequence.

(i) If $\frac{a-b}{b-c} = \frac{a}{a}$, then a, b, c are in A.P.

(ii) If $\frac{a-b}{b-c} = \frac{a}{b}$, then a,b, c are in G.P.

(iii) If $\frac{a-b}{b-c} = \frac{a}{c}$, then a, b, c are in H.P.

6. The arithmetic mean of a set of any n numbers $a_1, a_2, \ldots, a_n$ is $A = \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n}$.

7. The n A.M.'s between a and b are

$$a + \frac{b-a}{n+1},\quad a + \frac{2(b-a)}{n+1},\quad \ldots\ldots,\quad a + \frac{n(b-a)}{n+1}$$

8. The sum of all the n A.M.'s inserted between a and b is equal to n times the single A.M. between a and b.

9. The geometric mean of a set of any n positive numbers $a_1, a_2, \ldots, a_n$ is
$G = (a_1a_2 \ldots a_n)^{1/n}$

10. The n G.M.'s between a and b are

$$G_1 = a\left(\frac{b}{a}\right)^{1/n+1},\qquad G_2 = a\left(\frac{b}{a}\right)^{2/n+1}, \ldots\ldots,$$

$$G_n = a\left(\frac{b}{a}\right)^{n/n+1}$$

11. The product of all the n G.M.s between a and b is equal to the nth power of the single GM between a and b.

12. The harmonic mean of a set of any n numbers $a_1, a_2, \ldots, a_n$ is $H = \left[\frac{\left\{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right\}}{n}\right]^{-1}$

13. The n H.M.'s between a and b are given by

$$\frac{1}{H_1} = \frac{1}{a} + \frac{(a-b)}{(n+1)ab},$$

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{(n+1)ab}, \ldots\ldots,$$

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$$

14. The sum of reciprocals of n Harmonic means between two given numbers is n times the reciprocal of single H.M. between them.

15. The sum of n terms of an A.G.P. is

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)} - \frac{[a+(n-1)d]r^n}{(1-r)},$$

$r \ne 1$

www.pdfworld.net

www.pdfworld.net

and the sum to infinity is

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}.$$

16. (i) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

(ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$

(iv) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

17. (sum of given numbers)^2
= sum of squares of numbers + 2 (sum of products of numbers taking two at a time)

18. If the first differences are constant, then assume the general term as $t_n = an + b$.
If the second differences are constant, then assume the general term as $t_n = an^2 + bn + c$.
If the third differences are constant, then assume the general term as $t_n = an^3 + bn^2 + cn + d$.
If the first differences are in G.P. with common ratio r, then assume the general term as $t_n = ar^n + b$.
If the second differences are in G.P. with common ratio r, then assume the general term as $t_n = ar^n + bn + c$.
If the third differences are in G.P. with common ratio r, then assume the general term as $t_n = ar^n + bn^2 + cn + d$.

19. If a, b are positive, then $A \geq G \geq H$. Also, equality occurs if and only if a = b.

20. Let a, b be positive real numbers and α, β be positive rational numbers such that $\alpha + \beta = 1$. Then $\alpha a + \beta b \geq a^\alpha . b^\beta$, and equality occurs if and only if a = b.

21. Theorem on Weighted Means

If $(a_1, a_2, \ldots, a_n)$ be a set of n positive numbers and $(q_1, q_2, \ldots, q_n)$ be a set of n positive rational numbers then $\left(\frac{\Sigma q_i a_i}{\Sigma q_i}\right)^{\Sigma q_i} > \prod a_i^{q_i}$
where i = 1, 2, ... , n and a's are not all equal.

22. mth Powers Theorem

If $(a_1, a_2, \ldots, a_n)$ be a set of positive numbers and all a's are not equal, then $\frac{\Sigma a_i^m}{n} \gtrless \left(\frac{\Sigma a_i}{n}\right)^m$
according as m does not or does lie between 0 and 1.

23. Cauchy-Schwartz inequality

If a, b, c and x, y, z are any real numbers (positive, zero or negative), then
$(ax + by + cz)^2 \leq (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$,
with equality if a : b : c :: x : y : z.

24. Tchebycheff's Inequalities

If $a_1, a_2, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots, b_n$ are any real numbers such that $a_1 \leq a_2 \leq \ldots \leq a_n$ and $a_1 \leq a_2 \leq \ldots \leq a_n$, then $n(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \geq (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n)$

25. RMS-AM inequality

$$\sqrt{\frac{(a_1^2 + \ldots + a_n^2)}{n}} \geq \frac{a_1 + \ldots + a_n}{n}$$

26.

(i) If $x + y + z + \ldots + w = c$ then $xyz \ldots w \leq \left(\frac{c}{n}\right)^n$.

(ii) If xyz .... w = c, then

$$\left(\frac{x + y + z + \ldots + w}{n}\right)^n \geq c,$$

assuming all quantities as positive.

www.pdfworld.net

www.pdfworld.net

[Objective Exercises]

Single Correct Answer Type for JEE Main

1. If S_r denotes the sum of r terms of an A.P. and $\frac{S_a}{a^2} = \frac{S_b}{b^2} = c$. then S_c is
(A) c^3 (B) c/ab
(C) abc (D) a + b + c

2. Which one of the following is true?
(A) The sequence given by log 2, log4, log8, log16, log32,.... is arithmetic.
(B) Two different arithmetic sequences cannot have the same fifth term.
(C) The sum of an arithmetic sequence can never be negative.
(D) It is not possible to find the 51st term of an arithmetic sequence without knowing all 50 of the preceding terms.

3. If x, y, z are real numbers satisfying the expression
$25(9x^2 + y^2) + 9z^2 - 15(5xy + yz + 3zx) = 0$,
then x, y, z are in
(A) A.P. (B) G.P.
(C) H.P. (D) None of these

4. Let a_n be the nth term of an A.P. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, the common difference of the A.P. is
(A) $\alpha - \beta$ (B) $\beta - \alpha$
(C) $(\alpha - \beta)/2$ (D) $(\alpha - \beta)/100$

5. The largest term common to the sequence 1, 11, 21, 31, ... to 100 terms and 31, 36, 41, 46,..... to 100 terms is
(A) 381 (B) 471
(C) 281 (D) none of these

6. If S_1, S_2 and S_3 denote the sum of first n_1, n_2 and n_3 terms respectively of an A.P., then
$\frac{S_1}{n_1}(n_2 - n_3) + \frac{S_2}{n_2}(n_3 - n_1) + \frac{S_3}{n_3}(n_1 - n_2) =$
(A) 0 (B) 1
(C) $S_1S_2S_3$ (D) $n_1n_2n_3$

7. If S_n denotes the sum to n terms of an A.P. and $p^2S_p = pS_{p^2}$ for each $p \in N$ then S_n is
(A) depends on d only
(B) divisible by n^2
(C) divisible by n
(D) divisible by (n – 1)

8. If $a_1, a_2, a_3 \ldots a_n$ be in A.P. where $a_1 = 0$ and common difference $\neq 0$, then
$$\frac{a_3}{a_2} + \frac{a_4}{a_3} + \frac{a_5}{a_4} + \ldots + \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_{n-2}}\right)$$
is equal to
(A) $\frac{a_{n-1}}{a_2} - \frac{a_2}{a_{n-1}}$ (B) $\frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$
(C) $\frac{-a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$ (D) none of these

9. If a, b, c be the p^{th}, q^{th} & r^{th} terms respectively of an A.P. and G.P. both, then the product of the roots of the equation $a^b b^c c^a x^2 - abcx + a^cb^ac^b = 0$ is equal to
(A) – 1 (B) 1
(C) 2 (D) (b – c) (c – a) (a – b)

10. If $m = \sum_{r=0}^{\infty} a^r$, $n = \sum_{r=0}^{\infty} b^r$, where $0 < a, b < 1$, then the quadratic equation whose roots are a and b is
(A) $mnx^2 + (m + n - 2mn)x + mn - m - n + 1 = 0$
(B) $mnx^2 + (2mn - m - n)x + mn - m - n + 1 = 0$
(C) $mnx^2 + (2mn + m + n)x + mn + m + n + 1 = 0$
(D) $mnx^2 - (2mn + m + n)x + mn + m + n + 1 = 0$

11. If $(r)_n = \underbrace{rrr.....r}_{n}$, $a = (6)_n$, $b = (8)_n$ and $c = (4)_{2n}$, then
(A) $a^2 + b + c = 0$ (B) $a^2 + b - c = 0$
(C) $a^2 + b - 2c = 0$ (D) $a^2 + b - ac = 0$

www.pdfworld.net

www.pdfworld.net

12. A particle begins at the origin and moves 1 unit right and reaches P_1 ; 1/2 unit up and reaches P_2; 1/4 unit right and reaches P_3 ; 1/8 unit down and reaches P_4 ; 1/16 unit right and reaches P_5 and so on. If $P_n = (x_n, y_n)$ then $\lim_{n\to\infty} P_n$ is

(A) (2, 3) (B) (4/3, 2/5)
(C) (2/5, 1) (D) none of these

13. Which one of the following is true:

(A) $\frac{n!}{(n-1)!} = \frac{1}{n-1}$

(B) The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,.... can be defined recursively using $a_0 = 1$, $a_1=1$, $a_n = a_{n-2} + a_{n-1}$, where $n \geq 2$.

(C) The same sum cannot be expressed by varying the limits of summation.

(D) As more terms are added, the sequence whose general term is given by $a_n = \frac{1}{n!}$ has a sum that is getting farther and farther from e.

14. The value of $\sum_{n=0}^{m} \log \frac{a^{2n-1}}{b^{m-1}}$ is $(a \neq 0, 1; b \neq 0, 1)$

(A) $m \log\left(\frac{a^{2m}}{b^{m-1}}\right)$ (B) $\log\left(\frac{a^{2m}}{b^{m-1}}\right)$

(C) $\frac{m}{2}\log\left(\frac{a^{2m}}{b^{2m-2}}\right)$ (D) $\frac{m}{2}\log\left(\frac{a^{2m}}{b^{m+1}}\right)$

15. $\sum_{r=1}^{n} r^2 - \sum_{m=1}^{n}\sum_{r=1}^{m} r$ is equal to

(A) 0 (B) $\frac{1}{2}\left(\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right)$

(C) $\frac{1}{2}\left(\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r\right)$ (D) none of these

16. The coefficient of x^n in the expansion of $(1 + x + 2x^2 + ... + nx^n)^2$ $(|x| < 1)$ is

(A) $\frac{n(n+1)}{2}$ (B) $\frac{n(n+1)(2n+1)}{6}$

(C) $\frac{n(n^2+11)}{6}$ (D) none of these

17. Find the sum of the infinite terms of series $1 + \frac{1}{5} + \frac{3}{5^2} + \frac{5}{5^3} + ... \text{ to } \infty$ is

(A) 3/8 (B) 8/3
(C) 11/8 (D) none of these

18. A man is to recieve Rs. P at the end of the first year, Rs. 2P at the end of second year, Rs. 3P at the end of third year, and so on. Assuming the process continues indefinitely if the rate of interest is Rs. r per rupee and it is compounded annually, then present value of the income is

(A) $\frac{P(1-r)}{r^2}$ (B) $\frac{P(1+r)}{r^2}$

(C) $\frac{P(1+r)}{r}$ (D) none of these

19. If $\frac{a+be^y}{a-be^y} = \frac{b+ce^y}{b-ce^y} = \frac{c+de^y}{c-de^y}$, then a, b, c, d are in

(A) A.P. (B) G.P.
(C) A.G.P. (D) None

20. If a, b, c are in G.P., x and y be the A.M.s between a, b and b, c respectively, then $\left(\frac{a}{x} + \frac{c}{y}\right)\left(\frac{b}{x} + \frac{b}{y}\right)$ is equal to

(A) 2 (B) –4
(C) 2 (D) None

21. If $0 < x < \frac{\pi}{4}$, then the minimum value of $(\sin x + \cos x + \text{cosec } 2x)^3$ is

(A) 27 (B) 13.5
(C) 6.75 (D) none of these

22. If x, y, z are positive then the minimum value of $[a^{\log y - \log z} + a^{\log z - \log x} + a^{\log x - \log y}]$ is

(A) 3 (B) 1
(C) 9 (D) 16

23. Sum to 16 terms $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$ is

www.pdfworld.net

www.pdfworld.net

(A) 442 (B) 446
(C) 440 (D) 448

24. The value of $\sum_{r=1}^{n}\left\{(2r-1)a+\frac{1}{b^r}\right\}$ is equal to

(A) $an^2+\frac{b^{n-1}-1}{b^{n-1}(b-1)}$ (B) $an^2+\frac{b^n-1}{b^n(b-1)}$

(C) $an^3+\frac{b^{n-1}-1}{b^n(b-1)}$ (D) None of these

25. The sum of the series $\sum_{r=1}^{n}\frac{(r^2+r-1)}{(r+2)!}$ is

(A) $\frac{1}{2}-\frac{(n+1)}{(n+2)!}$ (B) $\frac{1}{2}+\frac{(n+1)}{(n+2)!}$

(C) $1-\frac{(n+1)}{(n+2)!}$ (D) None of these

26. The sum of all the products of the first n positive integers taken two at a time is

(A) $\frac{1}{24}(n-1)\,n(n+1)(3n+2)$

(B) $\frac{1}{48}(n-2)(n-1)n^2$

(C) $\frac{1}{6}n(n+1)(n+2)(n+5)$

(D) None of these

27. Let $t_r=2^{r/2}+2^{-r/2}$. Then $\sum_{r=1}^{15}t_r^2$ is equal to

(A) $\frac{2^{21}-1}{2^{10}}+20$ (B) $\frac{2^{21}-1}{2^{10}}+19$

(C) $\frac{2^{21}-1}{2^{20}}+1$ (D) None of these

28. The sum of the series $1.1 + 2.3 + 4.5 + 8.7 +$ ..... to n terms is

(A) $3-2^{n+1}+(2n+1)\,2^n$
(B) $3+2^{n+1}+(2n+1)\,2^n$
(C) $3+2^{n-1}+(2n-1)\,2^n$
(D) $3-2^{n+1}+(2n-1)\,2^n$

29. Let $S_n(1\le n\le 9)$ denote the sum to n terms of the series $1 + 22 + 333 +$ ... $+ 999999999$, then for $2\le n\le 9$.

(A) $S_n-S_{n-1}=\frac{1}{9}(10^n-n^2+n)$

(B) $S_n=\frac{1}{9}(10^n-n^2+2n-2)$

(C) $9(S_n-S_{n-1})=n(10^n-1)$

(D) None of these

30. If $\frac{b+c}{a+d}=\frac{bc}{ad}=3\left(\frac{b-c}{a-d}\right)$ then $\pi^{\frac{1}{a}-1}$, $\pi^{\frac{1}{b}-1}$, $\pi^{\frac{1}{c}-1}$, $\pi^{\frac{1}{d}-1}$ are in

(A) A.P. (B) G.P.
(C) H.P. (D) None of these

31. If $a>0, b>0, c>0$ and $s=a+b+c$, then $\left[\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)-\frac{9}{2s}\right]$ is

(A) <0 (B) >3
(C) >0 (D) None of these

32. If a, b , x, y are positive rational numbers such that $\frac{1}{x}+\frac{1}{y}=1$, then $\left[\frac{a^x}{x}+\frac{b^y}{y}-ab\right]$ is

(A) <0 (B) >1
(C) ≥ 0 (D) None

33. Sum to infinite terms of the series

$\left[\frac{1}{5}-\frac{2}{7^2}+\frac{3}{5^3}-\frac{4}{7^4}+\text{.....}\right]$ is

(A) $\frac{211}{1152}$ (B) $\frac{220}{1811}$

(C) $\frac{2}{311}$ (D) None of these

34. If in a series $t_n=\frac{n}{(n+1)!}$ then $\sum_{n=1}^{20}t_n$ is equal to

(A) $\frac{20!-1}{20!}$ (B) $\frac{21!-1}{21!}$

(C) $\frac{1}{2.(20-1)!}$ (D) None of these

www.pdfworld.net

www.pdfworld.net

35. If S_n denote the sum of the cubes of the first n natural numbers and s_n denote the sum of the first n natural numbers. Then $\sum_{r=1}^{n} \frac{S_r}{s_r}$ is

(A) $\frac{n(n+1)(n+2)}{6}$ (B) $\frac{n(n+1)}{2}$

(C) $\frac{n^2+3n+2}{6}$ (D) None of these

36. If $a_1, a_2, \ldots\ldots, a_n$ are n non-zero real numbers such that $(a_1^2 + a_2^2 + \ldots + a_{n-1}^2)(a_2^2 + a_3^2 + \ldots + a_n^2) \le (a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n)^2$

(A) A.P. (B) G.P.

(C) H.P. (D) None of these

37. The sum of the series $1^2 - 2 \cdot 2^2 + 3^2 - 2 \cdot 4^2 + 5^2 - 2 \cdot 6^2 + \ldots$; when n is odd is given by $\frac{-n(n+1)(n-4)}{6}$ then sum when n is even is given by

(A) $\frac{n(n+1)(n-4)}{6}$ (B) $\frac{n(n+1)(n+5)}{6}$

(C) $\frac{n(n-1)(n+3)}{6}$ (D) None of these

38. In the arithmetic progression whose common difference is non-zero, the sum of first 3n terms is equal to the sum of next n terms. Then the ratio of the sum of the first 2n terms to the next 2n terms is

(A) 1/10 (B) 10

(C) 5 (D) 1/5

39. If $u_n = \sum_{n=0}^{n} \frac{1}{2^n}$. Then $\sum_{n=1}^{n} u_n$ is

(A) $2^{-n} + 2n - 1$ (B) $2^n - 2n + 1$

(C) $2^{-n} - 2n + 1$ (D) None of these

40. The value of

$$\lim_{n\to\infty}\left[\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^3} + \frac{9}{1^2+2^2+3^2+4^2} + \ldots\ldots\text{to n terms}\right] \text{ is}$$

(A) 6 (B) 7

(C) 8 (D) 9

41. The value of the expression $[a^2(1+b^2) + b^2(1+c^2) + c^2(1+a^2) - 6abc]$ is

(A) ≤ 0 (B) < 0

(C) > 0 (D) ≥ 0

42. Let r be the common ratio of the G.P. $a_1, a_2, a_3, \ldots\ldots, a_n$, then $\frac{1}{a_1^m + a_2^m} + \frac{1}{a_2^m + a_3^m} + \ldots + \frac{1}{a_{n-1}^m + a_n^m}$ is

(A) $\frac{1 - r^{m(1-n)}}{a_1^m(r^m - r^{-m})}$ (B) $\frac{1 + r^{m(1-n)}}{a_1^m(r^m - r^{-m})}$

(C) $\frac{1 + r^{m(1-n)}}{a_1^m(r^m + r^{-m})}$ (D) None of these

43. If $A = 1 + r^a + r^{2a} + r^{3a} + \ldots\ldots$ to inf. and $B = 1 + r^b + r^{2b} + r^{3b} + \ldots\ldots$ to inf. then a/b is equal to

(A) $\log_{(1-B)}(1-A)$ (B) $\log_{\left(\frac{B-1}{B}\right)}\left(\frac{A-1}{A}\right)$

(C) $\log_B A$ (D) None of these

44. If a, b, c, d are in G.P, then $(a^3 + b^3)^{-1}$, $(b^3 + c^3)^{-1}$, $(c^3 + d^3)^{-1}$ are in

(A) A. P. (B) G.P.

(C) A.G.P. (D) None of these

45. If the sum of the series $\sum_{n=1}^{\infty} r^n$, $|r| < 1$, is s, then $\sum_{n=1}^{\infty} r^{2n}$ is

(A) s^2 (B) $2s / (s^2 - 1)$

(C) $s^2 / (2s + 1)$ (D) None of these

46. If x, y, z > 0 and x + y + z = 1, then $\frac{xyz}{(1-x)(1-y)(1-z)}$ is necessarily

(A) ≤ 8 (B) $\le \frac{1}{8}$

(C) $< \frac{1}{8}$ (D) None of these

47. The value of the recurring decimal 0.10898989 ....is

www.pdfworld.net

www.pdfworld.net

(A) $\frac{1089}{10000}$ (B) $\frac{1079}{9900}$

(C) $\frac{1069}{9800}$ (D) $\frac{1099}{9800}$

48. The sum to n terms of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$ is

(A) $\frac{n(n+1)(n+2)}{12}$

(B) $\frac{n(n+1)^2(n+2)^2}{12}$

(C) $\frac{n^2(n+1)(n+2)}{12}$

(D) $\frac{n(n+1)^2(n+2)}{12}$

49. Let S_n, $n \geq 1$, be the sets defined as $S_1 = \{0\}$, $S_2 = \left\{\frac{3}{5}, \frac{5}{2}\right\}$, $S_3 = \left\{\frac{8}{3}, \frac{11}{3}, \frac{14}{3}\right\}$, $S_4 = \left\{\frac{15}{4}, \frac{19}{4}, \frac{23}{4}, \frac{27}{4}\right\}$ and so on, then the sum of the elements of S_{20} is

(A) 589 (B) 609

(C) 189 (D) 209

50. The value of $\frac{1}{1.3.5} \frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \frac{1}{9.11.13}$ equals

(A) $\frac{70}{249}$ (B) $\frac{53}{249}$

(C) $\frac{35}{429}$ (D) $\frac{35}{249}$

Multiple Correct Answer Type for JEE Advanced

51. For the A.P. given by $a_1, a_2, ..., a_n, ...$, the equations satisfied are

(A) $a_1 + 2a_2 + a_3 = 0$

(B) $a_1 - 2a_2 + a_3 = 0$

(C) $a_1 + 3a_2 - 3a_3 - a_4 = 0$

(D) $a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 = 0$

52. The sum of the first three terms of the G.P. in which the difference between the second and the first term is 6 and the difference between the fourth and the thrd term is 54, is

(A) 39 (B) -10.5

(C) 27 (D) -27

53. If $\cos(\theta - \alpha)$, $\cos\theta$, $\cos(\theta + \alpha)$ are in H.P. and $\cos\alpha \neq 1$, then the angle α cannot lie in the

(A) I quadrant (B) II quadrant

(C) III quadrant (D) IV quadrant

54. If a, b, c are in H.P., then :

(A) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

(B) $\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}$

(C) $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in G.P.

(D) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

55. If a, b, c are in A.P. & a^2, b^2, c^2 are in H.P., then :

(A) $a = b = c$ (B) $a^2 = b^2 = \frac{c^2}{2}$

(C) a, b, c are in G.P. (D) $-\frac{a}{2}$, b, c are in G.P.

56. If the first & the $(2n + 1)^{th}$ terms of an A.P., a G.P. & an H.P. of positive terms are equal and their $(n + 1)^{th}$ terms are a, b & c respectively, then :

(A) $a = b = c$ (B) $a \geq b \geq c$

(C) $a + c = 2b$ (D) $ac = b^2$.

57. If the arithmetic mean of two positive numbers a & b $(a > b)$ is twice their geometric mean, then a : b is :

(A) $2 + \sqrt{3} : 2 - \sqrt{3}$

(B) $7 + 4\sqrt{3} : 1$

www.pdfworld.net

www.pdfworld.net

(C) $1 : 7 - 4\sqrt{3}$

(D) $2 : \sqrt{3}$

58. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then

(A) $a + c = b + d$

(B) $e = 0$

(C) a, b = –2/3, c – 1 are in A.P.

(D) c/a is an integer

59. If b_1, b_2, b_3 $(b_i > 0)$ are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$, holds is given by

(A) $r > 3$ (B) $0 < r < 1$

(C) $r = 3.5$ (D) $r = 5.2$

60. If the quadratic equation,

$(b^2 + c^2)x^2 - 2(a + b)cx + (c^2 + a^2) = 0$

has equal roots, then

(A) a, b, c are in G.P.

(B) a, b, c are in A.P.

(C) a, c, b are in G.P.

(D) a, c, b are in A.P.

61. The sum of the first three consecutive terms of an A.P. is 9 and the sum of their squares is 35. Then sum to n terms of the series is :

(A) $n(n + 1)$ (B) n^2

(C) $n(4 - n)$ (D) $n(6 - n)$

62. The sum to n terms of the series

$$\frac{1}{\sqrt{3}+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{15}} + ...$$

(A) $\frac{\sqrt{3+4n}-\sqrt{3}}{4}$ (B) $\frac{n}{\sqrt{3+4n}-\sqrt{3}}$

(C) less than n (D) greater than $\sqrt{n}/2$

63. If $\sum_{k=1}^{n}\left(\sum_{m=1}^{n} m^2\right) = an^4 + bn^3 + cn^2 + dn + e$ then -

(A) $a = \frac{1}{12}$ (B) $b = \frac{1}{6}$

(C) $d = \frac{1}{6}$ (D) $e = 0$

64. If the triplets log a, log b, log c and (log a–log 2b)(log 2b – log 3c), (log 3c – log a) are in arithmetic progression then

(A) $18(a + b + c)^2 = 18(a^2 + b^2 + c^2) + ab$

(B) a, b, c are in G.P.

(C) a, 2b, 3c are in H.P.

(D) a, b, c can be the lengths of the sides of a triangle

65. If S_n denotes the sum of first n terms of an Arithmetic progression and a_n denotes the n^{th} term of the same A.P. Given $S_n = n^2p$; $S_k = k^2p$; where k, p, $n \in N$ and $k \neq n$ then

(A) $a_1 = p$

(B) common difference = 2p

(C) $S_p = p^3$

(D) $a_p = 2p^2 - p$

66. If x, y, z are positive real numbers, such that $x + y + z = a$, then

(A) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{a}$

(B) $(a - x)(a - y)(a - z) \geq 8xyz$

(C) $(a - x)(a - y)(a - z) \leq \frac{8}{27}a^3$

(D) none of these

67. If the roots of the equation, $x^3 + px^2 + qx - 1 = 0$ from an increasing G.P. where p and q are real, then

(A) $p + q = 0$

(B) $p \in (-3, \infty)$

(C) one of the roots is unity

(D) one root is smaller than 1 and one root is greater than 1

68. Which of the cubic polynomials given below do not have their roots in arithmetic progression?

(A) $x^3 + 3x^2 - 2x - 1$ (B) $x^3 + 3x^2 - 2x - 2$

(C) $x^3 + 3x^2 - 2x - 3$ (D) $x^3 + 3x^2 - 2x - 4$

69. If $a = \underbrace{111.....1}_{55 \text{ times}}$, $b = 1 + 10 + 10^2 + 10^3 + 10^4$ and $c = 1 + 10^5 + 10^{10} + ...+10^{50}$ then

www.pdfworld.net

www.pdfworld.net

(A) b, $\frac{a}{2}$, c are in A.P.

(B) b, $\sqrt{a}$, c are in G.P.

(C) a is a prime number

(D) a is a composite number

70. Let a_1, a_2, a_3 ..... and b_1, b_2, b_3 ..... be arithmetic progressions such that $a_1 = 25$, $b_1 = 75$ and $a_{100} + b_{100} = 100$. Then

(A) the difference between successive terms in progression 'a' is opposite of the difference in progression 'b'.

(B) $a_n + b_n = 100$ for any *n*.

(C) $(a_1 + b_1), (a_2 + b_2), (a_3 + b_3)$, ... are in A.P.

(D) $\sum_{r=1}^{100}(a_r + b_r) = 10000$

Comprehension - 1

The real roots of the equation
$2x^3 - 19x^2 + 57x + k = 0$
are the first three terms of a geometric progression.

71. The value of k equals

(A) 216 (B) 108

(C) –54 (D) –108

72. If the geometric progression is increasing then the sum of its first n terms equals

(A) $\left(\frac{3}{2}\right)^n - 1$ (B) $4\left(\left(\frac{3}{2}\right)^n - 1\right)$

(C) $6\left(\left(\frac{3}{2}\right)^n - 1\right)$ (D) $4(2^n - 1)$

73. If the geometric progression is decreasing then the sum of its infinite number of term is

(A) 27/2 (B) 9

(C) 9/2 (D) 12

Comprehension - 2

Let equation $x^3 + px^2 + qx - q = 0$ where $p, q \in R - \{0\}$ has 3 real roots α, β, γ in H.P., the

74. (9p + 2q) has the value equal to

(A) 9 (B) –18

(C) –27 (D) 1

75. Minimum value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ is

(You may use the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$ for any $a, b, c \in R$)

(A) $\frac{1}{3}$ (B) 1

(C) $\frac{4}{3}$ (D) 3

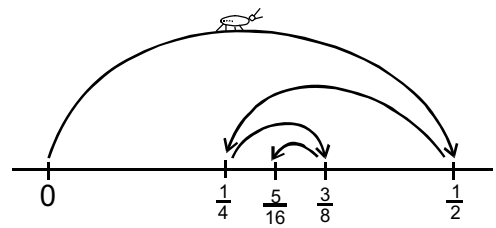
76. $\frac{p}{q}$ has the minimum value equal to

(A) $-\frac{1}{2}$ (B) $-\frac{1}{3}$

(C) $-\frac{1}{4}$ (D) –1

Comprehension - 3

A flea, to amuse itself, jumps $\frac{1}{2}$ meter to the right, then $\frac{1}{4}$ meter to the left, then $\frac{1}{8}$ meter to the right, then $\frac{1}{16}$ meter to the left, and so on, as shown in figure. On the nth jump the flea travels $1/2^n$ meter, but continues alternating right and left. The flea starts at the number 0 on the number line.



77. After n jumps, where n is odd the flea is at the number

www.pdfworld.net

www.pdfworld.net

(A) $\frac{1}{3}\left(1+\frac{1}{2^n}\right)$ (B) $\frac{2}{3}\left(1+\frac{1}{2^n}\right)$

(C) $\left(1-\frac{1}{2^n}\right)$ (D) none of these

78. After n jumps, where n is even, the flea is at the number

(A) $\frac{2}{9}\left(1-\frac{1}{2^{n+1}}\right)$ (B) $\frac{1}{3}\left(1-\frac{1}{2^n}\right)$

(C) $\frac{2}{9}\left(1+\frac{1}{2^{n-1}}\right)$ (D) none of these

79. As the flea keeps on jumping, what single number does it keep jumping over?

(A) $\frac{3}{8}$ (B) $\frac{1}{9}$

(C) $\frac{1}{3}$ (D) none of these

Comprehension - 4

N, the set of natural numbers, is partitioned into subsets $S_1 = \{1\}$. $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, $S_4 = \{7, 8, 9, 10\}$,.....

80. Last term of S_{50} is
(A) 25550 (B) 1275
(C) 1225 (D) 2450

81. Sum of the elements in the subset S_{50} is
(A) 62525 (B) 62500
(C) 62450 (D) 62475

82. First term of subset S_7 is
(A) 21 (B) 22
(C) 23 (D) 24

Comprehension - 5

Four different integers form an increasing A.P. One of these numbers is equal to the sum of the squares of the other three numbers. Then

83. The smallest number is :
(A) – 2 (B) 0
(C) –1 (D) 2

84. The common difference of the four numbers is
(A) 2 (B) 1
(C) 3 (D) 4

85. The sum of all four numbers is
(A) 10 (B) 8
(C) 2 (D) 6

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

86. **Assertion (A) :** The leftmost three digits of the number $1^1 + 2^2 + 3^3 + ... + 999^{999} + 1000^{1000}$ is 100

Reason (R) : The first 999 terms of the above series will not contribute the leftmost three digits of the sum. The leftmost three digits come from the term 1000^{1000} only.

87. **Assertion (A) :** If $x_1, x_2, ..., x_n \geq -1$ and $\sum_{i=1}^{n} x_i^3 = 0$ then $\sum_{i=1}^{n} x_i^3 \leq \frac{n}{3}$.

Reason (R) : The inequality $x^3 - \frac{3}{4}x + \frac{1}{4} \geq 0$ holds for all $x \geq -1$. Hence $\sum_{i=1}^{n}\left(x_i^3 - \frac{3}{4}x_i + \frac{1}{4}\right) \geq 0$ which implies $\sum_{i=1}^{n} x_i \leq \frac{n}{3}$

88. **Assertion (A) :** If $|x| < 1$ then

$$\lim_{m\to\infty} \sum_{r=0}^{m} \frac{x^{r+1}}{(r+1)(r+2)} = 1 + \frac{(1-x)}{x} \ln(1-x)$$

Reason (R) : $\sum_{m=0}^{\infty}\left(\frac{x^{m+1}}{m+1} - \frac{x^{m+1}}{m+2}\right)$

$$= 1 + \left(1-\frac{1}{x}\right)\sum_{m=0}^{\infty}\frac{x^{m+1}}{m+1}$$

$$= 1 + \left(1-\frac{1}{x}\right)\ln\frac{1}{1-x}, |x| < 1.$$

www.pdfworld.net

www.pdfworld.net

89. Assertion (A) : $\sum_{n=1}^{\infty} \frac{3^{-n}}{n} = \ln(3/4)$.

Reason (R) : We integrate the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x| < 1$ from 0 to $\frac{1}{3}$ to get $\sum_{n=1}^{\infty} \frac{3^{-n}}{n}$.

90. Assertion (A) : Let a sequence of rational numbers be defined by $x_0 = 10$, $x_1 = 29$ and $x_{n+2} = \frac{19x_{n+1}}{94x_n}$ for $n \geq 0$. The value of $\sum_{n=0}^{\infty} \frac{x_{6n}}{2^n} = 20$.

Reason (R) : $x_{n+3} = \frac{19^2}{94^2 x_n} \Rightarrow x_{n+6} = x_n$.

$\Rightarrow \sum_{n=0}^{\infty} \frac{x_{6n}}{2^n} = \sum_{n=0}^{\infty} \frac{10}{2^n} = 20$

91. Assertion (A) : Let $f(x) = 1 - (x + x^2 + ...+x^n) + (x + x^2 + ... + x^n)^2 = ... + (-1)^n(x + x^2 + ... +x^n)$. The coefficient of $x^r (2 \leq r \leq n)$ in $f(x)$ is 2

Reason (R) : Since $x + x^2 + ... + x^n = \frac{x(1-x^n)}{1-x}$ and $f(x)$ is a geometric series, its sum is $\frac{1-x}{1-x^{n+1}}$. Hence the coefficient of x^r in $f(x)$ is same as that in $(1-x)(1 + x^{n+1} + x^{2n+2} + ...)$.

92. Assertion (A) : a, b c are in H.P. and $a > b > c > 0$ then value of $\frac{2a-b}{2a+b} + \frac{2c-b}{2c+b}$ lies between $\frac{2}{3}$ and 1.

Reason (R) : $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. and $\frac{1}{a} < \frac{1}{b} < \frac{1}{c}$ or $\frac{b}{a}, 1, \frac{b}{c}$ are in A.P. > or $(1-d), 1, (1+d)$ are $\frac{b}{a}, 1, \frac{b}{c}$ where $0 < d < 1$.

93. Assertion (A) : If a sequence (a_n) which satisfies $a_{n+1} = \frac{a_n^2 - a_{n-1} + 2a_n}{a_{n-1} + 1}$ $a_1 = 1$ and $a_9 = 7$, then the value of a_5 is 5.

Reason (R) : Let $b_n = a_n - 1$. Then the sequence (b_n) is a G.P. from which $b_5^2 = b_1 b_9 \Rightarrow b_5^2 = 16$.

94. Assertion (A) : The sum $\sum_{1=1}^{\infty} \frac{ai}{a^i}$ can be found only when $a > 1$, as $\left(\frac{a}{1-a}\right)^2$.

Reason (R) : An A.G.P. with infinite number of terms can be summed when the common ratio is less than 1 in magnitude.

95. Assertion (A) : The value of $\sum_{n=2}^{17} \frac{n^2+n+1}{n(n^2-1)(n+2)}$ is $\frac{593}{969}$.

Reason (R) : Since $\frac{n^2-n+1}{n(n^2-1)(n+2)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n+2}\right)$, the summation becomes $\frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{18} - \frac{1}{19}\right)$.

Match the Columns for JEE Advanced

96.

Column-I	Column-II
(A) The sequence a, b, 10, c, d is an arithmetic progression. The value of a + b + c + d	(P) 10
(B) The sides of right triangle form a three term geometric sequence. The shortest side has length 2. The length of the hypotenuse is of the form $a + \sqrt{b}$ where $a \in N$ and $\sqrt{b}$ is a surd, then $a^2 + b^2$ equals	(Q) 20

www.pdfworld.net

www.pdfworld.net

(C) The sum of first three consecutive numbers of an infinite G..P. is 70, if the two extremes be multiplied each by 4, and the mean by 5, the products are in A.P. The first term of the G.P. is — (R) 26

(D) The diagonals of a parallelogram have a measure of 4 and 6 metres. They cut off forming an angle of 60°. If the perimeter of the parallelogram is $2\left(\sqrt{a}+\sqrt{b}\right)$ where a, b $\in$ N then (a + b) equals — (S) 40

97.

Column - I	**Column - II**
(A) The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2-(4+\sqrt{5})x+8+2\sqrt{5}=0$ is	(P) 2
(B) Let $a_1, a_2, ..., a_{10}$, be in A.P. and $h_1, h_2, ..., h_{10}$ be in H.P. If $a_1 = h_1 = 2$ & $a_{10} = h_{10} = 3$ then $a_4\, h_7$ is	(Q) 3
(C) The number of integer values of m, for which the x coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is	(R) 4
(D) Between 2 and 5 six geometric means are inserted. If their product can be expressed as $(10)^n$ then the value of n equals.	(S) 6

98.

Column-I	**Column-II**
(A) The equation $x^{x\sqrt{x}}=\left(x\sqrt{x}\right)^x$ has two solutions in positive real numbers x. One obvious solution is x = 1. The other one is x =	(P) 8/3
(B) Suppose a triangle ABC is inscribed in a circle of radius 10 cm. If the perimeter of the triangle is 32 cm then the value of $\sin A + \sin B + \sin C$ equals	(Q) 9/4
(C) Sum of infinite terms of the series $1+\frac{3}{4}+\frac{7}{16}+\frac{15}{64}+\frac{31}{256}+....$ equals	(R) 5/4
(D) The sum of $\sum_{r=1}^{\infty}\left(\frac{r+3}{r(r+1)(r+2)}\right)$ equals	(S) 8/5
	(T) 7/4

99.

Column-I	**Column-II**
(A) If three unequal numbers a, b, c are A.P. and b-c, c-b a-b are in G.P., then $\frac{a^3+b^3+c^3}{3abc}$ is equal to	(P) $\frac{1}{3}$

www.pdfworld.net

www.pdfworld.net

(B) Let x be the arithmetic mean and y,z be two geometric means between any two positive numbers, then $\frac{y^3+z^3}{xyz}$ is equal to — (Q) 1

(C) If a, b, c be three positive number which form three successive terms of a G.P. and $c > 4b - 3a$, then the common ratio of the G.P. can be equal to — (R) 2

(D) $\lim_{n\to\infty} \tan\left\{\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{2r^2}\right)\right\}$ is equal to — (S) 3

100.

Column-I	Column-II
(A) If a, b, c be positive numbers then $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ must be greater than or equal to	(P) 4
(B) If h be the H.M. and g be the G.M. of two positive numbers a and b such that h : g = 4 : 5, then a/b can be equal to	(Q) 9
(C) If $S = \sum_{r=0}^{\infty} \frac{1}{2^r}$ and $S - S_{n+1} > 10^{-3}$ then n is less than or equal to	(R) 10
(D) If $(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots\ldots\ldots\ldots(1+x^{128}) = \sum_{r=0}^{n} x^r$ then n is equal to	(S) 255

Review Exercises for JEE Advanced

1. Write the first five terms of each sequence whose general term is given.

(i) $a_n = \frac{3.5\ldots\ldots\ldots\ldots(2n+1)}{2.4\ldots\ldots(2n)}$

(ii) $a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ 2a_n + 5 & \text{if } a_n \text{ is odd and } a_1 = 9 \end{cases}$

(iii) $a_n = \frac{1+(-1)^{n+1}}{2i^{n-1}}$ $(i = \sqrt{-1})$

2. Prove that if a, b, and c form an arithmetic progression then the numbers $a^2 + ab + b^2$, $a^2 + ac + c^2$, and $b^2 + bc + c^2$ also form an arithmetic progression.

3. Solve $1+3+5+\ldots+(1+2x) + \left(\frac{7}{2}+5+\frac{13}{2}+\ldots+\frac{7+3x}{2}\right) = 105$

4. Prove that the cube of any whole number n is the sum of n terms of an arithmetic progression in which

(i) $a_1 = 1, d = 2(n+1)$

(ii) $a_1 = n, d = 2n$

(iii) $a_1 = n^2 - n + 1, d = 2$

5. The sum of three numbers forming a geometric progression is equal to 56. If we subtract 1, 7, 21 from these numbers, respectively, then the newly obtained numbers will form an arithmetic progression. Find these numbers.

www.pdfworld.net

www.pdfworld.net

6. The sides of a triangle are in G.P. and its largest angle is twice the smallest one. Prove that the common ratio of the G.P. lies in the interval $(1, \sqrt{2})$.

7. If S_p denote the sum of the series $1 + r^p + r^{2p} + ...$ ad inf., and s_p the sum of the series $1 - r^p + r^{2p} - ....$ ad inf., prove that $S_p + s_p = 2S_{2p}$.

8. Given an arithmetic and a geometric progression. The first two terms of the geometric progression coincide respectively with the first and the second terms of the geometric progression, and the third term the geometric progression exceeds the third term of the arithmetic progression by 8. Find these progressions.

9. It is given that the numbers $3, 3\log_y x, 3\log_z y, 7\log_x z$ form an arithmetic progression. Prove that $x^{18} = y^{21} = z^{28}$.

10. Find the sum $nx + (n-1)x^2 + ... + 2x^{n-1} + x^n$.

11. The number sequence 1, 4, 10, 19, ... satisfies the condition that the differences of two subsequent terms form an arithmetic progression. Find the nth term and the sum of the first n terms of this sequence.

12. Prove that : $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{199} - \frac{1}{200} = \frac{1}{101} + \frac{1}{102} + \frac{1}{103} + ... + \frac{1}{200}$

13. Using the definition of the arithmetic mean $\bar{x}$ of a set of n numbers $x_1, x_2, ... x_n$.

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, prove that

(i) $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$

(ii) $\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$

14. Show that $\frac{n-1}{n+1} + 3\left(\frac{n-1}{n+1}\right)^2 + 5\left(\frac{n-1}{n+1}\right)^3 + ... \text{ upto inf.} = \sum_{r=1}^{n-1} r$.

15. Show that whether n is odd or even,
$1^4 - 2^4 + 3^4 - 4^4 + ... \text{ to } (n-1) \text{ terms} = \frac{1}{2}(-1)^n n(n-1)(n^2 - n - 1)$.

16. Sum the series to n terms :
$(n^2-1)^2 + 2(n^2-2^2) + 3(n^2-3^2) + ...$

17. Evaluate $\sum_{1}^{n}\left\{\sum_{1}^{n}(2^n + 3n)\right\}$.

18. Find the constant term in the product f(x). g(x) of the functions :
$f(x) = 1 + 2x + 3x^2 + ... + (n+1)x^n$
$g(x) = 1 + \frac{1}{3x} + \frac{1}{9x^2} + ... + \frac{1}{3^n.x^n}$, $x \neq 0$

19. Prove that the sum of the numbers in any group in the following, is the square of an odd integer:
(1); (1, 3, 5) ; (1, 3, 5, 7, 9); .....

20. Find the sum of the products of $1, \frac{1}{2}, \frac{1}{2^2}, ..., \frac{1}{2^{10}}$ taking two at a time.

21. Find the sum to 2n terms of the series
$5^3 + 4.6^3 + 7^3 + 4.8^3 + 9^3 + 4.10^3$

22. Find the sum of the following series:
(i) $1 + (1+2)x + (1+2+3)x^2 + ...$ to n terms.
(ii) $2.1^2x + 2^2.2^2x^2 + 2^3 . 3^2x^3 + ... + 2^n . n^2x^n$.

23. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... \text{ upto } \infty = \frac{\pi^2}{6}$, then find

(i) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... \text{ upto } \infty$

(ii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... \text{ upto } \infty$

24. Show that
$(1+5^{-1})(1+5^{-2})(1+5^{-4})(1+5^{-8}) .... (1+5^{-2^n}) = \frac{5}{4}(1-5^{-2^{n+1}})$

25. One side of an equilateral triangle is 24 cm. The mid-points of its sides are joined to form another triangle whose mid-points are in term

www.pdfworld.net

www.pdfworld.net

joined to form still another triangle. This process continues indefinitely. Find the sum of the perimeters of all such triangles.

26. Find the sum of n terms of the series
$\frac{4}{5}+\frac{4.7}{5.8}+\frac{4.7.10}{5.8.11}+\ldots\ldots$

27. Find the sum of the infinite series :
$1^2-\frac{2^2}{5}+\frac{3^2}{5^2}-\frac{4^2}{5^3}+\frac{5^2}{5^4}-\frac{6^2}{5^5}+\ldots..$

28. Find the sum of n terms of the series
$\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\ldots\ldots$

29. Prove for any two positive numbers $a \neq b$ and a positive integer n, $ab^n < \left(\frac{a+nb}{n+1}\right)^{n+1}$.

30. If a, b, c are positive fractions and $a+b+c=1$, prove that $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq 64$.

31. If a, b and c are positive real numbers such that $a+b+c=1$, prove that
$\frac{b(1-b)}{ac}+\frac{c(1-c)}{ab}+\frac{a(1-a)}{bc} \geq 6$.

32. If a, b, c are three positive quantities whose sum is equal to their product then prove that $a+b+c \geq 3\sqrt{3}$. Also prove that if $a+b+c=3\sqrt{3}$ then $a=b=c$.

33. If $4a+3b+2c=5$ and $a>0, b>0, c>0$ then for what values of a, b, c the value of $a^2b^3c^4$ will be the greatest?

34. If $a+b+c=1$ and $a>0, b>0, c>0$ then prove that $ab^2c^3 \leq \frac{1}{432}$.

35. Find the greatest value of the product of three positive numbers if the sum of their products taking two at a time is 12.

Target Exercises for JEE Advanced

1. Show that square of any even natural number $(2n)^2$ is equal to the sum of n terms of some series of integers in A.P.

2. The sixth term of the arithmetic progression is equal to 3, and the common difference exceeds 1/2. At what value of the difference of the progression is the product of the first, the fourth and the fifth term the largest?

3. Find the sum of all non-reducible fractions with the denominator 3 contained between the numbers 4 and 8.

4. If the nonzero numbers $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ are in AP, prove that
$\frac{1}{a_1a_2a_3}+\frac{1}{a_2a_3a_4}+\ldots\ldots+\frac{1}{a_{n-2}a_{n-1}a_4}$
$=\frac{1}{2(a_2-a_1)}\left(\frac{1}{a_1a_2}-\frac{1}{a_{n-1}a_n}\right)$

5. Prove that (666 ..... to n digits)^2 + (888 ..... to n digits) = (444 ..... to 2n digits).

6. Show that the sum of the products of every pair of the squares of the first n natural numbers is
$\frac{1}{360}n(n^2-1)(4n^2-1)(5n+6)$

7. Prove that no G.P. can have three of its terms (not necessarily consecutive) as three consecutive non-zero integers.

8. Solve the following system of equations:
$\frac{x}{y}=\frac{y}{z}=\frac{z}{u}=\frac{u}{s}=\frac{s}{t}$,
$x=8u,\ \ x+y+z+u+s+t=15\frac{3}{4}$

9. An AP and a GP each has 'a' as the first term and 'b' as the second term with $0<b<a$. If S denotes the sum to infinity of the GP, prove that the sum of the first n terms of the AP is $na-\frac{n(n-1)a^2}{2S}$.

10. If $a_1, a_2, a_3 \ldots\ldots a_{2n}$ are positive real numbers in GP, show that
$\sqrt{a_1a_2}+\sqrt{a_3a_4}+\sqrt{a_5a_6}+\ldots\ldots\ldots+\sqrt{a_{2n-1}a_{2n}}$
$=\sqrt{a_1+a_3+\ldots+a_{2n-1}}\ \sqrt{a_2+a_4+\ldots+a_{2n}}$.

www.pdfworld.net

www.pdfworld.net

11. If S be the sum of the n consecutive integers beginning with a, and t the sum of their squares, show that $nt - S^2$ is independent of a.

12. Find the sum of n terms of the series $(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)+....$ where $a \neq 1$, $b \neq 1$, and $a \neq b$

13. Find the sum of n terms of the series

$$\frac{1^2}{2.3}.4+\frac{2^2}{3.4}.4^2+\frac{3^2}{4.5}.4^3+\frac{4^2}{5.6}.4^4+...$$

14. (i) Find the coefficient of x^{99} in the product $(2x-1)(2x-3)(2x-5)....(2x-199)$.

(ii) Find the coefficient of x^n in the continued product $(x+1)(2x+1)(4x+1)....(2^nx+1)$.

15. In an AP of 2n terms the middle pair of terms are $\alpha+\beta$ and $\alpha-\beta$. Show that the sum of the cubes of the terms of the AP is $2n\alpha\{\alpha^2+(4n^2-1)\beta^2\}$

16. Find the positive real number x such that x, [x] and x – [x] are in geometric progression, where [ ] is the greatest integer function.

17. If S_n represents the sum to n terms of a G.P. whose first term & common ratio are a & r respectively, then prove that $S_1+S_2+S_3+.....$

$$+S_{2n-1}=\frac{an}{1-r}-\frac{ar(1-r^{2n})}{(1-r)^2(1+r)}.$$

18. Prove that in the sequence of numbers 49, 4489, ... in which every number is made by inserting 48 in the middle of the previous number as indicated, each number is the square of an integer.

19. If $t_1=1$, $t_r-t_{r-1}=2^{r-1}$, $r\geq 2$. Then find $\sum_{r=1}^{n} t_r$.

20. Prove that the sum of n terms of the series, $3.1^2+2^2+3.3^2+4^2+3.5^2+6^2+.....$ is $n(n+1)(4n+5)/6$, when n is odd. Hence find the sum of first n terms when n is even.

21. Show that for any positive integer n there exists positive integers i and j such that :

$$\frac{1}{n}=\frac{1}{i(i+1)}+\frac{1}{(i+1)(i+2)}+.....+\frac{1}{j(j+1)}$$

22. If two positive numbers are such that their arithmetic mean is 2k times their geometric mean then show that the numbers are in the ratio $(k\pm\sqrt{4k^2-1^2})^2$

23. Find the value of the sum $\sum_{r=1}^{n}\sum_{s=1}^{n}\delta_{rs}2^r3^s$ where δ_{rs} is zero if $r\neq s$ & δ_{rs} is one if $r=s$.

24. Show that m^r, where $r\geq 2$ can be represented as a sum of m consecutive odd integers.

25. A positive number is such that its fractional part, its integral part and the number itself constitute the first three terms of a geometrical progression . Show that the n^{th} term and the sum of the first n terms of the G.P. are, $2^{n-2}.\cos^{n-2}\frac{\pi}{5}$ and $2^n.\cos^n\frac{\pi}{5}-1$ respectively.

26. Find the sum of the cubes of the terms of an A.P., and show that it is exactly divisible by the sum of the terms.

27. Find the sum to n terms :
$1^2+3^2+6^2+10^2+15^2+....$

28. Find the sum of n terms of the series

$$\frac{1}{3!}+\frac{5}{4!}+\frac{11}{5!}+.....+\frac{n^2+n-1}{(n+2)!}$$

29. Prove that $1.2^2.3^3.....n^n<\left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}$ for $n\geq 2$, $n\in N$

30. Let a, b, c be real numbers with $0<a,b,c<1$ and $a+b+c=2$. Prove that

$$\frac{a}{1-a}.\frac{b}{1-b}.\frac{c}{1-c}\geq 8.$$

31. Prove that $n\{(n+1)^{1/n}-1\}<S_n<(n-1)\left\{1+\frac{1}{n-1}-\frac{1}{n^{1/(n-1)}}\right\}$ when

$S_n=1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}$ and $n>2$.

www.pdfworld.net

www.pdfworld.net

32. If $a^2 + b^2 = 1$, prove that

$$\left(a^2 - \frac{1}{a^2}\right)^2 + \left(b^2 - \frac{1}{b^2}\right)^2 \geq \frac{9}{2}.$$

33. Prove that for positive quantities a, b, c we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

34. If a, b, c and d are positive, show that

$$\frac{1}{b+c+d} + \frac{1}{c+d+a} + \frac{1}{d+a+b} + \frac{1}{a+b+c} \geq \frac{16}{a+b+c+d}$$

35. If a, b, c are the sides of a triangle, prove that $\left(\frac{a+b+c}{3}\right)^2 \geq \left(\frac{4\Delta}{\sqrt{3}}\right)$, where Δ is the area of the triangle.

Previous Years' Questions (JEE Advanced)

A. Fill in the blanks

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is ..... **[IIT - 1984]**

2. The sum of the first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ...$ is $n(n+1)^2/2$, when n is even. When n is odd, the sum is ..... **[IIT - 1988]**

3. Let the harmonic mean and geometric mean of two positive numbers be in the ratio 4 : 5. Then the two numbers are in the ratio..... **[IIT - 1992]**

4. For any odd integer $n \geq 1$, $n^3 - (n-1)^3 + ... + (-1)^{n-1} 1^3 =$..... **[IIT - 1996]**

5. Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers. The $\frac{y^3 + z^3}{xyz}$ = ... **[IIT - 1997]**

6. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then A = ... and B = ... **[IIT - 1997]**

B. Multiple Choice Questions with ONE correct answer

7. The third term of a geometric progression is 4. The product of the first five terms is **[IIT - 1982]**
(A) 4^3 (B) 4^5
(C) 4^4 (D) None of these

8. The rational number, which equals the number $2.\overline{357}$ with recurring decimal is **[IIT - 1983]**
(A) $\frac{2355}{1001}$ (B) $\frac{2379}{997}$
(C) $\frac{2355}{999}$ (D) None of these

9. If a, b, c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in **[IIT - 1985]**
(A) A.P. (B) G.P.
(C) H.P. (D) None of these

10. Sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + ...$ is equal to **[IIT - 1988]**
(A) $2^n - n - 1$ (B) $1 - 2^{-n}$
(C) $n + 2^{-n} - 1$ (D) $2^n + 1$

11. If $\ln(a+c)$, $\ln(a-c)$, $\ln(a-2b+c)$ are in A.P., then **[IIT - 1994]**
(A) a, b, c are in A.P. (B) a^2, b^2, c^2 are in A.P.
(C) a, b, c are in G.P. (D) a, b, c are in H.P.

12. Let n(> 1) be a positive integer. Then the largest integer m such that $(n^m + 1)$ divides $S = 1 + n + n^2 + ... + n^{127}$ is **[IIT - 1995]**
(A) 127 (B) 63
(C) 64 (D) 32

www.pdfworld.net

www.pdfworld.net

13. Let $a_1, a_2, ...a_{10}$ be in A.P. and $h_1, h_2, ...h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h = {}_{10} = 3$, then a_4h_7 is **[IIT - 1999]**

(A) 2 (B) 3
(C) 5 (D) 6

14. The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$ is **[IIT - 1999]**

(A) 2 (B) 4
(C) 6 (D) 8

15. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then **[IIT - 2000]**

(A) $a = \frac{4}{7}, r = \frac{3}{7}$ (B) $a = 2, r = \frac{3}{8}$
(C) $a = \frac{3}{2}, r = \frac{1}{2}$ (D) $a = 3, r = \frac{1}{4}$

16. Le α, β be the roots of $x^2 - x + p = 0$ and γ, δ the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., then the integral values of p and q respectively, are **[IIT - 2001]**

(A) –2, –32 (B) –2, 3
(C) –6, 3 (D) –6, – 32

17. Let the positive numbers a, b, c, d be in A.P. Then abc, abd, acd, bcd are **[IIT - 2001]**

(A) NOT in A.P./G.P./ H.P.
(B) in A.P.
(C) in G.P. (D) in H.P.

18. If the sum of the first 2n terms of the A.P. 2, 5, 8,.....is equal to the sum of the first n terms of the A.P. 57, 59, 61,....., then n equals **[IIT - 2001]**

(A) 10 (B) 12
(C) 11 (D) 13

19. The number of solutions of $\log_4(x - 1) = \log_2(x - 3)$ is **[IIT - 2001]**

(A) 3 (B) 1
(C) 2 (D) 0

20. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. if $a < b < c$ and $a + b + c = 3/2$, then the value of a is **[IIT - 2002]**

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{2\sqrt{3}}$
(C) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ (D) $\frac{1}{2} - \frac{1}{\sqrt{2}}$

21. An infinite G.P. has first term 'x' and sum '5', then x belongs to **[IIT - 2004]**

(A) $x < -10$ (B) $-10 < x < 0$
(C) $0 < x < 10$ (D) $x > 10$

22. In the quadratic equation $ax^2 + bx + c = 0$, $\Delta = b^2 - 4ac$ and $\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3$, are in G.P. where α, β are the root of $ax^2 + bx + c = 0$, then **[IIT - 2005]**

(A) $\Delta \neq 0$ (B) $b\Delta = 0$
(C) $c\Delta = 0$ (D) $\Delta = 0$

23. Let $a_1, a_2, a_3, ...$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is **[IIT - 2012]**

(A) 22 (B) 23
(C) 24 (D) 25

C. Multiple Choice Questions with ONE or MORE THAN ONE correct answers

24. If the first and the (2n – 1) terms of an A.P., a G.P. and an H.P. are equal and their n-th terms are a, b and c respectively, then **[IIT - 1988]**

(A) $a = b = c$ (B) $a \geq b \geq c$
(C) $a + c = b$ (D) $ac - b^2 = 0$.

25. For $0 < \phi < \pi/2$, if

$x = \sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$,

$z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$ then :

[IIT - 1993]

(A) $xyz = xz + y$ (B) $xyz = xy + z$
(C) $xyz = x + y + z$ (D) $xyz = yz + x$

26. Let T_r be the r^{th} term of an A.P., for r = 1, 2, 3,..... If for some positive integers m, n we have

$T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals **[IIT - 1998]**

www.pdfworld.net

www.pdfworld.net

(A) $\frac{1}{mn}$ (B) $\frac{1}{m}+\frac{1}{n}$

(C) 1 (D) 0

27. If x > 1, y > 1, z > 1 are in G.P., then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in **[IIT - 1999]**

(A) A.P. (B) H.P.

(C) G.P. (D) None of these

28. For a positive integer n, let a(n) **[IIT - 1999]**

$= 1 + \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\ldots \frac{1}{(2^n)-1}$. Then

(A) $a(100) \le 100$ (B) $a(100) > 100$

(C) $a(200) \le 100$ (D) $a(200) > 100$

29. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then **[IIT - 2008]**

(A) $\frac{1}{PS}+\frac{1}{ST}<\frac{2}{\sqrt{QS\times SR}}$

(B) $\frac{1}{PS}+\frac{1}{ST}>\frac{2}{\sqrt{QS\times SR}}$

(C) $\frac{1}{PS}+\frac{1}{ST}<\frac{4}{QR}$

(D) $\frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$

30. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is : **[IIT - 2009]**

(A) $\frac{n(4n^2-1)c^2}{6}$ (B) $\frac{n(4n^2+1)c^2}{3}$

(C) $\frac{n(4n^2-1)c^2}{3}$ (D) $\frac{n(4n^2+1)c^2}{6}$

D. Subjective Problems

31. The harmonic mean of two numbers is 4. Their arithmetic mean A and the geometric mean G satisfy the relation. $2A + G^2 = 27$. Find the two numbers. **[IIT - 1979]**

32. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120°, and the common difference is 5°. Find the number of sides of the polygon. **[IIT - 1980]**

33. If $a_1, a_2, \ldots\ldots\ldots\ldots, a_n$ are in arithmetic progression, where $a_i > 0$ for all i, Show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots\ldots$$

$$+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_{n-2}}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$ **IIT - 1982]**

34. Does there exist a geometric progression containing 27, 8 and 12 as three of its terms? If it exits, how many such progressions are possible? **[IIT - 1982]**

35. Find three numbers a, b, c between 2 and 18 such that (i) their sum is 25 (ii) the numbers 2, a, b are consecutive terms of an A.P. and (iii) the numbers b, c, 18 are consecutive terms of a G.P. **[IIT - 1983]**

36. If n is a natural number such that

$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \ldots\ldots p_k^{\alpha_k}$ and $p, p_2, \ldots\ldots\ldots\ldots$, p_k are distinct primes, then show that in ℓn $n \ge k \ln 2$ **[IIT - 1984]**

37. If $\log_3 2, \log_3(2^x - 5)$, and $\log_3\left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, determine the value of x. **[IIT - 1990]**

38. Let p be the first of the n arithmetic means between two numbers and q the first of n harmonic means between the same numbers. Show that q does not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$. **[IIT - 1991]**

39. If $S_1, S_2, S_3, \ldots\ldots\ldots, S_n$ are the sums of infinite geometric series whose first terms are 1, 2, 3,..... n and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ .....

www.pdfworld.net

www.pdfworld.net

$\frac{1}{n+1}$ respectively, then find the value of $S_1^2 + S_2^2 + S_3^2 + ... + S_{2n-1}^2$. **[IIT - 1991]**

40. The fourth power of the common difference of an arithmetic progression with integer entris is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer. **[IIT - 2000]**

41. Let a_1, a_2,..... be positive real numbers in geometric progression. For each n, let A_n, G_n, H_n be respectively, the arithmetic mean, geometric mean, and harmonic mean of a_1, a_2,....., a_n. Find an expression for the geometric mean of G_1, G_2, ..., G_n in terms of A_1, A_2,.....,A_n, H_1, H_2,.....,H_n. **[IIT - 2001]**

42. Let a, b be positive real numbers. If a, A_1, A_2,b are in arithmetic progression, a, G_1, G_2, be are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}.$$

[IIT - 2002]

43. If a, b, c are in A.P., a^2, b^2, c^2 are in H.P., then prove that either a = b = c or a, b, $-\frac{c}{2}$ form a G.P. **[IIT - 2003]**

44. If a, b, c are positive real numbers, then prove that :
$\{(1+a)(1+b)(1+c)\}^7 > 7^7 a^4b^4c^4$
[IIT - 2004]

45. A cricketer plays n(> 1) matches and scores $k(2^{n+1-k})$ runs in his kth match ($1 \le k \le n$). If the total rns scored by him in the n matches is $\frac{1}{4}(n+1)(2^{n+1} - n - 2)$, find the value of n. **[IIT - 2005]**

46. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ...(-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n \ \forall\ n \ge n_0$.
[IIT - 2006]

E. Comperehension # 1

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is (2r – 1). Let $T_r = V_{r+1} - V_r - 2$ and $Q_t = T_{r+1} - T_r$ for r = 1, 2,.....

47. The sum $V_1 + V_2 + ... + V_n$ is **[IIT - 2007]**

(A) $\frac{1}{12}n(n+1)(3n^2 - n + 1)$

(B) $\frac{1}{12}n(n+1)(3n^2 + n + 2)$

(C) $1/2n(2n^2 - n + 1)$

(D) $1/3(2n^3 - 2n + 3)$

48. T_r is always **[IIT - 2007]**

(A) an odd number

(B) an even number

(C) a prime number

(D) a composite number

49. Which one of the following is a correct statement? **[IIT - 2007]**

(A) Q_1, Q_2, Q_3,.... are in A.P., with common difference 5

(B) Q_1, Q_2, Q_3,.... are in A.P., with common difference 6

(C) Q_1, Q_2, Q_3,.... are in A.P., with common difference 11

(D) $Q_1 = Q_2 = Q_3$=....

Comprehension # 2

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For n ≥ 2, Let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

www.pdfworld.net

www.pdfworld.net

50. Which one of the following statement is correct? **[IIT - 2007]**
(A) $G_1 > G_2 > G_3 > ....$
(B) $G_1 < G_2 < G_3 < ....$
(C) $G_1 = G_2 = G_3 = ...$
(D) $G_1 < G_3 < G_5 < ....$ and $G_2 > G_4 > G_6 > ....$

51. Which one of the following statement is correct? **[IIT - 2007]**
(A) $A_1 > A_2 > A_3 > .....$
(B) $A_1 < A_2 < A_3 < .....$
(C) $A_1 > A_3 > A_5 > .....$and $A_2 < A_4 < A_6 < .....$
(D) $A_1 < A_3 < A_5 < .....$and $A_2 > A_4 > A_6 > .....$

52. Which one of the following statement is correct? **[IIT - 2007]**
(A) $H_1 > H_2 > H_3 > .....$
(B) $H_1 < H_2 < H_3 < .....$
(C) $H_1 > H_3 > H_5 > .....$and $H_2 < H_4 < H_6 < .....$
(D) $H_1 < H_3 < H_6 < .....$and $H_2 > H_4 > H_6 > .....$

F. Assertion (A) & Reason (R)

53. Suppose for distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$. **[IIT - 2008]**
Assertion (A) : The number b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.
Reason (R) : The numbers b_1, b_2, b_3, b_4 are in H.P.

G. Integer Answer Type

54. Let S_k, k = 1, 2,...., 100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is 1/k. Then the value of $\frac{100^2}{100\,!} + \sum_{k=1}^{100} \left|(k^2 - 3k + 1)S_k\right|$ is

55. Let $a_1, a_2, a_3, ...a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for k = 3,4, ...,11.
If $\frac{a_1^2 + a_2^2 + ...a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + ...a_{11}}{11}$ is equal to

56. The minimum value of the sum of real numbers a–5, a–4, 3a–3, 1, a8 and a10 with a > 0 is **[IIT - 2011]**

57. Let $a_1, a_2, a_3, ..., a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_1$, $1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let m = 5n. If $\frac{S_m}{S_m}$ does not depend on n, then a_2 is **[IIT - 2011]**

ANSWERS

Practice Problems—A

1. (i) 4, 7, 10,..... (ii) 2, 5, 10, 17, ...
(iii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...$ (iv) $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, ...$
2. 15
3. 1, 2, 3/2, 5/3, 8/5.
4. (i) $0, \frac{1}{9}, \frac{2}{27}, \frac{1}{27}, \frac{4}{243}$
(ii) –2, 4, –6, 8, –10
(iii) $1, \frac{7}{6}, 1, \frac{5}{6}, \frac{19}{27}$
5. (i) finite (ii) infinite
(iii) finite
6. (i) –2, 1, 4, 7 (ii) 1, 1, 2, 3
7. $a_3 = 3$, $a_6 = 7/6$, $a_8 = 9/8$.
8. (a) $\frac{1}{2^n}$ (b) $\frac{n+1}{n}$
(c) $\frac{1}{n(n+1)}$ (d) n^2
(e) $\frac{10n}{3^n}$ (f) $\frac{2n+1}{2n+3}$

Practice Problems—B

1. (i) $\frac{25}{12}$ (ii) 288
(iii) 3
2. (i) –1 + 1 + 3 + 5 + 7
(ii) 0, 1/2, 2/3, 3/4, 4/5
3. (i) $\sum_{j=3}^{7}(12 - 3j)$ (ii) $\sum_{j=0}^{9} 2(3)^{j+1}$
(iii) $\sum_{j=0}^{10}(j^2 - 4j + 3)$

www.pdfworld.net

www.pdfworld.net

4. (i) 220 (ii) 304

5. (i) $\sum_{i=1}^{8} \frac{(-1)^{n+1}}{2^{n-1}}$ (ii) $\sum_{i=1}^{4} \frac{x}{x+i}$

(iii) $\sum_{i=1}^{\infty} 8i$

6. 4

Practice Problems—C

1. No. **2.** Yes.
3. 2, 4, 6, 8 or 8, 6, 4, 2 **4.** 3, 5, 7, 9
6. $1325\sqrt{3}$ **7.** $\sqrt{3}$
8. 5 **9.** 1, –2, –5,.....
10. 612 **11.** n = 19
12. 1, 10, double answer because sum of 9 terms beginning with the second term is zero.
13. yes, $t_1 = 5$ and $d = 4$ **14.** 6r–1
15. $a_n = 8n - 4$ **16.** 630, 135, 765
17. 4 : 3

Practice Problems—D

5. $(a_1 = 2, d = -3)$, $(a_1 = -10, d = -3)$.

6. 1/3, 2/3, 1 **7.** $-\frac{5}{4}$

Practice Problems—E

2. 32, 16, 8, 0 and 2, 6, 18, 30.
5. 24, 27, 30, ..., 54 and 24, 24, 24,....
6. $\frac{2}{3}[1-(-1)^n 2^n]$ **8.** 9
9. $a2^{n-1}$; Yes **10.** No.
11. 3, –6, 12, –24. **12.** n = 6
13. 3, 7, 11 or 12, 7, 2

Practice Problems—F

1. $\frac{5}{12}$ **2.** 3
3. 16, 8, 4, 2 .... or 48, –24, 12, – 6....
4. $a_4 = \frac{3}{16}$; $r = \frac{1}{4}$ **5.** 1
6. $\frac{S^2}{2S-1}$ **7.** $\frac{41}{24}$
8. $-1 < x < 3$ **10.** $4+3\sqrt{2}$
11. (i) 15 (ii) {3, 4} or {48, 1/4}
(iii) $\{a^2, 1/a^2\}$ (iv) 4
12. 2, 6, 18

Practice Problems—G

1. $\frac{4}{9}$ **2.** $\frac{1}{11}$
3. 6 **8.** 1, 3 or 3, 1

Practice Problems—H

3. 1, 3, 5, ... **6.** $-7, \frac{7}{2}, \ldots\ldots, \frac{7}{32}$
8. 6 and 24 **9.** 4 : 9

Practice Problems—I

1. $\frac{1}{(1-x)^3}$
2. $2^{-n} + 2n - 1$
3. $4p^2a + \frac{2}{9}\left(1 - \frac{1}{2^{2p}}\right)$
4. (i) $\frac{2nx^n(x-1) - (x+1)(x^n-1)}{(x-1)^2}$

(ii) $\frac{2^n + (-1)^{n+1}(6n+1)}{9.2^{n-1}}$

Practice Problems— J

1. $\frac{1}{12}n(3n^3 + 10n^2 + 21n + 38)$,
2. 286
3. 1275
4. 4300.
7. n^4
8. $4^{n+1} - 4 - n(n+1)(n^2 - n - 1)$
9. $\frac{1}{12}n(n+1)^2(n+2)$.
11. (i) $\frac{1}{2}n(6n^2 - 3n - 1)$;

(ii) $\frac{1}{8}n(n-1)(9n^2 - 9n - 2)$, where $n \geq 2$.

12. 220

www.pdfworld.net

www.pdfworld.net

Practice Problems—K

1. $(4^{n-1}+3^{n-1})x^{n-1}$; $\frac{1-4^n x^n}{1-4x}+\frac{1-3^n x^n}{1-3x}$
2. $(1+3^{n-1}-2^{n-1})x^{n-1}$;

   $\frac{1-x^n}{1-x}+\frac{1-3^n x^n}{1-3x}-\frac{1-2^n x^n}{1-2x}$
3. (2^n-1) ; $2(2^n-1)-n$
4. $n^2(n+1)$; $\frac{1}{12}n(n+1)\,n+2)(3n+1)$
5. 3.2^n+n+2 ; $6(2^n-1)+\frac{n(n+5)}{2}$
6. $3^{n-1}+n$; $\frac{3^n+n^2+n-1}{2}$
7. $S=\frac{1+(1-x^{-1/3})(1-y^{-1/4})}{\{1-(1-x^{-1/3})(1-y^{-1/4})\}^2}$

Practice Problems—L

1. $\frac{1}{24}-\frac{1}{6(3n+1)(3n+4)}$; $\frac{1}{24}$
2. $\frac{3}{4}-\frac{2}{n+2}+\frac{1}{2(n+1)(n+2)}$; $\frac{3}{4}$
3. $\frac{1}{4}n(n+1)(n+8)(n+9)$
4. $\frac{1}{2}-\frac{1.3.5.....(2n+1)}{2.4.6.....(2n+2)}$
5. $34-(4n^2+12n+17)/2^{n-1}$.
6. (i) $1-\frac{1}{n+1}.\frac{1}{3^n}$ (ii) $(n+1)!-1$

   (iii) $\frac{29}{36}-\frac{1}{n+2}-\frac{1}{2(n+1)(n+2)}-\frac{1}{3(n+1)(n+2)(n+3)}$

   (iv) $\frac{(n-1)n(n+1)(n+2)}{6(2n+1)}$

   (v) $\frac{1}{2}\left[1-\frac{1}{13.5....(2n+1)}\right]$
7. $11\frac{9}{4}$
8. $1-\frac{1}{n+1}.\frac{1}{2^n}$

Practice Problems—M

1. $\frac{n(n+1)(12n^3+33n^2+37n+8)}{60}$
2. $\frac{n(n+1)(n+8)(n+9)}{4}$
3. $\frac{n(n+1)(n+2)(n+3)(2n+3)}{10}$
4. $\frac{11}{18}$
5. $\frac{7}{288}-\frac{3n+7}{6(2n+2)(2n+4)(2n+6)}$; $\frac{7}{288}$

Practice Problems—N

1. $2n^2-n+1$
2. $\frac{n}{3}(4n^2+6n-1)$
3. 3.2^n-n+1
4. $n^3-(n+1)^2$; $\frac{n}{12}(3n^3+2n^2-15n-26)$.
5. $\frac{1}{4}3^{n-1}-\frac{1}{2}n^2+2n-\frac{3}{4}$;

   $\frac{1}{8}(3^n-1)-\frac{n}{12}(2n^2-9n-2)$.

Objective Exercises

1. A	**2.** A	**3.** A
4. D	**5.** A	**6.** A
7. C	**8.** B	**9.** B
10. A	**11.** B	**12.** B
13. B	**14.** C	**15.** C
16. C	**17.** C	**18.** B
19. B	**20.** C	**21.** B
22. A	**23.** B	**24.** B
25. A	**26.** A	**27.** D
28. D	**29.** C	**30.** B
31. C	**32.** C	**33.** A
34. B	**35.** A	**36.** B
37. B	**38.** D	**39.** A

www.pdfworld.net

www.pdfworld.net

40. A **41.** D **42.** A
43. B **44.** B **45.** C
46. B **47.** B **48.** D
49. A **50.** C **51.** BD
52. AB **53.** AD **54.** ABCD
55. AC **56.** BD **57.** ABC
58. ABCD **59.** ABCD **60.** CD
61. BD **62.** ABCD **63.** AC
64. BD **65.** ABCD **66.** ABC
67. ACD **68.** ABC **69.** BD
70. ABCD **71.** C **72.** B
73. A **74.** C **75.** A
76. B **77.** A **78.** B
79. C **80.** B **81.** A
82. B **83.** C **84.** B
85. C **86.** A **87.** A
88. A **89.** D **90.** A
91. D **92.** A **93.** D
94. D **95.** C

96. (A) – S ; (B) – R; (C) – S; (D) – R
97. (A) – R ; (B) – S; (C) – P; (D) – Q
98. (A) – Q; (B) – S; (C) – P; (D) – R
99. (A) – R ; (B) – R ; (C) –P, R ; (D) –Q
100. (A) – P, Q ; (B) – P ; (C) – P, Q ; (D) – S

Review Exercises for JEE Advanced

1. (i) $\frac{3}{2},\frac{15}{8},\frac{35}{16},\frac{315}{128},\frac{693}{256}$
(ii) 9, 23, 51, 107, 219
(iii) 1, 0, –1, 0, 1,

3. 6.

5. 8, 16, 32

7. 6, –3, $\frac{3}{2}$,....

8. $\begin{cases}\text{AP } 2,6,10,....\\ \text{GP } 2,6,18,.....,\end{cases}$ $\begin{cases}\text{AP } 2,-2,-6,....\\ \text{GP } 2,-2,2.....,\end{cases}$

$\begin{cases}\text{AP } 1,1+2\sqrt{2},1+4\sqrt{2},....\\ \text{GP } 1,1+2\sqrt{2},9+4\sqrt{2},....\end{cases}$

$\begin{cases}\text{AP } 1,1-2\sqrt{2},1-4\sqrt{2},....\\ \text{GP } 1,1-2\sqrt{2},9-4\sqrt{2},....\end{cases}$

10. $\frac{x^2(x^n-1)}{(x-1)^2}-\frac{nx}{x-1}$

11. $\frac{n(n^2+1)}{2}$

16. $\frac{1}{4}n^2(n^2-1)$

17. $2^{n+2}-4-2n+\frac{n(n+1)(n+2)}{2}$

18. $\frac{9}{4}\left(1-\frac{1}{3^{n+1}}\right)-\frac{n+1}{2.3^n}$

20. $\frac{1}{3}-\frac{1}{2^{11}}+\frac{1}{3}.\frac{1}{2^{21}}$

21. $[n(10n^3+96n^2+343n+540)]$

22. (i) $\frac{1}{2}\left[\frac{2(1-x^n)}{(1-x)^3}-\frac{2nx^n}{(1-x)^2}-\frac{n(n+1)x^n}{1-x}\right]$

(ii) $-\frac{x(1-x)}{(1+x)^3}$

23. (i) $\frac{\pi^2}{8}$ (ii) $\frac{\pi^2}{12}$

25. 144 cm.

26. $\frac{4.7.....(3n+4)}{2.5.....(3n+2)}-2$

27. $12\frac{25}{54}$

28. $\frac{n(n+2)}{(n+1)^2}$

33. $a=\frac{5}{18}, b=\frac{5}{9}, c=\frac{10}{9}$

35. 8

Target Exercises for JEE Advanced

2. 2.4
3. 17
8. (8, 4, 2, 1, 1/2, 1/4)

12. $\frac{1}{(a-b)}\left\{\frac{a^2(1-a^n)}{1-a}\right\}-\frac{1}{(a-b)}\left\{\frac{b^2(1-b^n)}{1-b}\right\}$

www.pdfworld.net

www.pdfworld.net

13. $\frac{n-1}{n+2}\cdot\frac{4^{n+1}}{3}+\frac{2}{3}$

14. (i) $-2^{99}\cdot 100^2$ (ii) $2^{\frac{n^2+n+2}{2}}\cdot\left(1-\frac{1}{2^{n+1}}\right)$

16. $\frac{\sqrt{5}-1}{2}$

19. $2^{n+1}-n-2$

23. $(6/5)(6^n-1)$

26. $\frac{n}{2}\left(2a+(n-1)d\right)\left\{a^2+(n-1)ad+\frac{n(n-1)}{2}d^2\right\}$

27. $\frac{1}{20}n^5+\frac{1}{4}n^2(n+1)^2-\frac{1}{24}n(n+1)(2n+1)$
$+\frac{1}{8}n(n+1)-\frac{1}{20}n$

28. $\frac{1}{2}-\frac{1}{n+2}\cdot\frac{1}{n!}$

Previous Years' Questions (JEE Advanced)

1. 3050 **2.** $\frac{n^2(n+1)}{2}$ **3.** 4 : 1 or 1 : 4 **4.** $\frac{1}{4}(n+1)^2(2n-1)$

5. 2	**6.** –3, 77	**7.** B
8. C	**9.** A	**10.** C
11. D	**12. C**	**13.** D
14. B	**15.** D	**16.** A
17. D	**18.** C	**19.** B
20. D	**21.** B	**22.** C
23. D	**24.** BD	**25.** BC
26. C	**27.** B	**28.** AD
29. BD	**30. C**	

31. 3 and 6 or 6 and 3 **32.** 9

33. yes infinite **34.** 5, 8, 12 **37.** 3

39. $\frac{n(2n+1)(4n+1)-3}{3}$

41. $G=(A_1A_2.....A_nH_1H_2.....H_n)^{1/2n}$

45. 7	**46.** 6	**47.** B
48. D	**49.** B	**50.** C
51. A	**52.** B	**53.** C
54. 4	**55.** 0	**56.** 8

57. 3 or 9

pdfworld.net

www.pdfworld.net

www.pdfworld.net

4 Permutation and Combination

4.1 INTRODUCTION

The main subject of this chapter is counting. Given a set of objects the problem is to arrange a subset according to some specification or to select a subset as per some specification. We shall actually be interested in the number of such possible arrangements or selections. The idea of this chapter is to examine certain aspects of the question "how many"?

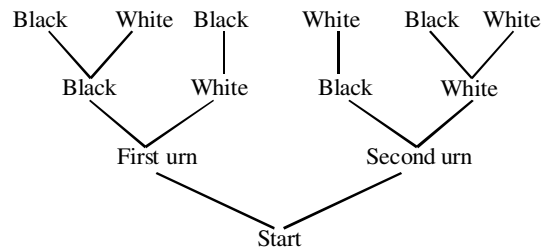
Combinatorial problems is the name attached to problems in which it is required to compute by how many methods (ways) it is possible to realize this or that requirement, to fulfil a certain condition, to make this or that choice.

First of all we shall understand the fundamental principles of counting which we intuitively use in our everyday life.

Tree Diagram

Consider two urns : first containing 2 black and 1 white balls and second containing 1 black and 2 white balls. Let us find the number of ways of drawing a ball and then draw a second ball without replacing the first ball.

This is a three stage process; first we select an urn out of two urns, then draw a ball and then draw a second ball. The tree of logical possibilities is shown in the figure. We note that six is the correct number of logical possibilities.



The reason for this is : If we choose the first urn (which contains two black balls and one white ball) and draw from it a black ball, then the second draw may be of either colour; however, if we draw a white ball first, then the second ball drawn is necessarily black. Similar remarks apply if the second urn is chosen.

www.pdfworld.net

www.pdfworld.net

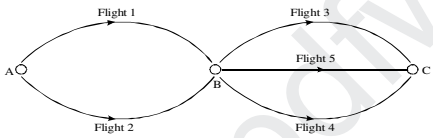
4.2 FUNDAMENTAL PRINCIPLES OF COUNTING

Multiplication Principle

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then total number of different ways of simultaneous occurrence of both the events in the definite order is $m \times n$.

Two events are **independent** events if neither influences the outcome of the other. Here the two events must be independent because the multiplication principle is not necessarily valid in situations where the second event is dependent on, or restricted by, the first.

Suppose there are two flights (Flight Nos. 1 and 2) in the morning from place A to place B and three flights (Flight Nos. 3, 4 & 5) in the evening from place B to place C. In how many ways can one fly from place A to place C via place B?



We reason as follows, there are two ways of flying from A to B. For each such choice of flight, there are three ways of flying from B to C. Flight 1 can be followed up by anyone of the three flights 3, 4 or 5 and similarly flight 2 can be followed up by any one of 3, 4 or 5. Thus, there are $2 \times 3 = 6$ ways of flying from A to C.

The most important point to note here is the fact that the 2nd part of the journey was totally independent of the 1st part of the journey. In other words, the choice of the flight from B to C had nothing to do with the choice of the flight from A to B.

➤ **Example 1.** There are 25 mathematics books and 24 physics books on a library shelf. In how many ways can we choose one mathematics and one physics book?

➤ ***Solution*** Choosing a mathematics book is Event E Choosing a physics book is Event F.

Event E can happen in any one of 25 ways. Event F can happen in any one of 24 ways. What physics book we choose is independent of what maths book we choose and viceversa. So Events E and F are independent. Thus the multiplication rule applies and the two books can together be chosen in $25 \times 24 = 600$ ways.

➤ **Example 2.** A dice is a six-faced cube, with the faces reading 1, 2, 3, 4, 5 and 6. When two dices are thrown we add the digits they show on top and take that sum as the result of the throw. In how many different ways the first throw of the 2 dices shows a total of 5, and second throw of the 2 dices shows a total of 4.

➤ ***Solution*** Event E (the first throw resulting in 5) can happen in one of four ways : 3 + 2; 4 + 1; 2 + 3; 1 + 4.

Event F (the second throw resulting in 4) can happen in one of three ways : 2 + 3; 1 + 3; 3 + 1.

The two events can together happen in $4 \times 3 = 12$ ways.

➤ **Example 3.** A firm is about to take on a new sales manager and a new receptionist. There are 6 applicants for the director's job and 10 for that of the receptionist. How many ways can the pair be chosen?

➤ ***Solution*** Any of the six applicants for the sales manager's job could be linked with any of the ten possible receptionists. This gives $6 \times 10 = 60$ possible combinations. The two new employees can be chosen in 60 ways.

If n independent events occur, with

m_1 ways for event 1 to occur.

m_2 ways for event 2 to occur,

.

.

.

and m_n ways for event n to occur,

then there are $m_1 \cdot m_2 \cdot \cdots \cdot m_n$ different ways for all n events to occur.

Suppose a hospital uses a light signal system for letting its doctors know when they are needed

www.pdfworld.net

www.pdfworld.net

somewhere. Each set of lights has four colours which may be lit up four, three, two or one at a time. On seeing his or her particular combination of colours, the doctor is expected to report to a control office. How many doctors could each have his or her own combination of lights?

Each of the four lights could be dealt with in two ways – each could be on or off. So the total number of ways the signals may be set is $2 \times 2 \times 2 \times 2 = 16$. But this includes the case where all the lights are off, which would be no use as a signal, so we are left with 15 distinct combinations. Thus 15 doctors could each be given a personal colour combination.

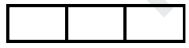
Unfortunately, the hospital has more than 15 doctors. So it decides that each of the fifteen patterns of lights can appear in two different ways – steady or flickering. How many doctors can the system cope with now ?

Each of the fifteen patterns can be dealt with in either of two ways. This allows the system to signal $15 \times 2 = 30$ different doctors.

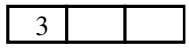
Filling Places

In how many different ways can three people queue up at a bus stop?

One method for tackling the problem above (and other like it) is to say that we have three places in the queue to fill – one for each person

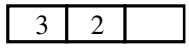


In the first place we can put person A, or B, or C. So that place can be dealt with in three ways.

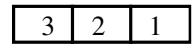


Once we have filled this first place (with either A, B or C) in how many different ways can we fill the second place?

Having put one of the three people in the first place, we have a choice of two only for the second place.

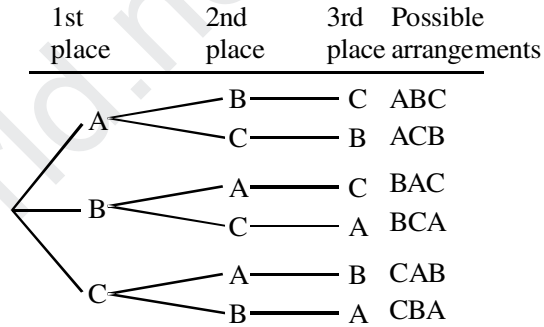


So we've chosen one of the three passengers for our first place and one of the remaining two for our second place. How many ways can we fill the third place? There is only one way of filling the third place (with the one passenger remaining who has not yet been given a place).



There are 3 ways of dealing with the first place; then, when that has been done, 2 ways of dealing with the second place, leaving 1 way of dealing with the 3rd place. So there are $3 \times 2 \times 1 = 6$ ways of dealing with the series of choices. 6 different queues could be formed.

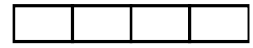
We can illustrate this with a tree diagram:



Note that the same three passengers appear in each possible arrangement. Only the order is different.

➤ **Example 4.** The Khanna family (Dad, Mum, Sia, and Jeet) move to a new district, and they decide that each person will register with a different dentist. There are 6 dentists in the district, and we'll need to know how many ways this family can register.

➤ ***Solution*** There are four places to fill. We need a different dentist for each member of the Khanna family, and we have 6 dentists to choose from



The first person to choose (e.g. Mum) can have any of the 6 dentists. This leaves 5 for the second person (e.g. Dad) to choose from. The third person (say Sia) can then choose from 4, leaving any of the remaining 3 for Jeet.

www.pdfworld.net

www.pdfworld.net

Mum	Dad	Sia	Jeet
6	5	4	3

Thus there are $6 \times 5 \times 4 \times 3 = 360$ ways of registering 4 dentists from a group of 6 dentists.

STUDY TIP As you can confirm for yourself, it wouldn't matter which member of the family was given first, second, third and fourth choice. The same number of orders would arise.

➤ **Example 5.** Five children are running a race. How many different ways could they finish the race?

➤ ***Solution*** If we think in terms of five places to be filled, the first place can be filled by any of the five children. Once that has been done, the second place can be filled by any of the remaining four children, and so on until there is only one child left to fill the fifth place :

5	4	3	2	1

And $5 \times 4 \times 3 \times 2 \times 1 = 120$. So the five children could finish the race in any of 120 different orders. But suppose we have prizes for the first three children only. How many ways can the first three places be filled?

The first three places can be filled in 60 different ways. With three places, the first can be filled by any of the five children; the second by any of the remaining four and the third by any of the other three children :

5	4	3

Hence the prizes can be given in $5 \times 4 \times 3 = 60$ ways.

➤ **Example 6.** How many integers are there less than 1000, ending with 3, 6 or 9 ?

➤ ***Solution*** We shall consider three blank spaces

(ordered from left to right) to represent numbers less than 1000.

Two-digit and one digit numbers will have zeros in the first place and first two places respectively. The number zero is 000. So the 1000 numbers less than 1000 (including zero) are obtained by filling up the three blank spaces in all possible ways (precisely, $10 \times 10 \times 10 = 10^3$ ways) by the digits 0, 1, 2, 3, ..., 9. Since we want only those numbers that are ending with 3, 6, or 9 we reason as follows. The first place has all the 10 choices; the second place also has 10 choices; whereas the third place has only 3 choices, viz., 3, 6 and 9. Thus the three blank spaces can be filled with $10 \times 10 \times 3 = 300$. So the required no of integers is 300.

➤ **Example 7.** A restaurant offers a choice of 3 salads, 5 main dishes, and 2 desserts. Use the fundamental principle of counting to find the number of different 3-course meals that can be selected.

➤ ***Solution*** Three events are involved: selecting a salad, selecting a main dish, and selecting a dessert. The first event can occur in 3 ways, the second event can occur is 5 ways, and the third event can occur in 2 ways; thus, there are $3 \cdot 5 \cdot 2 = 30$ possible meals.

➤ **Example 8.** A teacher has 5 different books that he wishes to arrange in a row. How many different arrangements are possible?

➤ ***Solution*** Five events are involved : selecting a book for the first spot, selecting a book for the second spot, and so on. For the first spot, the teacher has 5 choices. After a choice has been made, the teacher has 4 choices for the second spot. Continuing in this manner, there are 3 choices for the third spot, 2 for the fourth spot, and 1 for the fifth spot. By the multiplication rule, there are $5 \times 4 \times 3 \times 2 \times 1$ or 120 different arrangements.

➤ **Example 9.** Suppose the teacher in the above example wishes to place only 3 of the 5 books in a row. How many arrangements of 3 books are possible?

➤ ***Solution*** The teacher still has 5 ways to fill the first spot, 4 ways to fill the second spot, and 3 ways to fill the third. Since only 3 books will be used, there are only 3 spots to be filled instead of 5, with $5 \times 4 \times 3 = 60$ arrangements.

➤ **Example 10.** Four persons enter a railway carriage in which there are six seats. In how many ways can they take their places?

➤ ***Solution*** The first person may seat himself in 6 ways; and then the second person in 5; the third in 4, and the fourth in 3; and since each of these ways may

www.pdfworld.net

www.pdfworld.net

be associated with each of the others, the required answer is $6 \times 5 \times 4 \times 3 = 360$.

➤ **Example 11.** Four speakers expressed their desire to take the floor at a meeting. In how many ways is it possible to arrange them in the list of speakers?

➤ ***Solution*** The first speaker can be singled out in four ways, the second, obviously, in three ways. Now only two men are left and consequently, there are two ways to fill in the place in the list of speakers.

And finally, there is no choice for the fourth speaker; he will be the last to take the floor. Since every method of choosing the first speaker can be combined with every method of choosing the second speaker and with either of the two possible ways of choosing the third speaker, the number of ways of forming the list of speakers is equal to $4 \times 3 \times 2 \times 1 = 24$

➤ **Example 12.** There are four possible routes from police HQ to the bus station. Another four from the bus station to the bank, and four more from the bank to police HQ. How many ways can a police car patrol a circuit from police HQ to bus station to bank and back to HQ ?

➤ ***Solution*** The patrol car can deal with the trip to the bus station in any of 4 ways; once it has done that it can deal with the trip from bus station to bank in any of another 4 ways; and then the trip back to HQ can also be done in any of 4 ways. Hence, there are $4 \times 4 \times 4 = 64$ combined routes.

➤ **Example 13.** Four prisoners are being moved by train from one jail to another. If the train has eight coaches and each prisoner must travel in a different coach, in how many possible ways can the men travel on the train ? How many ways would there be if they did not have to travel in different coaches ?

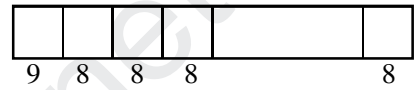
➤ ***Solution*** There are $8 \times 7 \times 6 \times 5 = 1680$ ways for the four prisoners each to travel on a different coach of the 8 coach train. But if they don't have to travel in different coaches there are $8 \times 8 \times 8 \times 8 = 4096$ ways for them to travel.

➤ **Example 14.** Certain registration numbers are formed from three different letters, chosen from the first ten letters of the alphabet, followed by a three digit number which must not begin with zero. Calculate how many registration numbers can be formed.

➤ ***Solution*** There are ten choices for the first letter, 9 for the second, and eight for the third. Since the number cannot begin with 0, we have nine choices for the first digit, but ten for the second and for the third. So we can form $10 \times 9 \times 8 \times 9 \times 10 \times 10 = 648000$ registration numbers.

➤ **Example 15.** How many numbers of n digits can be made with the non-zero digits in which no two consecutive digits are the same ?

➤ ***Solution*** There are nine non-zero digits, namely 1, 2, 3,..... and 9.



In order to make an n-digit number we have to fill n places by using the nine digits. As no two consecutive digits are to be the same, a digit used in a place cannot be used in the next place but it can be used again in the place coming after the next place.
So the first place can be filled in 9 ways;
the second place can be filled in 8 ways
(rejecting the digit used in the first place)
the third place can be filled in 7 + 1 , i.e., 8 ways
(rejecting the digit used in the second place but including the digit used in the first place)

and so on.

$\therefore$ The required number of numbers

$= 9 \times (8 \times 8 \times 8 \times$ .... to n-1 factors$) = 9 \times 8^{n-1}$.

Addition Principle

If an event can occur in 'm' different ways, and another event can occur in 'n' different ways, then exactly one of the events can happen in m + n ways.

➤ **Example 16.** There are 30 persons in a group of which 20 are boys and 10 are girls. In how many ways can a group leader be selected if the group leader can either be a boy or a girl.

➤ ***Solution*** Using the sum rule, the number of ways in which either a boy or a girl can be selected for the group leader = 20 + 10 = 30.

www.pdfworld.net

www.pdfworld.net

Some problems can be solved most readily by considering separate cases.

➤ **Example 17.** How many of the first 1000 positive integers have distinct digits ?

➤ ***Solution*** Setting aside the integer 1000, whose digits are not distinct, the others can be separated into three types :

Integers with one digit : 1, 2, 3, ... , 9;

Integers with two digits : 10, 11, 12, ..., 99;

Integers with three digits : 100, 101, 102, ... , 999.

The number of three digits integers with distinct digits is 648, as shown in the previous example. A similar argument shows that there are 81 two digit integers and of course 9 one digit integers that meet the specification of distinct digits. Hence the answer is

$$648 + 81 + 9 = 738$$

The idea used here is the addition principle. If the things to be counted are separated into cases, the total number is the sum of the numbers in the various cases.

➤ **Example 18.** How many numbers between 10 and 10,000 can be formed by using the digits 1, 2, 3, 4, 5 if

(i) no digit is repeated in any number.

(ii) digits can be repeated.

➤ ***Solution*** (i) Number of two digit numbers $= 5 \times 4 = 20$

Number of three digit numbers $= 5 \times 4 \times 3 = 60$

Number of four digit numbers $= 5 \times 4 \times 3 \times 2 = 120$

Total $= 20 + 60 + 120 = 200$

(ii) Number of two digit numbers $= 5 \times 5 = 25$

Number of three digit numbers $= 5 \times 5 \times 5 = 125$

Number of four digit numbers $= 5 \times 5 \times 5 \times 5 = 625$

Total $= 25 + 125 + 625 = 775$

Complementary Principle

If a work is to be done under some restriction then the number of ways to do the work

= (the number of ways to do the work without restriction) – (the number of ways to do the work under opposite restriction).

Practice Problems [A]

1. A boy has five pairs of trousers and six shirts. His girl friend has four skirts and seven tops. Who has the most choice of what to wear?
2. A room has six doors. In how many ways is it possible to enter by one door and leave by another?
3. How many different types of homes are available if a builder offers a choice of 5 basic plans, 3 roof styles, and 2 exterior finishes ?
4. A tyre store carries 8 different sizes of tyres, each in both tube and tubeless variety, each with either nylon or rayon cord, and each with white sidewalls or plain black. How many different kinds of tyres does the store have ?
5. A mail order company offers 23 styles of ladies' slippers. If each style were available in 12 lengths, 3 widths and 6 colors, how many different kind of ladies' slippers would the warehouse have to keep in stock?
6. From the digits 0, 1, 2, 3, 4, 5, 6 how many four digit numbers with distinct digits can be constructed? How many of these are even numbers ?
7. How many of the integers (whole numbers) between 10,000 and 100,000 have no digits other than 6, 7, or 8 ? How many have no digits other than 6, 7, 8 or 0 ?
8. How many solutions in positive integers x and y are there of the equation $x + y = 100$? By a solution, we mean an ordered pair (x, y) that satisfies the equation. How many solutions are there in non-negative integers ?
9. Find the number of terms in the product $(a + b + c)(d + e + f)(p + q + r + s)(x + y + u + v + w)$
10. A menu lists a choice of soup or orange juice for an appetizer, a choice of egg, chicken or

www.pdfworld.net

www.pdfworld.net

fish for the dinner, and a choice of ice-cream or cake for dessert. A complete dinner consists of one choice in each case. Draw the tree for the possible complete dinners and answer the following:

(i) How many different complete dinners are possible?

(ii) How many complete dinners are there which have chicken for the dinner?

(iii) How many complete dinners are there available for a man who will eat cake only if he had egg?

4.3 FACTORIAL

An efficient way of writing a product of several consecutive integers is the factorial notation. The number n! (read as "n-factorial") is defined as follows :

For any positive integer n

$n! = n(n-1)(n-2)....(3)(2)(1)$,

and $0! = 1$

For instance, $4! = 4.3.2.1 = 24$.

(i) Factorials of negative integers are not defined.

(ii) $n! = n\,.\,(n-1)!$

(iii) $(2n)! = 2^n\,.\,n!\,[1\,.\,3\,.\,5\,.\,7\,...\,(2n-1)]$

(iv) Whereas the reader is asked to evaluate or compute such numbers as 5! and 8!, he would not be expected to compute (say) 20!. If such a number is the answer to a question, it should be left in precisely that form.

➤ **Example 1.** Simplify the expression $\frac{n!}{(n-3)!} + \frac{2n!}{4(2n-3)!}$

➤ ***Solution*** $\frac{n!}{(n-3)!} + \frac{2n!}{4(2n-3)!} = n(n-1)(n-2) + 2n(2n-1)(2n-2)/4$

$= n(n-1)(n-2) + n(n-1)(2n-1)$

$= n(n-1)(n-2+2n-1) = 3n(n-1)^2$.

➤ **Example 2.** Simplify the expression

$$\frac{3.6.9.12....(3n)}{3^n}$$

➤ ***Solution*** $\frac{3.6.9.12....(3n)}{3^n} = \frac{3^n.1.2.3.4....n}{3^n} = n!$

➤ **Example 3.** Find the sum of n terms of the series whose n^{th} term is $n!\,.\,n$.

➤ ***Solution*** The required sum

$= (1)\,! + 2(2)\,! + 3\,(3)\,! + ... + n\,n\,!$

$= (2-1)\,1\,! + (3-1)\,2\,! + (4-1)\,3\,! + ... + \{(n+1)-1\}\,n!$

$= (2!-1\,!) + (3!-2!) + (4!-3!) + ... + [(n+1)\,!-n!]$

$= (n+1)! - 1 \qquad (\,n\,(n-1)\,! = n\,!\,)$

➤ **Example 4.** If $1.0! + 3.1! + 7.2! + 13.3! + 21.4! + ...$ upto $(n+1)$ terms $= 4000.\,(4000!)$, then find the value of n.

➤ ***Solution*** Clearly $t_r = (r^2 + r + 1).\,r!\,;\ r = 0, 1, 2, ..., n$

i.e. $t_r = [(r+1)^2 - r]\,r!$

$= (r+1)\,(r+1)\,r! - r.\,r! = (r+1).\,(r+1)! - (r+1-1).\,r!$

$= (r+2-1)\,(r+1)! - (r+1)!\,r! + r! = (r+2)\,(r+1)!$

$- (r+1)! - (r+1)! + r!$

$= (r+2)! - 2\,(r+1)! + r!$

$t_0 = 2! - 2.1! + 0!$

$t_1 = 3! - 2.2! + 1!$

$t_2 = 4! - 2.3! + 2!$

$t_3 = 5! - 2.4! + 3!$

.....

.....

$t_{n-1} = (n-1)! - 2n! + (n-1)!$

$t_n = (n+2)! - 2(n+1)! + n!$

$\Rightarrow \sum_{r=0}^{n} t_r = (n+2)! + (n+1)! - 2(n+1)! - 2.1! + 0! + 1!$

$= (n+2)! = (n+1)!\,(n+2-1) = (n+1)\,(n+1)!$

www.pdfworld.net

www.pdfworld.net

Given $(n + 1)(n + 1)! = 4000(4000)! \Rightarrow n + 1 = 4000 \Rightarrow n = 3999$.

4.4 PERMUTATION

Each of the arrangements in a definite order which can be made by taking some or all of a number of things is called a permutation.

Consider the arrangements of the objects A, B, C and D. These arrangements can be counted by considering 4 places and counting the number of ways to fill these places by the objects A, B, C & D.

Place Number :	1	2	3	4
Number of choices :	4	3	2	1

The first place can be filled by any one of the 4 objects A, B, C or D

The object kept at the first place can't be kept at the second place. The second place can be filled in 3 ways by one of the remaining 3 objects. The third place can be filled in 2 ways by one of the remaining 2 objects. The fourth place can be filled in 1 way.

$\therefore$ Total number of permutations = total number of ways of filling the four places $= 4 \times 3 \times 2 \times 1 = 4!$ (By the multiplication principle)

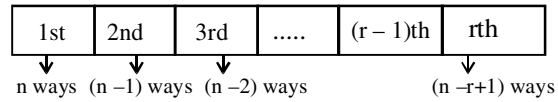
Similarly, to count the number of permutations of n different objects $a_1, a_2, \ldots a_n$, we consider n places and fill them one by one with the remaining objects.

Place Number:	1	2	3	..... n
Number of choices :	n	(n – 1)	(n – 2)	 1

Total number of permutations = total number of ways of filling the n places

$= n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!$

We often come across a situation in which all the given objects are not used in the arrangements. Suppose we are given n different objects $a_1, a_2, \ldots a_n$ and we have to count the number of arrangements taken r at a time. It is presumed that r does not exceed n, that is $r \le n$. Note that the n objects must be different. To count these arrangements we consider r places.



The first place can be filled up by any one of the n objects $a_1, a_2, a_3, \ldots, a_n$ in n ways.

The number of objects left after filling up the first place = n – 1

The second place can be filled up by any one of the remaining (n – 1) objects in (n – 1) ways.

The number of objects left after filling up the first and second places = n – 2

The third place can be filled up by any one of the remaining (n – 2) objects in (n – 2) ways.

.....

The number of ways of filling up rth place = n – (r – 1) = n – r + 1

By the multiplication rule of counting, first, second, third...., rth places can be together filled up in

$$n(n-1)(n-2)\ldots(n-r+1) \text{ ways}$$

$$= \frac{[n(n-1)(n-2)\ldots.(n-r+1)](n-r)\ldots.3.2.1}{(n-r)(n-r-1)\ldots.3.2.1}$$

$$= \frac{n!}{(n-r)!}$$

The number of permutations of n different objects taken r at a time is denoted by ^nP_r

where $^nP_r = n(n-1)(n-2)\ldots.(n-r+1)$, or $^nP_r = \dfrac{n!}{(n-r)!}$

Note that ^nP_r is the product of the r integers n – 1, n – 2, ..., n – r + 1.

In particular, $^nP_n = n(n-1)(n-2)\ldots.1$, or $^nP_n = n!$

For example, $^7P_7 = 7.6.5.4.3.2.1 = 7!$

$$^{10}P_4 = \frac{10!}{6!} = \frac{10.9.8.7.6.5.4.3.2.1}{6.5.4.3.2.1} = 10.9.8.7$$

$^{12}P_5 = 12.11.10.9.8.$

➤ **Example 1.** Suppose 8 people enter an event in a swim meet. In how many ways could the gold, silver, and bronze prizes be awarded?

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** The required number of ways is arrangement of 3 people out of 8 i.e.

$${}^8P_3 = \frac{8!}{5!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 8\cdot7\cdot6 = 336.$$

STUDY TIP The following two steps involved in the solution of a permutation problem :
Step 1 : Recognizing the objects and the places involved in the problem.
Step 2 : Checking whether the repetition of the objects is allowed or not.

➤ **Example 2.** A group of students studies seven disciplines. In what ways is it possible to arrange the timetable for Monday if on this day of the week four lessons in different subjects are to take place ?

➤ ***Solution*** The number of ways is equal to the number of permutations of 7 elements taken 4 at a time, i.e. to ${}^7P_4 = 7.6.5.4 = 840$.

➤ **Example 3.** How many integers between 100 and 999 inclusive consist of distinct odd digits ?

➤ ***Solution*** The odd digits are 1, 3, 5, 7, 9; the even digits are 0, 2, 4, 6, 8. An integer such as 723 is not to be counted because it contains the even digit 2; and an integer such as 373 is not to be counted because it does not have distinct digits. The question amounts to asking for the number of permutations of the five distinct digits 1, 3, 5, 7, 9, taken three at a time. The answer is

$${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5.4.3 = 60$$

➤ **Example 4.** How many six digit numbers multiple of 5 can be formed from the digits 1, 2, 3, 4, 5, 6, provided the numbers thus formed have no repeated digits.

➤ ***Solution*** In each of such numbers the digit 5 must occupy the last place. The remaining five places may be occupied by the remaining five digits arranged in any order. Consequently, the desired number of six digit numbers multiple of 5 is equal to the number of permutations of five elements taken all at a time, i.e. to 5! = 5. 4. 3. 2. 1 = 120

➤ **Example 5.** How many numbers of three digits can be formed using the digits 1, 2, 3, 4, 5, without repetition of digits. How many of these are even?

➤ ***Solution*** Three places are to be filled with 5 different objects.
$\therefore$ Number of ways $= {}^5P_3 = 5\times4\times3 = 60$
For the 2nd part, unit digit can be filled in two ways & the remaining two digits can be filled in ${}^4P_2$ ways.
$\therefore$ Number of even numbers $= 2\times{}^4P_2 = 24$.

➤ **Example 6.** How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if repetition of the digits is not allowed?

➤ ***Solution*** Every number between 100 and 1000 is a 3 digit number. We, first, have to count the permutations of 6 digits taken 3 at a time. This number would be ${}^6P_3$. But these permutations will include those also where 0 is at the 100's place. For example, 029, 047, ..., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers, we fix 0 at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is ${}^5P_2$. So

$$\text{The required number} = {}^6P_3 - {}^5P_2 = \frac{6!}{3!} - \frac{5!}{3!}$$

$$= 4\times5\times6 - 4\times5 = 100$$

➤ **Example 7.** If ${}^{k+5}P_{k+1} = \frac{11(k-1)}{2}\times{}^{k+3}P_k$, then find k .

➤ ***Solution*** ${}^{k+5}P_{k+1} = \frac{11(k-1)}{2}\times{}^{k+3}P_k$

$$\Rightarrow \frac{(k+5)!}{((k+5)-(k+1))!} = \frac{11(k-1)}{2}\times\frac{(k+3)}{((k+3)-k)!}$$

www.pdfworld.net

www.pdfworld.net

$$\Rightarrow \frac{(k+5)(k+4)(k+3)!}{4.3!} = \frac{11(k-1)}{2} \times \frac{(k+3)!}{3!}$$

$$\Rightarrow k^2 - 13k + 42 = 0 \quad \Rightarrow \quad k = 6, 7$$

CAUTION The above formula for ${}^nP_r$ cannot be used to solve all problems about permutations, because not all such problems admit as solutions all ordered arrangements of n distinct objects, r at a time. A problem can sometimes be solved by direct use of the multiplication principle, as the following examples illustrate.

➤ **Example 8.** How many 4-digit numbers can be formed from 3, 4, 5, 6, 7 without repetition, if the numbers must all be odd.

➤ ***Solution*** Clearly, there are four places to fill. But the last place must be filled with one three digits. The last place must be filled with either 3 or 5 or 7 otherwise, the resulting number will not be odd.
So, we have three ways of filling the last place.

1st	2nd	3rd	4th

Then any of the even digits, or any of the odd digits not used in the last place, can be put in the first place – 4 possibilities in all. Once this has been done, there are 3 digits left for the second place; and this leaves just 2 ways of filling the third place.

4	3	2	3

So, we can form $4 \times 3 \times 2 \times 3 = 72$ numbers.

➤ **Example 9.** Five friends are setting out on a journey in a 5-seater car. In how many ways can they seat themselves for the journey if only two of them can drive?

➤ ***Solution*** With five places (seats) to fill one of them (the driver's) can be filled in only two ways – so we deal with it first :

driver →

2	4	3	2	1

The remaining places can be filled in 4 ways, 3 ways, 2 ways and 1 way respectively. This gives $2 \times 4 \times 3 \times 2 \times 1 = 48$ different possible seating arrangements.

➤ **Example 10.** A pop-group's spot on a variety show is to consist of three vocal numbers and two instrumentals. In how many ways can the 'spot' be arranged so that it begins and ends with a vocal and neither instrumental follows directly after the other ?

➤ ***Solution*** The order must be as follows :
Vocal Instrumental Vocal Instrumental Vocal
since the group has 3 vocals to choose from, the first place can be filled in any of 3 ways, the third place can be filled in any of 2 ways and 1 vocal is left for the fifth :

3		2		1

With 2 instrumental numbers to choose from, the group can deal with the second place in either of 2 ways, leaving just 1 way of filling the fourth place :

3	2	2	1	1

And $3 \times 2 \times 2 \times 1 \times 1 = 12$ possible arrangements

➤ **Example 11.** Find the number of numbers between 300 and 3000 that can be formed with the digits 0, 1, 2, 3, 4 and 5, no digit being repeated in any number.

➤ ***Solution*** Any number between 300 and 3000 must be of 3 or 4 digits.

Case I. When the number is of 3 digits :

3 or 4 or 5		
↓ 3 ways		↓ ${}^5P_2$ ways

The hundreds' place can be filled up by any one of the three digits 3, 4 and 5 in 3 ways.
The remaining two places can be filled up by remaining five digits in ${}^5P_2$ ways.
∴ The number of numbers formed in this case

$$= 3 \times {}^5P_2 = 3 \times \frac{5!}{2!} = 60$$

www.pdfworld.net

www.pdfworld.net

Case II. When the number is of 4 digits :

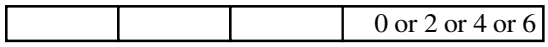
Number of numbers formed $= 2 \times {}^5P_3 = 2 \times \frac{5!}{1!} = 120$

$\therefore$ The required number = 60 + 120 = 180.

➤ **Example 12.** How many even numbers of four digits can be formed with the digits 0, 1, 2, 3, 4, 5 and 6 without repetition?

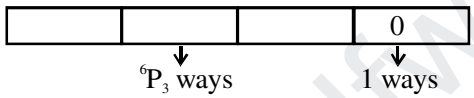
➤ ***Solution*** Each even number must have 0, 2, 4 or 6 in its units place.

Here total number of digits = 7.



When 0 occurs at units place there is no restriction on other places and when 2 or 4 or 6 occurs at units place there is restriction on thousands' place as 0 can not be put at thousands' place.

Case I. When 0 occurs at units place :

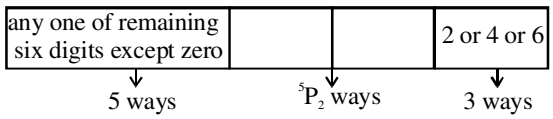


The units' place can be filled up by 0 in 1 way and remaining three places can be filled up by remaining 6 digits in $^6P_3$ ways.

$\therefore$ The number of numbers formed in this case

$= 1 \times {}^6P_3 = \frac{6!}{3!} = 120.$

Case II. When 0 does not occur at units place :



The units' place can be filled up by any one of the three digits 2, 4 and 6 in 3 ways.

The thousands' place can be filled up by any one of the remaining six digits excepts zero in 5 ways.

The remaining two places can be filled up by the remaining five digits in $^5P_2$ ways.

$\therefore$ The number of numbers formed in this case

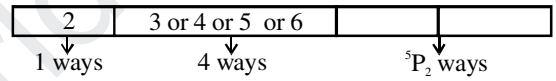
$= 5 \times 3 \times {}^5P_2 = 15 \times \frac{5!}{3!} = 300$

$\therefore$ The required number = 120 + 300 = 420

➤ **Example 13.** How many numbers of four digits greater than 2300 can be formed with the digits 0, 1, 2, 3, 4, 5 and 6; no digit being repeated in any number ?

➤ ***Solution*** Since the number must be of four digits and greater than 2300, when anyone of 3, 4, 5, 6 occurs at thousands' place the number will be definitely greater than 2300 but when 2 occurs at thousands' place there will be further restriction on hundreds' place to make the number greater than 2300.

Case I. When 2 occurs at thousands place :

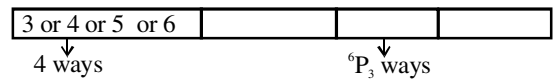


The thousands' place can be filled up by 2 in 1 way and hundreds' place can be filled up by any one of the four digits 3, 4, 5 and 6 in 4 ways.

The remaining two places can be filled up by the remaining five digits in $^5P_2$ ways.

$\therefore$ The number of numbers formed in this case $= 1 \times 4 \times {}^5P_2 = 4 \times \frac{5!}{3!} = 80$

Case II. When any of 3, 4, 5 and 6 occurs at thousands' place :



The thousands' place can be filled up by any one of the four digits 3, 4, 5 and 6 in 4 ways and the remaining three places can be filled up by the remaining six digits in $^6P_3$ ways.

$\therefore$ The number of numbers formed in this case

$= 4 \times {}^6P_3 = 4 \times \frac{6!}{3!} = 480$

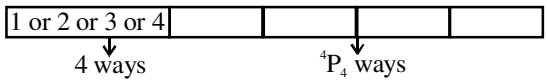
$\therefore$ The required number = 80 + 480 = 560

www.pdfworld.net

www.pdfworld.net

➤ **Example 14.** How many positive numbers can be formed by using any number of the digits 0, 1, 2, 3 and 4; no digit being repeated in any number?

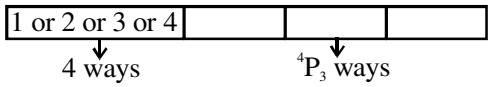
➤ ***Solution*** **Case I.** When number is of five digits :



The ten thousands' place can be filled up by any one of the four digits 1, 2, 3 and 4 in 4 ways and the remaining four places can be filled up by the remaining four digits in $^4P_4$ ways.

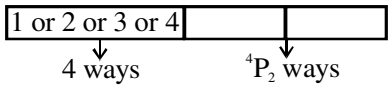
$\therefore$ The number of numbers formed in this case $= 4 \times {}^4P_4$

Case II. When number is of four digits.



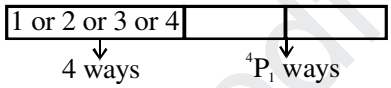
$\therefore$ The number of numbers formed in this case $= 4 \times {}^4P_3$

Case III. When number is of three digits.



$\therefore$ The number of numbers formed in this case $= 4 \times {}^4P_2$

Case IV. When number is of two digits :



Case V. When number is of one digit :

The number of positive numbers formed in this case $= 4$

$\therefore$ The required number

$= 4 \times {}^4P_4 + 4 \times {}^4P_3 + 4 \times {}^4P_2 + 4 \times {}^4P_1 + 4$

$= 96 + 96 + 48 + 16 + 4 = 260$

➤ **Example15.** How many numbers between 400 and 1000 can be made with the digits 2, 3, 4, 5, 6 and 0 repetition not allowed?

➤ ***Solution*** Any number between 400 and 1000 will be of three digits.

Since the number should be greater than 400, therefore, hundreds' place can be filled up by any one of the three digits 4, 5 and 6 in 3 ways.

The remaining two places can be filled up by the remaining five digits in $^5P_2$ ways.

$\therefore$ The required number $= 3 \times {}^5P_2 = 3 \times \dfrac{5!}{3!} = 60.$

➤ **Example16.** How many integers between 100 and 999 have distinct digits?

➤ ***Solution*** This is not simply $^{10}P_3$, the number of permutations of all ten digits taken three at a time because 086, for example, is not a number between 100 and 999. The digit 0 can be used in the units' place (as in 860), or in the tens' place (as in 806), but not in the hundreds' place. Consider three boxes to be filled by the digits of any of the integers under consideration.

There are nine choices for the digit in the hundreds' place, because 0 cannot be used. There are then nine choices for the digit in the tens' place, namely 0 together with the eight non-zero digits not used already. Similarly there are eight choices for the digit used in the unit's place. Hence the answer is 9.9.8 or 648.

➤ **Example17.** Of the 648 integers in the preceding problem, how many are odd numbers?

➤ ***Solution*** A number is odd if its units digit is odd, i.e. if the digit in the units' place is one of 1, 3, 5, 7, 9. So it is best to begin the argument by asking how many choices there are for the digit in the units place; the answer is five. Next, turn to the hundreds' place; there are eight digits from which a selection can be made, namely all the non-zero digits except the one already used in the units' place. Finally there are eight choices for the digit in the tens' place, so the answer is 5.8.8. or 320.

Practice Problems [B]

1. Evaluate the following :
   (i) 12!/10! (ii) 4! + 3! (iii) (4 + 3)!.
2. Simplify the following :
   (i) $\dfrac{7!-5!}{4!41}$ (ii) $\dfrac{(n+1)!}{(n-1)!}$ (iii) $\dfrac{(2n)!}{n!}$
3. In an experiment on social interaction, 6 people will sit in 6 seats in a row. In how many ways can this be done?
4. In how many ways can 3 men each be given a hotel room of his own if there are 8 free rooms available?

www.pdfworld.net

www.pdfworld.net

5. A man can cook 9 different dinners. How many different sequences of dinner could he cook for a week if he didn't prepare the same dinner twice?
6. A business school offers course in typing, shorthand, transcription, business english, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken?
7. In how many different ways can four passengers be seated in a railway coach compartment for four?
8. A trade union committee consists of 9 persons and has to elect a chairman, a vice-chairman and a treasurer. How many various combinations can there be?
9. How many elements should be taken so that the number of their permutations taken 4 at a time is 6 times greater than those taken 2 elements at a time?
10. Determine the number of permutations of the letters in the word ROCKET. How many permutations can be made so that the letter R is the first letter and ET are the last two letters?
11. In a club with 15 members, how many ways can a state of 3 officers consisting of president, vice-president, and secretary/treasurer be chosen?
12. In how many ways can 5 players be assigned to the 5 positions on a basketball team, assuming that any player can play any position ? In how many ways can 10 players be assigned to the 5 positions?
13. In how many ways can all of n distinct objects be put in k distinct boxes, not more than one in each box, if there are more boxes than things?

4.5 PERMUTATION WITH REPETITION

To find the number of permutations of n objects taken r at a time, when each object can be repeated any number of times.

In some situations repetition of an object is allowed i.e. a particular object can be used to fill the places again and again. To understand this concept let us consider the numbers, which can be made by using the number 1, 2, 3 & 4.

1111 1121 1131 4441
1112 1122 1132 4442
1113 1123 1133 4443
1114 1124 1134 4444

These arrangements can be counted as follows :

Place Number:	1	2	3	4
Number of choices:	4	4	4	4

Note that any place can be filled by any one of the four numbers, 1, 2, 3 & 4

Total number of arrangements $= 4 \times 4 \times 4 \times 4 = 4^4$.

In general, if we have n objects and we have to count the number of arrangements of the objects taken r at a time, then we consider r places

Place Number:	1	2	3	..... r
Number of choices :	n	n	n	.....n

Here any one of the n objects can be put in the first place; and, when the first place is filled, any one of the n objects can be put in the second place; and so on. The total number of arrangements

$$= \underbrace{n \times n \times n \times \ldots\ldots\ldots \times n}_{r \text{ times}} = n^r$$

Hence, the number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.

➤ **Example 1.** A child has four pockets and three marbles. In how many ways can the child put the marbles in its pockets?

➤ ***Solution*** The first marble can be put into the pocket in 4 ways, so can be the second. Thus the number of ways in which the child can put the marbles $= 4 \times 4 \times 4 = 64$ ways.

➤ **Example 2.** A number lock has four rings, each with ten different digits (0 to 9). How many different attempts to open the lock might one need to make in order to find the correct permutation?

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** We have four 'places' to fill (one for each ring) and 10 digits to choose from each time :

10	10	10	10	$\rightarrow 10^4 = 10\,000$

One might make 10000 different attempts.

➤ **Example 3.** The letters of the Morse code are sequences of dots and dashes. How many different letters can be formed by using 5 symbols ?

➤ ***Solution*** There are two elements, a dot and a dash. Since five symbols are used, the sample contains five elements which can be repeated. Thus, the number of different samples each of which represents a letter is equal to $2^5 = 32$.

➤ **Example 4.** How many 3-letter code words (permutations) can be made from the word ANCHOR if

(i) we can repeat letters ?

(ii) we cannot repeat letters ?

➤ ***Solution*** (i)

6	6	6

$6^3 = 216$ code words with repeats

(ii)

6	5	4

$6 \times 5 \times 4 = 120$ code words without repeats.

➤ **Example 5.** How many car registration plates can be made, using two letters followed by four digits (e.g. RJ 6610) ? (Zero is not allowed as the first of the four digits).

➤ ***Solution***

letters		digits			
26	26	9	10	10	10

We have a choice of 26 letters each time for the first two places. For the first digit, you have 9 choices only, because zero is not allowed. Thereafter, any of 10 digits can be chosen from each of the other places. The total number of permutations is : $26 \times 26 \times 9 \times 10 \times 10 \times 10 = 6084000$.

➤ **Example 6.** A student appears in an objective test which contains 10 multiple choice questions. Each question has four choices and one correct answer. What is the maximum number of different answers can the student give ? How will the answer change if each question may have more than one correct answers?

➤ ***Solution*** The first question has four possible answers and corresponding to any of these the second question can also have four answers. Continuing in this manner the total possible answers = $4 \times 4 \times 4 \times ....10$ times = 4^{10}.

Second part. Take any question suppose the choices are denoted by A, B, C, and D. Now the choice A is either correct or incorrect (two ways) similarly the other choices are either correct or incorrect. Thus this particular question can have $2 \times 2 \times 2 \times 2 = 16$ possible answers but this includes the case when all the four choices are incorrect. Thus the total number of answers = 15. Now as there are 10 questions and each question has 15 possible answers. Therefore the total number of answers = 15^{10}.

Practice Problems [C]

1. How many different outcomes are there in an experiment consisting of n tosses of a coin?
2. A letter lock consists of three rings each marked with fifteen different letters. Find in how many ways it is possible to make an unsuccessful attempt to open the lock?
3. There are stalls for 12 animals in a ship. In how many ways the shipload can be made if there are cows, calves and horses to be transported, animals of each kind being not less than 12?
4. In how many ways 5 delegates can be put in 6 hotels of a city if there is no restriction ?
5. In how many ways can a ten question multiple choice examination be answered if there are four choices a, b, c and d to each question ? If no two consecutive questions are answered the same way, how many ways are there?

pdfworld.net

www.pdfworld.net

www.pdfworld.net

6. A safe is locked by a device consisting of five disks with the digits 0, 1, 2,...., 9 on each of them. The safe gets unlocked by dialing a certain combination of digits. Will ten days be enough to open the safe if the "working day" lasts 13 hours and it takes five seconds to dial one combination of digits?

7. How many different telephone numbers are there if it is assumed that each number contains not more than seven digits (a telephone number may begin with a zero)?

8. In how many ways can three prizes be given away to 5 students when each student is eligible for any of the prizes?

9. In how many ways can 6 rings be worn on the four fingers of one hand?

10. In how many ways four friends can put up in 8 hotels of a town if
(i) There is no restriction?
(ii) No two friends can stay together?
(iii) All the friends do not stay in same hotel?

11. The members of a club are to vote "yes" or "no" on each of eight issues. In marking his ballot, a member has the option of abstaining on as many as seven of the issues, but he should not abstain in all eight cases. In how many ways can a ballot be marked?

12. In a school there are two prizes for excellence in Mathematics (1st and 2nd) two in Chemistry (Ist and IInd) and only one in Physics (1st). In how many ways can these prizes be awarded to 20 students?

13. How many different 4-letter radio station call letters can be made
(a) if the first letter must be K or W and no letter may be repeated;
(b) if repeats are allowed (but the first letter is K or W)?
(c) How many of the 4-letter call letters (starting with K or W) with no repeats end in R?

14. For many years, Delhi used 3 letters followed by 3 digits on its automobile license plates.
(a) How many different license plates are possible with this arrangement ?
(b) When the state ran out of new plates, the order was reversed to 3 digits followed by 3 letters. How many additional plates were then possible ?
(c) Several years ago, the plates described in (b) were also used up. The state then issued plates with 1 letter followed by 3 digits and then 3 letters. How many plates does this scheme provide ?

4.6 FURTHER ILLUSTRATIONS

Sum of natural numbers

To find the sum of all the numbers that can be formed using the digits $d_1, d_2, ..., d_n$ without repetition.
The sum of the digits in the units place of all numbers formed from the digits $d_1, d_2, ..., d_n$ taken all at a time and without repetition, is
$= (n-1)!\,(d_1 + d_2 + ... + d_n)$.
When d_1 is chosen in the units place, the other $n-1$ digits can be arranged in $(n-1)!$ ways so that d_1 occurs in the units place in $(n-1)!$ numbers. Similarly $d_2, d_3, ... d_n$ each occurs in the units place in $(n-1)$ numbers.
The digit d_1 occurs in the units place in $(n-1)!$ numbers, in the tens place in $(n-1)!$ numbers, and so on. Hence, sum of the digit d_1 only, is

$$= (n-1)!\,(10^0 d_1 + 10^1 d_1 + 10^2 d_1 + 10^2 d_2 + ... + 10^{n-1} d_1)$$

$$= (n-1)!\, d_1 \left(\frac{10^n - 1}{9}\right)$$

Similarly adding the contribution due to all the digits, we have
the sum of all the numbers that can be formed using the digits $d_1, d_2, ..., d_n$ without repetition, is

www.pdfworld.net

www.pdfworld.net

$= (n-1)!\,(d_1 + d_2 + \ldots\ldots + d_n)\left(\frac{10^n - 1}{9}\right)$.

➤ **Example 1.** Find the sum of the digits in the unit place of all numbers formed with the help of 3, 4, 5, 6 taken all at a time.

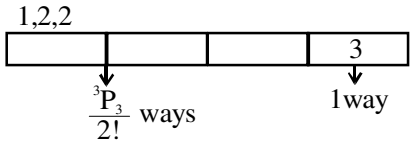
➤ ***Solution*** Sum of the digits in the unit place
$= (4-1)\,!\,(3+4+5+6)$
$= 6.18 \quad = 108.$

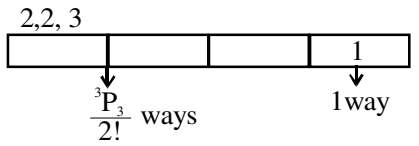
➤ **Example 2.** Find the sum of all the four digits numbers that can be formed with the digits 0, 1, 2 and 3.

➤ ***Solution*** The occurrence of zero at any place will not affect the sum of the digits at that place.
The number of numbers formed having 2 at units place $= 2 \times 2 \times 1 = 4$
The number of numbers formed having 2 at thousands' place $= 1 \times {}^3P_3 = 6$
Each of the digits 1, 2 and 3 will occur at units, tens and hundreds places 4 times and at thousands place 6 times.
$\therefore$ Sum of the digits at units, tens and hundreds places will be each $4(1 + 2 + 3) = 24$
And sum of the digits at thousands place $= 6(1+2+3) = 36$.
$\therefore$ The sum of the numbers formed $= 4(1 + 2 + 3)(1 + 10 + 100) + 6(1 + 2 + 3).\ 1000$
$= 24 \times 1 + 24 \times 10 + 24 \times 100 + 36 \times 1000 = 38664$

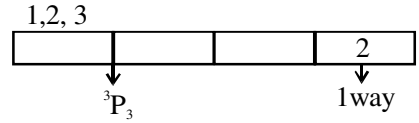
➤ **Example 3.** Find the sum of all the 4 digits numbers that can be formed with the digits 1, 2, 2 and 3.

➤ ***Solution***





Each of the digits 1 and 3 will occur at units, tens, hundreds and thousands place $\frac{{}^3P_3}{2!}$ i.e. 3 times



But 2 will occurs at each place ${}^3P_3$ i.e. 6 times

$\therefore$ Sum of the numbers formed $= 3(1+3)(1 + 10 + 100 + 1000) + 6.2.(1 + 10 + 100 + 1000) = 26664.$

➤ **Example 4.** Find the sum of all five digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 (repetition of digits not allowed).

➤ ***Solution*** The required sum

$= (5-1)\,!\,(1+2+3+4+5)\left(\frac{(10^5 - 1)}{9}\right)$

$= 24 \,.\, 15 \,.\, 11111 = 3999960.$

➤ **Example 5.** Prove that ${}^nP_r = r.{}^{(n-1)}P_{r-1} + {}^{(n-1)}P_r$

➤ ***Solution*** A common sense interpretation of the identity above is possible. Of the number of permutations of r things which may be made from n things, $r.{}^{(n-1)}P_{r-1}$ contain one specified thing and ${}^{(n-1)}P_r$ do not contain that specified thing and these two together give ${}^nP_r$.

➤ **Example 6.** Find the total number of signals that can be made by five flags of different colours when any number of them may be used.

➤ ***Solution*** Number of signals using 1 flag $= {}^5P_1$
Number of signals using 2 flags $= {}^5P_2$
.....
Number of signals using 5 flags $= {}^5P_5$
Hence, total number of signals that can be made $= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$.

➤ **Example 7.** There are counters available in 3 different colours (atleast four of each colour). Counters are all alike except for the colour. If 'm' denotes the number of arrangements of four counters if no arrangement consists of counters of same colour and 'n' denotes the corresponding figure when every arrangement consists of counters of each colour, then find the value of m/n.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** $m = 3^4 - 3 = 78$

$$n = 3^4 - \left[3\left(2^4 - 2\right) + 3\right] = 81 - 45 = 36$$

$$\text{Hence } \frac{m}{n} = \frac{78}{36} = \frac{13}{6}.$$

Ranking of a Word

➤ **Example 8.** If all the letters of the word 'QUEST' are arranged in all possible ways and put in dictionary order, then find the rank of the given word.

➤***Solution*** Number of words beginning with E = ${}^4P_4 = 24$

Number of words beginning with QE = ${}^3P_3 = 6$

Number of words beginning with QS = 6

Number of words beginning with QT = 6.

Next word is 'QUEST'

∴ Its rank is 24 + 6 + 6 + 6 + 1 = 43.

➤**Example 9.** The letters of the word OUGHT are written in all possible orders and these words rewritten out as in a dictionary. Find the rank of the word TOUGH in this dictionary.

➤ ***Solution*** Total number of letters in the word OUGHT is 5 and all the five letters are different. Alphabetical order of letters is G, H, O, T, U.

The number of words beginning with G = 4! = 24

The number of words beginning with H = 4! = 24

The number of words beginning with O = 4! = 24

The number of words beginning with TG = 3! = 6

The number of words beginning with TH = 3! = 6

The number of words beginning with TOG = 2! = 2

The number of words beginning with TOH = 2! = 2

There will be two words beginning with TOU and TOUGH is first among them

∴ The rank of word 'TOUGH' in the dictionary
= 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1 = 89th

➤**Example 10.** Find the number of 7 digit numbers the sum of whose digits is even.

➤***Solution*** A number of the seven digit will be of the form

$ax_1x_2x_3x_4x_5\,0$, $ax_1x_2x_3x_4x_5\,1$, $ax_1x_2x_3x_4x_5\,2$, $ax_1x_2x_3x_4x_5\,3$, ... $ax_1x_2x_3x_4x_5\,9$

where $a \in \{1, 2, 3, ...9\}$

and $x_1, x_2, x_3, x_4, x_5 \in \{0, 1, 2, 3, ... 9\}$

Since sum of the digits should be even, therefore,

if $a + x_1 + x_2 + x_3 + x_4 + x_5$ is an even number, then the digit at units place must be 0, 2, 4, 6, 8 and if $a + x_1 + x_2 + x_3 + x_4 + x_5$ is an odd number, then the digit at units place will be 1, 3, 5, 7, 9.

∴ The required number = $9 \times 10 \times 10 \times 10 \times 10 \times 10 \times 5$

$= 9 \times 10^5 \times 5 = 45 \times 10^5$

➤ **Example 11.** A new club flag is to be designed with 6 verical stripes using some or all of the colours yellow, green, blue and red. In how many ways can this be done so that no two adjacent stripes have the same colour ?

➤ ***Solution*** Let abcdef denote the 6 verical stripes in order from the left. Then a can be of any one of the 4 colours; then b can be of any one of the other 3 colours; then c can be again of any one of 3 colours since the colour used for a is now available. Similarly, there are 3 possible colours for each of d, e, f. Hence, there are 4×3^5 ways of designing the flag.

➤ **Example 12.** (i) How many different five digit numbers are there (leading zeros, e.g. 00144, not allowed)?

(ii) How many even 5 digit numbers are there ?

(iii) How many 5 digit numbers are there that are the same when the order of their digits is reversed (e.g. 14341)?

➤ ***Solution*** (i) A five digit number is of the form abcde where a, ..., e are the digits in it and a is the leading digit. Now a cannot be 0 and so can be chosen in 9 ways (any one of 1, ..., 9). Each of the remaining digits can be chosen in 10 ways since repetition is allowed. So in all 9×10^4 ways.

(ii) The number abcde is even if and only if the units digit e is even; so there are 5 choices for e : 0, 2, 4, 6, 8. so in all $9 \times 10^3 \times 5$ ways..

(iii) Two cases: either only a = 3 or exactly one of b, c, d, e = 3. There are 9^4 ways in the first case and $8 \times 9^3 \times 4$ in the second case (factor 4 for the 4 subcases b = 3 or c = 3 etc,); so in all $9^4 + 8 \times 9^3 \times 4$ ways.

www.pdfworld.net

www.pdfworld.net

(iv) By the given condition, the values of a, b, c fix the values of d, e : e = a, d = b. hence, the number of ways is 9×10^2

Practice Problems [D]

1. Find the number of positive integers with distinct digits.
2. Find the number of 9 digit numbers divisible by 9 which can be formed using the digits from 0 to 9 without repetition of digits.
3. Find the sum of all the four digit numbers that can be formed with the digits 3, 2, 3, 4.
4. Find the sum of all numbers greater than 10, 000 formed with the digits 0, 2, 4, 6 and 8; no digit being repeated in any number.
5. Find the sum of all the 4 digit numbers that can be formed with the digits 0, 2, 3 and 5.
6. The letters of the word 'RANDOM' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.
7. In how many ways can six persons A, B, C, D, E, F line up at a railway booking window? In how many of these A stands first? In how many of these A gets the ticket before B?
8. How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4, if : (i) repetition of digits is allowed? (ii) repetition of digits is not allowed?
9. How many five digits numbers divisible by 3 can be formed using the digits 0, 1, 2, 3, 4, 7 and 8 if each digit is to be used atmost once.
10. In how many ways can 6 boys and 5 girls be arranged for a group photograph if the girls are to sit on chairs in a row and the boys are to stand in a row behind them?
11. How many different numbers greater than 5000 can be formed with the digits 0, 1, 5, 9, no digit being repeated?
12. Find the number of numbers lying between 300 and 4000 that can be formed with the digits 0, 1, 2, 3, 4, 5; no digit being repeated.
13. How many five digits numbers are there whose sum of digits is odd?
14. If repetition of digits is not allowed how many numbers of four digits divisible by 5 can be formed with the digits 0, 4, 5, 6,7 ?
15. Find the number of numbers divisible by 25 that can be formed using only the digits 1, 2, 3, 4, 5 & 0 taken five at a time.

4.7 COMBINATION

Each of the groups or selections which can be made by taking some or all of a number of things, without reference to the order of the things in each group, is called a combination. If we select a group of items without worrying about what order they are in, then we are talking about a combination.

But when we start considering the order or arrangement within the group then we are dealing with a permutation.

Any combination of things can be permuted – reordered or rearranged – in several different ways. For instance, let's select the first three letters from the alphabet. We could write these letters down in six different ways :

ABC BCA CAB ACB BAC CBA

(i) How many combinations do we have here ?

(ii) How many permutations ?

(i) There is only one combination – the first three letters of the alphabet are the same three letters, no matter what order they are written in.

(ii) There are six permutations – each different order is a new permutation.

To get a new combination we must change atleast one of the items in the group – we could add a new item, remove one of the present items, or do both.

www.pdfworld.net

www.pdfworld.net

But to get a new permutation from a particular group (combination), we need only rearrange its items.

Which are more numerous, combinations or permutations ?

Permutations are more numerous. From each possible combination, more than one permutation can be made by rearranging its items.

Order counts in a permutation; in a combination it doesnot.

With n distinct objects, ${}^nC_r$ counts the number of ways of choosing r of them without regard to order. Any one of these choices is simply a collection of r objects. Such a collection can be ordered in r! different ways. Since to each combination there correspond r! permutations, there are r! times as many permutations as there are combinations:

$${}^nP_r = r!\,{}^nC_r \quad \text{or} \quad {}^nC_r = \frac{{}^nP_r}{r!}$$

But we know formula ${}^nP_r = \frac{n!}{(n-r)!}$, and hence we get the basic formula for ${}^nC_r$, ${}^nC_r = \frac{n!}{r!(n-r)!}$

For instance, the number of selections of 4 objects out of 6 is denoted by ${}^6C_4$

$\because$ The objects of each selection can be arranged in 4! ways

$\therefore$ The total number of permutation of A, B, C, D, E & F taken at a time

= 4! x number of selections of 4 objects out of A, B, C, D, E & F

$$\Rightarrow \quad {}^6C_4 = \frac{{}^6P_4}{4!} = \frac{{}^6P_4}{4!} = \frac{6!}{(6-4)!\times 4!} = 15$$

Earlier, we saw that there are 60 ways that at teacher can arrange 3 of 5 different books in a row. That is, there are 60 permutations of 5 things taken 3 at a time. Suppose now that the teacher does not wish to arrange the books in a row, but rather wishes to choose, without regard to order, any 3 of the 5 books to donate to a book sale to raise money for the school. In how many ways can this be done?

The number of combinations of 5 things taken 3 at a time is written ${}^5C_3$, or $\binom{5}{3}$, or C(5, 3).

$${}^5C_3 = \frac{5\cdot 4\cdot 3}{3!} = \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1} = 10$$

There are 10 ways that the teacher can choose 3 books for the book sale.

(i) The number of ways of selecting r things out of n things, is the same as the number of ways in which the remaining (n – r) can be selected and rejected.

(ii) The combination notation also represents the binomial coefficient. That is, the binomial coefficient ${}^nC_r$ is the combination of n elements chosen r at a time.

Comparision of permutation and combination

Permutations	**Combinations**
Different orderings or arrangements of the r objects are different permutations.	Each choice or subset of r objects gives one combination. Order within the group of r objects does not matter.
${}^nP_r = \frac{n!}{(n-r)!}$	${}^nC_r = \frac{n!}{(n-r)!\,r!}$
Clue words : arrangement, schedule, order	Clue words : group, committee, sample, selection, subset

www.pdfworld.net

www.pdfworld.net

➤ **Example 1.** A sales representative has 10 accounts in a certain city.

(a) In how many ways can 3 accounts be selected to call on ?

(b) In how many ways can calls be scheduled for 3 of the 10 accounts ?

➤ ***Solution*** (a) Within a selection of 3 accounts, the arrangement of the visits is not important, so there are

$${}^{10}C_3 = \frac{10!}{7!3!} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 120$$ ways to select 3 accounts.

(b) To schedule calls, the sales representative must order each selection of 3 accounts. Use permutations here, since order is important.

$${}^{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10\cdot 9\cdot 8 = 720$$

There are 720 different orders in which to call on 3 of the accounts.

CAUTION It should be stressed that not all counting problems lend themselves to either permutations or combinations. Whenever a tree diagram or the multiplication principle can be used directly, then use it.

➤ **Example 2.** How may matches will be played in a football championship of 16 participants if every two teams play with each other once?

➤ ***Solution*** The number of matches to be played is equal to the number of two element subsets contained in the set consisting of 16 elements i.e. their number is equal to

$${}^{16}C_2 = 120$$

➤ **Example 3.** Two ambulance attendants are needed in any emergency and there are six men of duty. In how many ways can the two be chosen ?

➤ ***Solution*** $${}^{n}C_r = \frac{n!}{(n-1)!r!} = \frac{6!}{4!2!}$$

$$= \frac{6.5.4!}{4!2!} = 15$$

There are 15 ways of choosing 2 men from 6.

➤ **Example 4.** In how many ways can 7 different books be given to two students if one student is to have 4 books and the other is to have 3 books ?

➤ ***Solution*** In selecting 4 books for one student you are leaving 3 for the other (or vice versa). So the number of ways of getting these combinations is

$${}^{7}C_4 = {}^{7}C_3 = \frac{7.6.5.4!}{4!3!} = 35$$

➤ **Example 5.** A teacher takes 3 children from her class to the zoo at a time as often as she can, but she does not take the same three children to the zoo more than once he finds that she goes to the zoo 84 times more than a particular child goes to the zoo. Find the number of children in her class.

➤ ***Solution*** Let number of children in the class be n. Since the teacher has to take 3 children with her everytime she goes to the zoo, hence the maximum number of times she can go to the zoo = maximum number of groups of 3 children she can form = ${}^{n}C_3$

Number of times a particular child can go to the zoo = Number of ways the other two children can be chosen out of remaining (n – 1) children = ${}^{n-1}C_2$

Therefore ${}^{n}C_3 - {}^{n-1}C_2 = 84 \quad \Rightarrow n = 60$

➤ **Example 6.** Prove that the product of r consecutive integers is divisible by r !

➤ ***Solution*** Let p be the product of r consecutive integers beginning with n.

Then $p = n(n+1)(n+2)\ldots.(n+r-1)$

Now $$\frac{p}{r!} = \frac{n(n+1)(n+2).....(n+r-1)}{r!}$$

$$= \frac{[1.2.3.....(n-1)][n(n+1)(n+2).....(n+r-1)]}{[1.2.3.....(n-1)]r!}$$

$$= \frac{(n+r-1)!}{(n-1)!r!} = {}^{n+r-1}C_r$$

= number of combinations of (n + r – 1) different things taken r at a time, which is an integer

$$\therefore \quad \frac{p}{r!} = \text{an integer}$$

STUDY TIP ${}^{n}C_r$ is a natural number.

$\therefore$ p is divisible by r!

➤ **Example 7.** Solve the equation, ${}^{x+3}P_2 = {}^{x+2}C_3 + 20$.

➤ ***Solution*** Simplifying we get, $x^3 - 3x^2 - 28x + 84 = 0$

www.pdfworld.net

www.pdfworld.net

Note x = 3 is a root $\Rightarrow (x-3)(x^2-28)=0$

$x = \pm 2\sqrt{7}$ is not possible

The solution is x = 3.

But what if we need more than two combinations?

Combining Combinations

➤ **Example 8.** Suppose we have 10 social workers and we wish them to work in groups of 2, 3 and 5. How many ways can we split them up into groups of the right size ?

➤ ***Solution*** Firstly, the group of 2 can be chosen in ${}^{10}C_2 = 45$ ways. Out of the remaining 8 workers we can select 3 in ${}^{8}C_3$ ways, leaving the remaining 5 workers to form the third group.

So the first group can be chosen in 45 ways and for each of these ways there are ${}^{8}C_3 = 56$ ways of forming the other two groups.

Since each of the 56 'split-ups' could be linked with any of the 45 selections that were possible originally, the total number of ways of forming all three groups is

$${}^{10}C_2 \times {}^{8}C_3 = 45 \times 56 = 2520$$

➤ **Example 9.** In this year's school play the cast consists of 5 members of a royal family, 15 soldiers, and 30 peasants. How many ways can the cast of 50 pupils be divided among these three groups?

➤ ***Solution*** There are ${}^{50}C_{30} \times {}^{20}C_{15}$ ways of making the three groups.

There are other equally correct answers such as

$${}^{50}C_5 \times {}^{45}C_{30} \text{ or } {}^{50}C_{15} \times {}^{35}C_5 \text{ etc.}$$

➤ **Example 10.** How many ways can 12 identical white and 12 identical black pawns be placed on the black squares of an 8 × 8 chessboard?

➤ ***Solution*** There are 32 black squares and of these 12 can be chosen to put the 12 white pawns in ${}^{32}C_{12}$ ways. Then out of remaining 20 black squares 12 can be chosen to put the 12 black pawns in ${}^{20}C_{12}$ ways. So the answer is ${}^{32}C_{12} \times {}^{20}C_{12}$.

➤ **Example 11.** How many ways are there to place 2 identical rooks in a common row or column of an 8 × 8 chessboard ?

➤ ***Solution*** Since the 2 rooks are identical, the order in which they are placed in a row (or column) is not in any row, 2 of the 8 squares can be chosen in ${}^{8}C_2$ ways. Similarly, 2 rooks can be placed in a column in $8 \times {}^{8}C_2$ ways.

So the total number is $2 \times 8 \times {}^{8}C_2$ ways. Similarly, 2 rooks can be placed in a column in $8 \times {}^{8}C_2$ ways. So the total number is $2 \times 8 \times {}^{8}C_2$.

➤ **Example 12.** How many ways are there to place 2 identical kings on an 8 × 8 chessboard so that the kings are not in adjacent squares?

➤ ***Solution*** A row can be chosen in 8 ways.

In a row, there are in all ${}^{8}C_2$ pairs of squares of which 7 are pairs of adjacent squares.

So there are $[{}^{8}C_2 - 7]$ pairs of places in any of which the two identical kings can be placed.

Hence the kings can be placed in a row in $8 \times [{}^{8}C_2 - 7]$ ways and in the same number of ways in a column. So the total number of ways is $2 \times 8 \times [{}^{8}C_2 - 7]$.

➤ **Example 13.** All possible two factor products are formed from the numbers 1, 2, 3, ..., 100. How many numbers out of the total obtained are multiples of 3 ?

➤ ***Solution*** The total number of two factor products $= {}^{100}C_2$

Out of the numbers 1, 2, 3,...., 100;

the multiples of 3 are 3, 6, 9, ..., 99; i.e. there are 33 multiples of 3,

and therefore there are 67 non multiples of 3.

$\therefore$ The number of two factor products which are not multiples of $3 = {}^{67}C_2$.

$\therefore$ The required number $= {}^{100}C_2 - {}^{67}C_2$

$= 4950 - 2211 = 2739.$

Alternatively, the number of two factor products formed when both factors are multiples of $3 = {}^{33}C_2$ and the number of two factor products formed when only one factor is a multiple of $3 = {}^{33}C_1 \times {}^{67}C_1$.

In either case the product is a multiple of 3.

$\therefore$ The required number $= {}^{33}C_2 + 33 \times 67$

$= 528 + 2211 = 2739.$

Quite often the number of ways of dealing with a problem is found by combining the ways of forming two different combinations. For instance, suppose

www.pdfworld.net

www.pdfworld.net

we have to form a committee of two boys and three girls from a group of 4 boys and 6 girls.

1. Form a committee of 2 boys and 3 girls from a group of 4 boys and 6 girls.
2. Choose the boys in ${}^4C_2 = 6$ ways.
3. Choose the girls in ${}^6C_3 = 20$ ways.
4. This gives $6 \times 20 = 120$ ways of combining the two combinations.

➤ **Example 14.** A basketball coach must select two attackers and two defenders from among three attackers and five defenders. How many different combinations of attackers and defenders can he select ?

➤ ***Solution*** He can select in ${}^3C_2 \times {}^5C_2 = \dfrac{5.4.3.2.1}{2.2.1}$

= 30 different combinations.

A soccer team of 11 players is to be chosen from 30 boys, of whom 4 can play only in goal, 12 can play only as forwards and the remaining 14 in any of the other positions. If the team is to include five forwards and of course, one goalkeeper, in how many ways can it be made up? There are :

${}^4C_1$ ways of choosing the goalkeeper

${}^{12}C_5$ ways of choosing the forwards

and ${}^{14}C_5$ ways of choosing the other 5 players.

That is, ${}^4C_1 \times {}^{12}C_5 \times {}^{14}C_5$ combinations altogether.

➤ **Example 15.** A cricket team of 11 men is to be chosen from 15 men. Four of the 15 can bowl and two others can keep wicket. If the team is must include only one wicket keeper and atleast three bowlers, in how many ways can it be chosen?

➤ ***Solution*** Notice that once you have considered the number of ways of forming a 3-bowler team you still have to consider the number of ways of forming a team with more than three bowlers.

We had to select 1 wicket keeper, 3(or 4) bowlers, and 7(or 6) other players from 2 wicket-keepers, 4 bowlers and 9 other players.

We can select a team with 3 bowlers in

$${}^2C_1 \times {}^4C_3 \times {}^9C_7 \text{ ways}$$

We can select a team with 4 bowlers in

$${}^2C_1 \times {}^4C_4 \times {}^9C_6 \text{ ways}$$

Thus the number of ways of selecting a team with 3 or 4 bowlers is

$$({}^2C_1 \times {}^4C_3 \times {}^9C_7) + ({}^2C_1 \times {}^4C_4 \times {}^9C_6)$$

$$= \left(2 \times 4 \times \frac{9.8.7!}{7!2!}\right) + \left(2 \times 1 \times \frac{9.8.7.6!}{6!3!}\right)$$

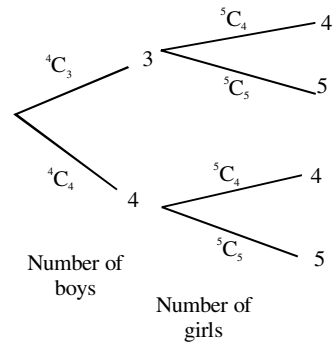
= 456 different teams could be formed.

➤ **Example 16.** A group of students consists of 4 boys & 5 girls. Find the number of ways of selecting a team of atleast 3 boys & 4 girls.

➤ ***Solution*** Every branch of above tree diagram gives us the constitutions of the team which are as follows :

Constitution of team		Number of ways of choosing
	3 Boys and 4 girls	${}^4C_3 \times {}^5C_4$
or	3 Boys and 5 girls	${}^4C_3 \times {}^5C_5$
or	4 Boys and 4 girls	${}^4C_4 \times {}^5C_4$
or	4 Boys and 5 girls	${}^4C_4 \times {}^5C_5$

Total number of ways of choosing a team



$$= {}^4C_3 \times {}^5C_4 + {}^4C_3 \times {}^5C_5 + {}^4C_3 \times {}^5C_4 + {}^4C_4 \times {}^5C_5$$

$$= 4 \times 4 + 4 \times 1 + 1 \times 5 + 1 \times 1 = 30$$

www.pdfworld.net

www.pdfworld.net

Standard Formulae

1. ${}^{n}C_{r} = {}^{n}C_{n-r}$

Combinatorial Argument :

Choosing r things out of in given things ≡ choosing (n – r) things out of given n things because when r things are chosen, n – r things are automatically left out.

If ${}^{n}C_{x} = {}^{n}C_{y}$ then either x = y or x + y = n

2. ${}^{n+1}C_{r} = {}^{n}C_{r} + {}^{n}C_{(r-1)}$

${}^{n+1}C_{r}$	=	${}^{n}C_{r}$	+	${}^{n}C_{r-1}$
Choosing any r object out of (n + 1) given objects	≡	Choosing r objects always exluding one particular object out of (n + 1) objects	or	Choosing r objects always including one particular object out of (n + 1) object

➤ **Example 17.** Evaluate ${}^{n}C_{r} + \sum_{j=0}^{3} {}^{n+j}C_{r+1+j}$

➤ ***Solution*** $({}^{n}C_{r} + {}^{n}C_{r+1}) + {}^{n+1}C_{r+2} + {}^{n+2}C_{r+3} + {}^{n+3}C_{r+4}$

$= ({}^{n+1}C_{r+1} + {}^{n+1}C_{r+2}) + {}^{n+2}C_{r+3} + {}^{n+3}C_{r+4}$

$= ({}^{n+2}C_{r+2} + {}^{n+2}C_{r+3}) + {}^{n+3}C_{r+4}$

$= {}^{n+3}C_{r+3} + {}^{n+3}C_{r+4}$

$= {}^{n+4}C_{r+4}$

3. $r.\,{}^{n}C_{r} = n.\,{}^{n-1}C_{r-1}$

Choosing r MP's from n citizens and Choosing 1 PM from r choosen MP's $r \times {}^{n}C_{r}$	≡	Choosing 1 PM (n ways) from n citizens and Choosing remaining (r – 1) MP's from (${}^{n-1}C_{r-1}$ ways) remaining (n – 1) citizens $n \times {}^{n-1}C_{r-1}$

www.pdfworld.net

www.pdfworld.net

4. The greatest value of ${}^nC_r$ is ${}^nC_{n/2}$ if n is even and ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$ if n is odd.

Proof : $\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n!}{(r+1)!(n-r-1)!} \times \frac{r!(n-r)!}{n!}$

$= \frac{n-r}{r+1}$

Case I : n is even ${}^nC_{r+1} > {}^nC_r$ if $n-r > r+1$

if $n-1 > 2r$ if $r < \frac{n}{2} - \frac{1}{2}$

$\because r \in I \quad \therefore {}^nC_r < {}^nC_{r+1}$ upto $r = \frac{n}{2} - 1$

Hence ${}^nC_0 < {}^nC_1 < {}^nC_2 < .... < {}^nC_{\frac{n}{2}-1}$

$> {}^nC_{\frac{n}{2}+1} > {}^nC_{\frac{n}{2}+2} > ....$

$\therefore$ The greatest values of ${}^nC_r = {}^nC_{n/2}$

Case II : n is odd ${}^nC_{r+1} \geq {}^nC_r$ if $r \leq \frac{n-1}{2}$

equality holds for $r = \frac{n-1}{2}$

$\therefore {}^nC_0 < {}^nC_1 < {}^nC_2 < .... < {}^nC_{\frac{n+1}{2}}$

$> {}^nC_{\frac{n+3}{2}} > ..... > {}^nC_n$

$\therefore$ The greatest value is ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$

➤ **Example 18.** A class has 21 students. The class teacher has been asked to make groups of r students each and go to zoo taking one group at a time. Find the size of group for which the teacher goes the maximum number of times to the zoo.

➤ ***Solution*** Number of groups of r students each $= {}^{21}C_r$

Number of times the teacher can go to the zoo = Number of groups she can form $= {}^{21}C_r$

$\therefore$ It is maximum when $r = \frac{21-1}{2}$ or $\frac{21+1}{2}$

i.e. r = 10 or 11.

5. ${}^nC_0^2 + {}^nC_1^2 + ... + {}^nC_n^2 = {}^{2n}C_n$

Combinatorial Argument

We consider a class containing n boys & n girls.

Total number of ways of choosing n students $= {}^{2n}C_n$.

n students can also be choosen in one of the following ways :

0 boy & n girls : ${}^nC_0 \times {}^nC_n = {}^nC_0 \times {}^nC_0$

$= {}^nC_0^2$ ways

1 boy & n – 1 girls : ${}^nC_1 \times {}^nC_{n-1}$

$= {}^nC_1 \times {}^nC_1 = {}^nC_1^2$ ways

2 boys & n – 2 girls : ${}^nC_2 \times {}^nC_{n-2}$

$= {}^nC_2 \times {}^nC_2 = {}^nC_2^2$ ways

.....

n boy & 0 girls : ${}^nC_n \times {}^nC_0 = {}^nC_n \times {}^nC_n = {}^nC_n^2$ ways

Total number of ways $= {}^nC_0^2 + {}^nC_1^2 + ... + {}^nC_n^2$ ways

6. ${}^nC_r \cdot {}^rC_s = {}^nC_s \cdot {}^{n-s}C_{r-s}$ $(n \geq r \geq s)$

Proof : LHS

$= \frac{n!}{r!(n-r)!} \times \frac{r!}{s!(r-s)!} = \frac{n!}{s!(r-s)!(n-r)!}$

Combinatorial Argument :

Choosing r MP's ($ {}^nC_r$ ways) and Choosing s ministers out of MP's ${}^nC_r \times {}^6C_5$	≡	Choosing s ministers ($ {}^nC_s$ ways) and Choosing remaining (r – s) MP's out of ($ {}^{n-s}C_{r-s}$ ways) remaining (n – s) citizens ${}^nC_s \times {}^{n-s}C_{r-s}$

www.pdfworld.net

www.pdfworld.net

7. $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \dfrac{n-r+1}{r}$

➤**Example 19.** If ${}^{n+1}C_{r+1} : {}^{n}C_{r} : {}^{n-1}C_{r-1} = 11 : 6 : 3$ then find the value of nr.

➤ ***Solution*** Given $\dfrac{{}^{n+1}C_{r+1}}{{}^{n}C_{r}} = \dfrac{11}{6}$

$\Rightarrow \dfrac{\frac{n+1}{r+1} \times {}^{n}C_{r}}{{}^{n}C_{r}} = \dfrac{11}{6} \quad \Rightarrow \quad 6n + 6 = 11r + 11$

$\Rightarrow 6n - 11r = 5$ ....(1)

Also, $\dfrac{{}^{n}C_{r}}{{}^{n-1}C_{r-1}} \Rightarrow \dfrac{\frac{n}{r} \times {}^{n-1}C_{r-1}}{{}^{n-1}C_{r-1}}$

$= \dfrac{6}{3} \Rightarrow n = 2r$ ....(2)

From (1) & (2); r = 5 and n = 10

$\Rightarrow$ nr = 50

➤ **Example 20.** If ${}^{n}C_{r-1} = 36$, ${}^{n}C_{r} = 84$ and ${}^{n}C_{r+1} = 126$, then find r.

➤***Solution*** Here $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \dfrac{84}{36}$

$\Rightarrow \quad \dfrac{n-r+1}{r} = \dfrac{7}{3} \quad \Rightarrow \quad 10r - 3n = 3$ ...(1)

and $\dfrac{{}^{n}C_{r+1}}{{}^{n}C_{r}} = \dfrac{n-(r+1)+1}{(r+1)} = \dfrac{126}{84}$

$\Rightarrow \quad \dfrac{n-r}{r+1} = \dfrac{3}{2} \quad \Rightarrow \quad 5r - 2n = -3$

or $10r - 4n = -6$ ...(2)

Subtracting (2) from (1), we get n = 9

from (1), $10r - 27 = 3 \quad \therefore \quad r = 3.$

➤ **Example 21.** Prove that, ${}^{n+m}C_{r} = {}^{n}C_{0} \cdot {}^{m}C_{r} + {}^{n}C_{1} \cdot {}^{m}C_{r-1} + {}^{n}C_{2} \cdot {}^{m}C_{r-2} + \ldots\ldots + {}^{n}C_{r} \cdot {}^{m}C_{0}$.

➤ ***Solution*** It is Vandermonde's identity.

Let there be 'n' boys and 'm' girls . We want to choose a team of 'r' persons, the boy-girl proportion allowed to be all possible . The 'r' persons can be chosen from n + m persons in ${}^{n+m}C_{r}$ ways. Also, number of ways ; when no boy and 'r' girls $= {}^{n}C_{0} \cdot {}^{m}C_{r}$

when 1 boy and r – 1 girls $= {}^{n}C_{1} \cdot {}^{m}C_{r-1}$

Similarly 'r' boys and no girls $= {}^{n}C_{r} \cdot {}^{m}C_{0}$

Hence, total $= {}^{n}C_{0} \cdot {}^{m}C_{r} + {}^{n}C_{1} \cdot {}^{m}C_{r-1} + \ldots + {}^{n}C_{r} \cdot {}^{m}C_{0} = {}^{m+n}C_{r}$

Alternatively : Compare the Coefficient of x^r in $(1 + x)^{m+n}$ and $(1 + x)^{m} \cdot (1 + x)^{n}$]

STUDY TIP In a standard pack of playing cards, there are 52 cards. These are divided into 4 suits of 13 cards each : spades, hearts, diamonds and clubs. Each card has a rank. The ranks of the 13 cards in each suit are 2, 3, 4,.... 10, Jack, Queen, King and Ace. Two or more pair is a set of two cards of the same kind (i.e. 2 twos or 2 eights or 2 kings etc.)

➤ **Example 22.** What is the number of ways of choosing 4 cards from a pack of 52 playing cards ? In how many of these

(i) four cards are of the same suit,
(ii) four cards belong to four different suits,
(iii) are face cards,
(iv) two are red cards and two are black cards,
(v) cards are of the same colour ?

➤ ***Solution*** There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore the required number of ways

$= {}^{52}C_{4} = \dfrac{52!}{4!48!} = \dfrac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4} = 270725$

(i) There are four suits : diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_{4}$ ways of choosing 4 diamonds. Similarly, there are ${}^{13}C_{4}$ ways of choosing 4 spades and ${}^{13}C_{4}$ ways of choosing 4 hearts. Therefore the required number of ways

$= 4 \cdot {}^{13}C_{4} = 4 \times \dfrac{13!}{4!\,9!} = 2860$

(ii) There are 13 cards in each suit.
Therefore, there are ${}^{13}C_{1}$ ways of choosing 1 card from 13 cards of diamond, ${}^{13}C_{1}$ ways of choosing 1 card from 13 cards of hearts, ${}^{13}C_{1}$ ways of choosing 1 card from 13 cards of clubs, ${}^{13}C_{1}$ ways of choosing 1 card from 13 cards of

www.pdfworld.net

www.pdfworld.net

spades. Hence, by multiplication principle, the required number of ways

$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in ${}^{12}C_4$ ways. Therefore, the required number of ways

$$= \frac{12!}{4!\,8!} = 495.$$

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways

$$= {}^{26}C_2 \times {}^{26}C_2 = \left(\frac{26!}{2!\,24!}\right)^2$$

$= (325)^2 = 105625$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ ways.
Therefore, the required number of ways

$$= {}^{26}C_4 + {}^{26}C_4 = 2 \times \frac{26!}{4!\,22!} = 29900$$

Practice Problems [E]

1. Which of the following everyday activities involve permutations, and which involve combinations only?
- (i) Choosing a cricket team from a class of 24 boys.
- (ii) Deciding which of the boys will bat first, second, and so on.
- (iii) Deciding which three books to borrow from a library.
- (iv) Planning the best sequence in which to decorate the rooms in my house.
- (v) Planting a row of rose-bushes so that red and yellow flowers alternate.

2. How many triangles can be constructed by connecting the vertices of an octagon.

3. How many ways can a traffic warden with 3 parking tickets left in his book, give them to three out of the seven cars he find that have over-parked ?

4. A party of eight people arrive at a river and a boatman agrees to take them across in two groups of four. In how many ways can the first boatload be chosen?

5. In a local police station, a different combination of three constables is selected daily for duty. If there are ten constables available and the system operates every day except Sunday, how many weeks will go by before the same three constables are on duty again together?

6. All possible two factor products are formed from the numbers 1, 2, 3, ..., 100. How many numbers out of the total obtained are multiples of 3?

7. Find the domain of definition of the function $f(x) = {}^{2x-8}C_{x+1}$ and the set of its values .

8. Solve the given inequality :
- (a) ${}^{10}C_{x-1} > 2\,{}^{10}C_x$
- (b) $8\,{}^{105}C_x < 3\,{}^{105}C_{x+1}$

9. Find the positive integral values of x such that

$${}^{x-1}C_4 - {}^{x-1}C_3 - \frac{5}{4}.(x-2)(x-3) < 0$$

10. Twelve people are waiting for a bus. When it comes it only has room for 6 on top and 4 downstairs . So in how many ways could you form the three groups (those who go upstairs, downstairs or wait) ?

11. There are m men and 2 women participating in a chess tournament. Every participant plays two games with every other participant. If the number of games played by the men between themselves exceeds by 66 the number of games played between men and women, then find the value of m and the total number of games played in the tournament.

12. A bag contains 5 black and 6 red balls, all balls being different. Determine the number of ways in which 2 black and 3 red balls can be selected.

13. A meeting of 40 people must choose a chairman, a secretary and 5 members of a committee. How many different committees can be formed?

14. A team consists of two house painters, three plasterers and one joiner. How many different

www.pdfworld.net

www.pdfworld.net

teams can be formed from a staff of fifteen house painters, ten plasterers and five joiners ?

15. Ten cards have been taken from a pack of 52. In how many cases will there be atleast one ace among the selected cards?

16. How many ways are there of choosing a hand of 6 cards containing an ace and a king of the same suit from a pack of 52 cards?

17. In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers ? Assume that the team of 11 players requires 5 batsmen, 3 all-rounders, 2 bowlers and 1 wicket keeper.

4.8 PERMUTATIONS OF ALIKE OBJECTS

To find the number of permutations of n objects taken all a time, when the objects are not all different.

Let the n objects be represented by letters; and suppose p of them to be a's, q of them to be b's, r of them to be c's, and so on.

Let P be the required number of permutations.

If in any one of the actual permutations we suppose that the a's are all changed into p letters different from each other and from all the rest; then, by changing only the arrangement of these p new letters, we should, instead of a single permutation have $\lfloor p$ different permutations.

Hence, if the a's were all changed into p letters different from each other and from all the rest the b's, c's, & c, being unaltered, there would be $P \times \lfloor p$ permutations.

Similarly, if in any one of the these new permutations we suppose that the b's are all changed into q letters different from each other and from all the rest, we should obtain $\lfloor q$ permutations by changing the order of these q new letters. Hence the whole number of permutations would now be $P \times \lfloor p \times \lfloor q$.

By proceeding in this way we see that if all the letters were changed so that no two were alike, the total number of permutations would be

$P \times \lfloor p \times \lfloor q \times \lfloor r \ldots.$

But the number of permutations all together of n different objects is $\lfloor n$.

Hence $P \times \lfloor p \times \lfloor q \times \lfloor r \ldots. = \lfloor n$

$$\therefore \quad P = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r \ldots\ldots}.$$

Thus, the number of permutations of n things taken all at a time when p of them are alike of one type, q of them are alike of another type, r of them are alike of a third type and the remaining n – (p + q + r) things are all different is $\frac{n!}{p!\,q!\,r!}$.

➤ **Example 1.** In how many ways can the letters of the word DADDY be permuted among themselves?

➤ ***Solution*** Here three letters are alike, viz., D, D and D. Thus of the 120 permutations which would otherwise have been there with D_1, D_2, D_3, Y and A, every set of permutations which only permute D_1, D_2 and D_3 among themselves will coalesce into a single permutation when D_1, D_2, D_3 become identical.

Thus for instance, the six permutations

D_1	D_2	A	D_3	Y
D_1	D_3	A	D_2	Y
D_2	D_3	A	D_1	Y
D_2	D_1	A	D_2	Y
D_3	D_2	A	D_1	Y

coalesce into a single permutation DADDY. The reduction therefore is by a factor of 1/3!.

Hence the answer is $1/6 \times (120) = 20$.

➤ **Example 2.** In how many ways can you permute the letters of the word VIVEKANANDA ?

➤ ***Solution*** There are 2 V's, 3 A's. 2N's and 1I, 1E, 1 K, 1D.

So the answer is $\frac{11!}{2!3!2!} = 1663200$.

➤ **Example 3.** A train time table must be compiled for various days of the week so that two trains a day depart for three days, one train a day for two days,

www.pdfworld.net

www.pdfworld.net

and three trains a day for two days. How many different time tables can be compiled ?

➤ ***Solution*** The number of trains a day (the digits 1, 2, 3) are three groups of like elements from which a sample must be formed. In the time table for a week the number 1 is repeated twice, the number 2 is repeated 3 times and the number 3 is repeated twice. The number of different time tables is equal to

$$\frac{7!}{2!3!2!} = 210$$

➤ **Example 4.** Find the number of six digit numbers not divisible by 5 that can be formed by using the digits of the number 121202.

➤ ***Solution*** The required number of numbers = Total number of arrangements – Number of arrangements starting with 0 – Number of arrangements ending with 0 $= \frac{6!}{3!\,2!} - \frac{5!}{3!\,2!} - \frac{5!}{3!\,2!} = 60 - 10 - 10 = 40.$

➤ **Example 5.** In a certain country, the numerals in car registration marks range from 1 to 999. Find the number of cases in which the first local car which you see while visiting that country have atleast two digits the same in its registration mark.

➤ ***Solution*** (i) For two digit number = 9 (11, 22,.....)

(ii) For three digit number all same = 9(111, 222,.....)

(iii) For 3 digit when 2 digit same and 1 different.

(a) $\times \times 0$ or $\times 0 \times$ $= 9 + 9 = 18$

(b) without zero ${}^3C_1 . 9 . 8 = 216$

Total = 216 + 36 + 9 = 261

➤ **Example 6.** Find the number of seven digit numbers which have exactly three 9's .

➤ ***Solution*** If 9 occupies the 1st position, then other two 9's can be filled in ${}^6C_2$ way and the remaining 4 places can be filled in $9^4 \Rightarrow {}^6C_2 . 9^4$.

If 9 does not occupy the 1st position then it can be filled in ${}^6C_3$ ways, and 1st place in 8 ways and remaining 3 places in 9^3 ways .

$\Rightarrow {}^6C_3 . 8 . 9^3$.

Total = ${}^6C_2 . 9_4 + {}^6C_3 . 8 . 9^3 = 215055$

➤ **Example 7.** In how many ways can 5 identical black balls, 7 identical red balls and 6 identical green balls be arranged in a row so that atleast one ball is separated from balls of the same colour ?

➤ ***Solution*** The required number of ways

= (the number of ways without restriction)

– (the number of ways when balls of each colours are consecutive)

$= \frac{18!}{5!7!6!} - 3!$ ($\because$ there are altogether 18 balls in which 5, 7 and 6 are identical; and considering balls of the same colour as one thing there are 3 things, there being no arrangement between balls of the same colour)$= \frac{18!}{5!6!7!} - 6.$

➤ **Example 8.** A dictionary is printed consisting of 7-lettered words that can be made with the letters of the word CRICKET. If the words are printed in the alphabetic order, as in an ordinary dictionary, find the position of the word CRICKET in that dictionary.

➤ ***Solution*** In alphabetic order the letters are C, C, E, I, K, R, T. Words beginning with C will be printed at first. Our word also begins with C. The number of words beginning with C and having C or E I or K in the second place = 4 × 5 !.

C	C	x	x	x	x	x
	E	x	x	x	x	x
	I	x	x	x	x	x
	K	x	x	x	x	x

C R	C	x	x	x	x
	E	x	x	x	x

C R I C	E	x	x
	K	E	T

The number of words beginning with CR and having C or E in the third place = 2 × 4 !.

The number of words beginning with CRIC and having E in the fifth place = 2 !.

www.pdfworld.net

www.pdfworld.net

After printing all these words, the word CRICKET will be printed.

$\therefore$ The required place of the word CRICKET

$= (4 \times 5! + 2 \times 4! + 2!) + 1 = 531.$

$\therefore$ The word will be printed in the 531th place.

➤ **Example 9.** Two players A and B play a series of games of chess. The winning player in any game gets 1 point while the losing player gets 0 point. The player who achieves 4 points first, wins the series. If no game ends in a draw, find the number of ways in which the series can be won by A.

➤ ***Solution*** A wins the series in the following ways : Note that in all these cases, the last match has been won by A.

(i) A wins all the games in a row. The number of ways = 1.

(ii) A wins the last game. Of the remaining 4 games played, A wins three while B wins one. The number of ways $= \frac{4!}{3!} = 4$

(iii) A wins the last game. Of the remaining 5 games played, A wins three while B wins two. The number of ways $= \frac{5!}{3!2!} = 10$

(iv) A wins the last game. Of the remaining 6 games played, A wins three while B wins three. The number of ways is $= \frac{5!}{3!3!} = 20$

Thus, total number of ways in which the series can be won by A $= 1 + 4 + 10 + 20 = 35.$

Practice Problems [F]

1. In how many ways can you permute the letters of the word CONSTITUTION ?
2. How many different permutations can be formed from the letters contained in the following words :
   (a) ZEBRA, (b) BAZAR,
   (c) SILICIC, (d) ABRACADABRA ?
3. In how many ways can we plant a line of 10 bulbs consisting of 5 roses, 3 daffodils, and 2 sunflowers ?
4. Find the sum of all the numbers that can be formed using all the digits 2, 3, 3, 4, 4, 4.
5. There are 4n things of which n are alike and all the rest different. Find the number of permutations of 4n things taken 2n at a time, each permutation containing the n like things.
6. How many ways of tossing 12 dices are there in which each of the values 2, 3, 4, 5, 6 occurs twice ?
7. We toss a coin and assume a head is a success and a tail is a failure. How many trials will lead to 52 successes out of 100 tosses of the coin ?
8. How many seven digits numbers can be formed with the digits 1, 2, 2, 2, 3, 3, 5 ? How many of them are odd ?
9. A biologist is studying patterns of male(M) and female(F) children in families. A family type is designated by a code; for example, FMM. denotes a family of three children of which the oldest is a female and the other two males. Note that FMM, MFM, and MMF are different types. How many family types are there among families with atleast one but not more than seven children ?
10. The letters of the word SURITI are written in all possible orders and are written down as in a dictionary. Find the rank of the work SURITI.

4.9 PERMUTATION UNDER RESTRICTIONS

The number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$.

The number of permutations of n different things, taken r at a time, when a particular thing is never taken in each arrangement is ${}^{n-1}P_r$.

➤ **Example 1.** Consider the set {a, b, c, d, e}. How many three letter words can be made out of them, with or without meaning ? How many of these will have

www.pdfworld.net

www.pdfworld.net

atleast one vowel in them? Answer these questions for both cases when repetitions of letters are allowed and when repetitions of letters are not allowed ?

➤ ***Solution*** **Case I.** Repetitions allowed.

Number of all three letter words = 5 × 5 × 5 = 125; because each space in the three letter word can be filled up by any of the five letters.

To count the number of such words with vowels in them, let us calculate the complementary number, viz., the number of words without any vowel in them. This number is 3 × 3 × 3 = 27, because each blank space can be filled up only by b, c or d. Hence the number of words which contain at least one vowel is 125 –27 = 98.

Case II. Repetitions not allowed.

Number of such 3-letter words

= Number of permutations of 3 objects out of 5 objects

$= {}^5P_3 = 5.4.3 = 60.$

Number of these words which have at least one vowel in them = $60 - {}^3P_3 = 60 - 6 = 54.$

➤ **Example 2.** How many different words can be formed with the letters of the word 'PENCIL' when vowels occupy even places.

➤ ***Solution*** There are 6 letters in the word PENCIL and no letter is repeated. There are two vowels E and I.

1st	2nd	3rd	4th	5th	6th

Even places are : 2nd, 4th and 6th

Now there are three places for two vowels

∴ 3 vowels can be arranged in $^3P_2 = 3! = 6$ ways.

The four consonants can be arranged in the remaining four places in 4! = 24 ways.

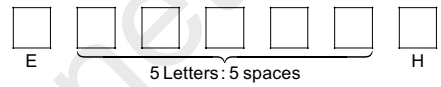
∴ The required number = 6 × 24 = 144

➤ **Example 3.** How many of the permutations of the word "ENGLISH" will (i) start with E ? (ii) end with H ? (iii) start with E and end with H ?

➤ ***Solution*** Thought this is not a direct application of the formula for n_r as in the previous example, a little study of the problem will show the connection. In part (i) we want the permutations to start with 'E'. In other words, of the 7 blank spaces we have to fill up with the letters of the word "ENGLISH", the first space allows no choice. It has to be filled up with 'E' only. So let us place 'E' in the first space. Having done this, we are left with only 6 letters and there are 6 places for them. This allows us 6! choices, since 6 letters can be permuted in 6! ways. Thus the answer to this part is 6! = 720

In part (ii) we have keep 'H' in the last blank space. Again this leaves a remainder of 6 letters and 6 places for them to go into. This can be done in 6! = 720 ways. So the answer to this part is also 720.

In part (iii) we have to keep 'E' in the first place and 'H' in the last place. Having done this, we are left with 5 letters and 5 spaces only:



this can be done in 5! ways. Each such way of arranging these 5 letters along with 'E' in the first place and 'H' in the last place gives us a required permutation for part (iii) of the problem. Hence the answer to this part is 5! = 120.

➤ **Example 4.** Three digit numbers in which the middle one is a perfect square are formed using the digits 1 to 9. Find their sum.

➤ ***Solution*** Middle place 1, 4 & 9

Two terminal positions 1, 2,....., 9

Hence total numbers = 9. 9. 3 = 243

For the middle place 1, 4 & 9 will come 81 times

$\Rightarrow$ sum = 81 × 10 (1 + 4 + 9) —— A

For units place each digit from 1 to 9 will appear 27 times

$\Rightarrow$ sum = 27 (1 + 2 +..... + 9) —— B

For hundreth's place, similarly sum = 27 × 10 (1 + 2 +..... + 9) —— C

A + B + C gives the required sum 134055.

String Method

Now, let's look at another kind of restriction that could be put upon the permutations. Consider the word NUMBER. We can easily see how many 6-letter code words can be made from this word. But

www.pdfworld.net

www.pdfworld.net

suppose it is asked how many permutations are there in which the two vowels (U and E) are next to each other ?

The best way of tackling this is to regard the two vowels as a single unit, UE (or EU). This now gives us five items to permute : N, UE, M, B and R. (This is like treating U and E as a single letter) This gives us $5 \times 4 \times 3 \times 2 \times 1 = 120$ permutations.

Altogether there will be $2 \times 120 = 240$ permutations in which the letters U and E are together (either UE or EU).

The number of permutations of n different things, taken all at a time, when m specified things always come together is $m ! \times (n - m + 1) !$.

➤ **Example 5.** A library has two books each having three copies and three other books each having two copies. In how many ways can all these books be arranged in a shelf so that copies of the same book are not separated.

➤ ***Solution*** Regarding all copies of the same book as one unit, we have only 5 units.

These 5 units can be arranged in 5! ways. But all copies of the same book being identical can be arranged in only one way.

$\therefore$ The required number = 5! = 120

➤ **Example 6.** In how many ways can 8 Indians, 4 Americans and 4 Englishmen be seated in a row so that all persons of the same nationality sit together.

➤ ***Solution*** Regarding all persons of the same nationality as one unit we have only three units.

These three units can be arranged in a row in 3! ways. But 8 Indians can be arranged among themselves in 8! ways, 4 Americans can be arranged among themselves in 4! ways and 4 Englishmen can be arranged among themselves in 4! ways.

$\therefore$ The required number = 3! 8! 4! 4!

➤ **Example 7.** How many different word can be formed with the letters of the word 'UNIVERSITY' so that all the vowels are together.

➤ ***Solution*** Total number of letters = 10; number of vowels = 4; I occurs twice.

Now, when 4 vowels are together, regarding the 4 vowels as one unit, we have only 6 + 1 = 7 letters. These 7 letters can be arranged in 7! ways.

Since I occurs twice, therefore, the four vowels can be arranged among themselves in $\frac{4!}{2!}$ ways.

$\therefore$ The required number = $7! \cdot \frac{4!}{2!} = 60480$

➤ **Example 8.** Five persons are to address a meeting. If a specified speaker is to speak before another specified speaker, find the number of ways in which this could be arranged. In how many of these arrangements will the first speaker come immediately before the second?

➤ ***Solution*** Let A, B be the corresponding specified speakers.

(i) Without any restriction the five persons can be arranged among themselves in 5! ways; but the number of ways in which A speaks before B and the number of ways in which B speaks before B is exactly equal to the number of ways in which B speaks before A.

$\therefore$ The required number of ways = $\frac{1}{2} \cdot 5! = 60.$

(ii) Regarding AB in that order as a single person, we can arrange them with the remaining three in 4! ways. Each of these arrangements corresponds to a way in which A speaks immediately before B.

$\therefore$ The required number of ways in this case = 4! = 24.

➤ **Example 9.** 10 IIT & 2 PET students sit in a row. Find the number of ways in which exactly 3 IIT students sit between 2 PET student.

➤ ***Solution*** 10 IIT students T_1, T_2,..... T_{10} can be arranged in 10 ! ways. Now the number of ways in which two PET student can be placed will be equal to the number of ways in which 3 consecutive IIT students can be taken i.e. in 8 ways and can be arranged in two ways The required number of ways = (10 !) (8) (2 !) = 16. 10 !.

Alternative

3 IIT student can be selected in $^{10}C_3$ ways. Now each selection of 3 IIT and 2 PET students in P_1 T_1 T_2 T_3 P_2 can be arranged in (2 !) (3 !) ways. Now

www.pdfworld.net

www.pdfworld.net

this group and the remaining 7 IIT students can be arranged in 8 ! ways.
The required number of ways = ${}^{10}C_3$ (2 !) (3 !) (8 !)

The number of permutations when all of certain given things are not to occur together.

In order to find the number of permutations when all of certain given things are not to occur together, find :

(i) the total number of arrangements when there is no restriction. Let this number be x.
(ii) Number of arrangements when all of the things (which are not to occur together) are together. Let this number be y.
(iii) The required number = x – y.

➤ **Example 10.** In a class of 10 students, there are 3 girls. In how many ways can they be arranged in a row such that all the three girls do not sit together.
➤ ***Solution*** We have 7 boys and 3 girls.
They can be seated in a row without restriction in 10! ways.
If the three girls are taken as one group, then the 7 boys and one group can be arranged in 8! ways while the girls can be arranged amongst themselves in 3! ways.
Thus, the number of ways in which the three girls sit together is

$= 8! \times 3!.$

Hence, the required number of ways

$= 10! - 8! \times 3!.$

➤ **Example 11.** Determine the number of permutations of n elements taken all at a time in which two given elements a and b are not adjacent.
➤ ***Solution*** Out of the total number n! of permutations we have to subtract the number of those in which the elements a and b are adjacent.
To form a permutation in which the elements a and b are adjacent we can take one of the permutations [whose number is (n – 2)!] containing the remaining n – 2 elements and add the two elements a and b to it so that they are adjacent.
This can be obviously done in 2(n – 1) ways (the factor 2 appears here because a and b can be interchanged).
Thus, the number of permutations in which a and b are adjacent is equal to 2(n – 2)! (n – 1), and the number we are interested in is equal to

$n! - 2(n-1)! = (n-1)!\,(n-2).$

Gap Method

The number of permutations when no two of a certain given things occur together

In order to find the number of permutations when no two of certain given things occur together.

(i) First of all put the m things on which there is no restriction in a line. These m things can be arranged in m! ways.
(ii) Then count the number of gaps between every two of m things on which there is no restriction including end positions. Number of such gaps will be (m + 1).
(iii) If m is the number of things on which there is no restriction and n the number of things, no two of which are to occur together, then the required number of ways = $m\,! \times {}^{m+1}C_n \times n\,!$

➤ **Example 12.** There are 10 candidates for an examination out of which 4 are appearing in Mathematics and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two mathematics candidates are together?
➤ ***Solution*** This question can be solved be gap method,
First, arrange the remaining candidates.
Here remaining candidates can be arranged in 6 ! ways.

x 0 x 0 x 0 x 0 x 0 x

x : places available for Mathematics candidates so that no two Mathematics candidates are together.
0 : places for others.

There are seven places available for Mathematics candidates.

Now 4 candidates can be placed in these 7 places in ${}^7P_4$ ways.
Hence the total number of ways
$= 6\,! \,.\, {}^7P_4 = 720 \times 840 = 604800.$

➤ **Example 13.** Of the 30! permutations of the integers 1, 2, 3,...., 30, how many have the property that multiples

www.pdfworld.net

www.pdfworld.net

of 3 are not in adjacent places, that is, no two of the integers 3, 6, 9, ..., 27, 30 are adjacent ?

➤ ***Solution*** Set aside the multiples of 3 momentarily; the other twenty integers can be permuted in 20! ways. Given any one of these permutations, there are 21 spaces between and at the ends of the integers. Choose 10 of these spaces to insert the multiples of 3; thus $^{21}C_{10}$ choices. But then the multiples of 3 can be inserted in 10! ways. So the answer is (20!). $^{21}C_{10}$. (10!) = (20!)(21!)/11!.

➤ **Example 14.** In how many ways can the letters AAABBCD be arranged so that

(i) the two B's are together but no two A's are together.

(ii) no two B's and no two A's are together.

➤ ***Solution*** (i) Considering the two B's as one single object, the letters BB, C, D can be arranged in 3! ways as shown below,

$\times BB \times C \times D \times$

while the three A's can be placed in any one of the four places marked by × in $^4C_3$ ways

Hence, the required number of arrangements

$= 3! \times {}^4C_3 = 24.$

(ii) The letters 2B's, 1C, 1D can be arranged in $\frac{4!}{2!}$ ways as

$\times B \times B \times C \times D \times$

while the three A's can be placed in any one of the five places marked by × in $^5C_3$ ways.

But this also includes those cases in which the 2B's are together.

Hence, the required number of arrangements

$= \frac{4!}{2!} \times {}^5C_3 - 24 = 120 - 24 = 96.$

➤ **Example 15.** Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

(i) do the words start with P

(ii) do all the vowels always occur together

(iii) do all the vowels never occur together

(iv) do the words begin with I and end in P ?

➤ ***Solution*** There are 12 letters, of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore the required number of arrangements = $\frac{12!}{3!4!2!} = 1663200$

(i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters. Therefore, the required number of words starting with P are = $\frac{11!}{3!2!4!} = 138600$

(ii) There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object (EEEEI) for the time being. This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2Ds, can be rearranged in $\frac{8!}{3!2!}$ ways. Corresponding to each of these arrangements, the 5 vowels E, E, E, E, I can be arranged in $\frac{5!}{4!}$ ways.

Hence the required number of arrangements = $\frac{8!}{3!2!} \times \frac{5!}{4!} = 16800$

(iii) The required number of arrangements
= The total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together.
= 1663200 –16800 = 1646400

(iv) Let us fix I and P at the extreme ends (I at the left end and P at the right end). We are left with 10 letters. Hence, the required number of arrangements

$= \frac{10!}{3!2!\,4!} = 12600$

The number of permutations when two types of things are to be arranged alternately

(i) If their numbers differ by 1 put the thing whose number is greater at first, third. fifth.... places etc. and other things at second, fourth, sixth.... places.

www.pdfworld.net

www.pdfworld.net

(ii) If the number of two types of things is same, consider two cases separately keeping first type of thing at first place, third, fifth places... etc and second type of things at first, third, fifth places.... and then add.

➤ **Example 16.** There are 5 boys and 3 girls. In how many ways can they be seated in a row so that all the three girls do not sit together.

➤ ***Solution*** Total number of persons = 5 + 3 = 8.
When there is no restriction they can be seated in a row in 8! ways.
But when all the three girls sit together, regarding the three girls as one unit
we have only 5 + 1 = 6 persons.
These 6 persons can be arranged in a row in 6! ways.
But the three girls can be arranged among themselves in 3! ways.
∴ The number of ways when three girls are together = 6! . 3!
∴ The required number of ways in which all the three girls do not sit together
= 8! – 6! . 3! = 6! (8.7 – 6) = 50.6! = 36000

➤ **Example 17.** In how many ways 4 boys and 4 girls can be seated in a row so that boys and girls are alternate?
Case I. When a boy sits at the first place : the arrangement will be of the form

B	G	B	G	B	G	B	G

Now there are four places namely Ist, 3rd, 5th and 7th for four boys, therefore four boys can be seated in 4! ways. Again there are four places namely 2nd, 4th, 6th and 8th for four girls, therefore four girls can be seated in 4! ways.
∴ The number of ways in this case = 4! 4!
Case II. When a girl sits at the first place the arrangement will be of the form

G	B	G	B	G	B	G	B

∴ The number of arrangements in this case = 4! 4!
∴ The required number = 4! 4! + 4! 4! = 2. 4! 4! = 1152

➤ **Example 18.** In how many ways 4 boys and 3 girls can be seated in a row so that they are alternate?
➤ ***Solution*** The arrangement will be the form

B	G	B	G	B	G	B

There are four places namely 1st, 3rd, 5th and 7th for four boys.
The four boys can be seated in 4! ways. Again there are three places namely 2nd, 4th and 6th for three girls. Therefore three girls can be seated in 3! ways.
∴ The required number = 4! 3! = 144

➤ **Example 19.** In how many ways can 4 boys and 3 girls be seated in a row so that boys and girls alternate.

➤ ***Solution*** B G B G B G B
The above arrangement is the only way the boys and girls can be seated.
Hence, the number of ways = 4! × 3!.

➤ **Example 20.** In how many ways can 4 boys and 4 girls be seated in a row so that boys and girls alternate .

➤ ***Solution*** **Case I.**
Boy occupies first place B G B G B G B G
The 4 boys and 4 girls have 4 places each to sit.
Thus, we have number of ways = 4! × 4!
Case II. Girl occupies first place G B G B G B G B
Again the number of ways = 4! × 4!
Hence, total number of ways = 2 × 4! × 4! = 1152.

➤ **Example 21.** How many words of three letters can be formed from the word 'Keppelin' a vowel being always in the middle ?

➤ ***Solution*** A three-letter word with e or i in the middle is to be formed. Excluding one 'e', there are 7 letters left of which 2 are p's
∴ when e is in the middle, the number of ways in which 2 different letters can occupy the end places is ${}^6C_2\ 2!$
And when e is in the middle, there is only one way in which 2 like letters can occupy the end places (pep).
∴ the number of three letter words with e in the middle = ${}^6C_2$. 2! + 1 = 31
Now, excluding 'i', there are 7 letters left, of which 2 are e's and 2 are p's .
∴ when i is in the middle, there are 2 ways in which 2 like letters can occupy the end places (pep, eie).
∴ the number of three letter words with i in the middle = ${}^5C_2$ 2! + 2 = 22
∴ the required number = 31 + 22 = 53.

www.pdfworld.net

www.pdfworld.net

➤ **Example 22.** How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places ?

The odd digits 1, 3, 3, 1 can be arranged in their four places in

$$\frac{4!}{2!2!} \text{ ways} \qquad ...(1)$$

The even digits 2, 4, 2 can be arranged in their three places in

$$\frac{3!}{2!} \text{ ways} \qquad ...(2)$$

Each of the ways in (1) can be associated with each of the ways in (2).

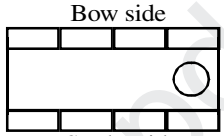
$$\text{Hence the required number} = \frac{4!}{2!\,2!} \times \frac{3!}{2!}$$

$$= 6 \times 3 = 18$$

➤ **Example 23.** An eight oared boat is to be manned by a crew chosen from 11 men of whom 3 can steer but cannot row and the rest cannot steer. In how many ways can the crew be arranged if two of the men can only row on bow side?

➤ ***Solution*** The total number of men = 11

The number of men who can only steer = 3



The number of other men = 8

The number of ways of selecting one man for steering out of 3 crew = $^3C_1$.

The number of ways in which two particular men who only row on bow side

can be arranged on bow side = $^4P_2$

The number of ways in which remaining 6 men can be arranged in remaining 6 places = 6!

$\therefore$ The required number = $^3C_1 \cdot {}^4P_2 \cdot 6!$

$= 3 \times 4 \times 3 \times 6! = 25920$

Alternative :

The number of ways in which one man out of 3 can be selected for steering = $^3C_1$

Since 2 particular persons are always to remain on bow sides therefore, 2 more persons for bow side can be selected out of remaining 6 persons in $^6C_2$ ways and 4 persons for stroke side can be selected out of remaining 4 in 1 way.

Now 4 persons on bow side can be arranged among themselves in 4! ways. Again 4 persons on stroke side can be arranged among themselves in 4! ways

$\therefore$ The required number = $^3P_1 \cdot {}^6C_2 \cdot {}^4C_4\ 4! \cdot 4!$

= 25920

➤ **Example 24.** How many seven letter words can be formed by using the letters of the word SUCCESS so that :

(a) the two C are together but no two S are together ?

(b) no two C and no two S are together ?

➤ ***Solution*** (a) Considering CC as a single object, U, CC, E can be arranged in 3 ! ways

$\times$ U $\times$ CC $\times$ E $\times$

Now the three S are to be placed in the four available places. Hence the required number of ways = 3 !. $^4C_3$ = 24.

(b) Let us first find the words in which no two S are together

(i) Arrange the remaining letters = $\frac{4!}{2!}$ = 12 ways

(ii) $\times$ U $\times$ C $\times$ C $\times$ E $\times$

There are five available places for three SSS.

Hence total no. of ways no two S together = $12 \times {}^5C_3$ = 120

$\therefore$ Hence no. of words having CC separated and SSS separated = 120 – 24 = 96.

Practice Problems [G]

1. In how many ways can 6 books be arranged on a shelf if (a) two particular books must be side by side and (b) if these two books must not be side by side ?
2. I have 2 copies of a statistics book, 2 copies of an algebra book, 3 copies of a calculus book, and 1 copy of a book on astronomy. In how many ways can I arrange these books on a shelf so that books of each topic are together ?

www.pdfworld.net

www.pdfworld.net

3. A collection of 30 volumes is on a book shelf. How many ways are there of arranging the series
(a) for volumes 1 and 2 to be side by side
(b) for volumes 3 and 4 not to be side by side

4. How many different four digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contains one digit 1?

5. In how many ways can the letters of the word PERMUTATIONS be arranged if the
(i) words start with P and end with S,
(ii) vowels are all together,
(iii) there are always 4 letters between P and S ?

6. In how many ways an the time-table for Monday be arranged if for this day five lessons are planned : in algebra, geometry, history, geography and literature, provided algebra and geometry do not immediately follow each other ?

7. How many five-digit numbers divisible by 4 can be formed by the digits 1, 2, 3, 4 and 5 if the
(i) digits can be repeated in the same number?
(ii) digits cannot be repeated in the same number?

8. How many five-digit numbers divisible by 6 can be made with the digits 0, 1, 2, 3, 4 and 5 if the digits cannot be repeated in the same number ?

9. How many five-digit telephone numbers with pairwise distinct digits can be composed ?

10. How many numbers of 5 digits divisible by 25 can be made with the digits 0, 1, 2, 3, 4, 5, 6 and 7?

11. Find the number of arrangements of the letters 'a b c d' in which neither a, b nor c, d come together.

12. A boat is to be manned by eight men, of whom 2 can only row on bow side and 1 can only row on stroke side. In how many ways can the crew be arranged ?

13. In a dinner party there are 10 Indians, 5 Americans and 5 Englishmen. In how many ways can they be arranged in a row so that all persons of the same nationality sit together ?

14. Six papers are set in an examination, 2 of them in mathematics. In how many different orders can the papers be given if two mathematics papers are not successive ?

15. In how many ways can the letters of the word 'PLANTAIN' be arranged so that the two 'A's do not come together ?

16. In how many different ways can the letters of the word 'SALOON' be arranged if the consonants and vowels must occupy alternate places ?

17. How many words can be formed by using the letters of the word 'BHARAT' ? How many of these words will not contain B and H together. How many of these start with B and end with T ?

18. There are nine different books on a shelf; four are red and five are green. In how many different orders is it possible to arrange the books on the shelf if
(a) there are no restrictions;
(b) the red books must be together and the green books together;
(c) the red books must be together whereas the green books may be, but need not be, together;
(d) the colors must alternate, i.e., no two books of the same color may be adjacent ?

4.10 NON-CONSECUTIVE SELECTION

The number of selections of r consecutive things out of n things in a row
$= n - r + 1$
The number of selections of r consecutive things out of n things along a circle

$$= \begin{cases} n, \text{ when } r < n \\ 1, \text{ when } r = n \end{cases}$$

For example, the selections of 2 consecutive things out of 5 things A, B, C, D, E
along a circle are : AB, BC, CD, DE, EA
The number of such selections = 5.
The selections of 3 consecutive things are :
ABC, BCD, CDE, DEA, EAB
The number of such selections = 5

www.pdfworld.net

www.pdfworld.net

The selection of 5 consecutive things out of 5 things along a circle is ABCDE

The number of such selection = 1

➤ **Example 1.** A straight is a five card hand containing consecutive values. (i) How many different straights are there? (ii) If the cards are not all from the same suit, then how many straights are there ?

➤ ***Solution*** (i) The number of types of cards (number of suits) = 4 (Heart, Club, Spade, Diamond)

The number of consecutive cards in each suit = 13

The number of ways of selecting 5 consecutive cards out of 13 from one suit

$= n - r + 1 = 13 - 5 + 1 = 9$

But one card of any denomination can be selected from any one of the 4 suits in 4 ways

$\therefore$ 5 cards of five different denominations can be selected from 4 suits in 4^5 ways

$\therefore$ The number of ways in which 5 cards of five consecutive denominations can be selected.

$= 9 \times 4^5 = 9 \times 1024 = 9216$

(ii) Total number of ways of selecting 5 consecutive cards = 9216

The number of ways of selecting 5 consecutive cards from one suit = 9

$\therefore$ The number of ways of selecting 5 consecutive cards from the same suit $= 4 \times 9 = 36$

$\therefore$ The required number $= 9216 - 36 = 9180$.

➤ **Example 2.** n different objects are arranged in a row. In how many ways can 3 objects be selected so that

(i) all the three objects are consecutive.

(ii) all the three objects are not consecutive.

➤ ***Solution*** (i) If the n objects are $a_1, a_2, a_3, \ldots a_n$ then possible selection will be

$a_1 a_2 a_3 , a_2 a_3 a_4 , a_3 a_4 a_5, \ldots\ldots a_{n-2}a_{n-1}a_n$

$\therefore$ The required number $= n - 3 + 1 = n - 2$

(ii) Total number of selection of 3 objects when there is no restriction $= {}^nC_3$

The number of selections of 3 objects which are consecutive $= n - 2$

$\therefore$ The required number $= {}^nC_3 - (n-2)$

$$= \frac{n(n-1)(n-2)}{6} - (n-2) = \frac{(n-2)}{6}(n^2 - n - 6)$$

$$= \frac{(n-2)(n-3)(n+2)}{6} = \frac{(n-3)(n^2-4)}{6}$$

➤ **Example 3.** There are n intermediate stations on a railway line from one terminus to another. In how many ways can the train stop at 3 of these intermediate stations if

(i) all the three stations are consecutive.

(ii) atleast two of the stations are consecutive.

➤ ***Solution*** Let the intermediate stations be $S_1, S_2, S_3, \ldots\ldots\ldots\ldots S_{n-2}, S_{n-1}, S_n$

(i) The number of triplets of consecutive stations, viz. $S_1S_2S_3$, $S_2S_3S_4$, $S_3S_4S_5$, ... $S_{n-2}S_{n-1}S_n$, is $(n-2)$

(ii) The total number of consecutive pair of stations, viz.

$S_1S_2, S_2S_3, \ldots\ldots\ldots\ldots, S_{n-1}S_n$ is $(n-1)$.

Each of the above pair can be associated with a third station in $(n-2)$ ways. Thus, choosing a pair of stations and any third station can be done in $(n-1)(n-2)$ ways. The above count also includes the case of three consecutive stations. However, we can see that each such case has been counted twice. For example, the pair S_4S_5 combined with S_6 and the pair S_5S_6 combined with S_4 are identical.

Hence, subtracting the excess counting, the number of ways in which three stations can be chosen so that at least two of them are consecutive

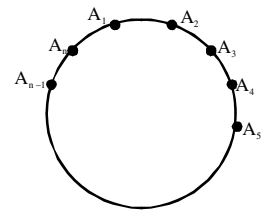
$= (n-1)(n-2) - (n-2) = (n-2)^2$

➤ **Example 4.** Triangles are formed from the vertices of an n-sided polygon. Find the number of triangles so that

(i) atleast one side of the triangle coincides with the side of the polygon.

(ii) no side of the triangle coincides with the side of the polygon.

➤ ***Solution*** (i) The total number of consecutive pair of vertices, viz.



www.pdfworld.net

www.pdfworld.net

$A_1A_2, A_2A_3, A_3A_4, \ldots\ldots\ldots\ldots, A_nA_1$, is n

Each of the above pair of vertices can be associated with a third vertex viz.

$A_1A_2A_3, A_1A_2A_4, A_1A_2A_5, \ldots\ldots\ldots\ldots, A_1A_2A_{n-1}$, in n – 3 ways.

Note that $A_1A_2A_n$ has not been counted since it will be counted when we write $A_n A_1 A_2$. Thus, choosing a pair of vertices and any third vertex can be done in n(n – 3) ways.

Hence, the number of triangles so that atleast one of its sides coincides with the sides of the polygon, is = n(n – 3)

(ii) Without restriction, the number of triangles = ${}^nC_3$. Hence, the number of triangles so that none of its side coincides with the sides of the polygon

$= {}^nC_3 - n(n-3)$

➤ **Example 5.** There are eight stations on a train route in the circular order. Find the number of selection of 3 stations such that no two stations are adjacent ?

➤ ***Solution*** If the stations $S_1, S_2, \ldots, S_8$ are arranged in a circular order and we have to select three stoppages then a particular station say S_1 will either be selected or not selected.

If S_1 is selected then S_2 and S_8 (which are adjacent to S_1) cannot be selected.

Thus we have a line S_3, S_4, S_5, S_6, S_7 from which we have to select 2 objects such that no two objects are adjacent which is equivalent to arranging 2 identical red balls and 5 – 2 = 3 identical white balls such that no two red balls are together. This is clearly ${}^4C_2$.

Now the first stopping station can be selected in ${}^8C_1 = 8$ ways. But a selection like S_1, S_3, S_7 will be counted thrice starting with S_1 or S_3 or S_7.

Thus the required number of selections is $({}^8C_1 \times {}^4C_2)/3 = 16$.

➤ **Example 6.** How many 7 digit numbers are there such that the digits are distinct integers taken from the set S = {1, 2, ..., 9} and such that the integers 5 and 6 do not appear consecutively in either order ?

➤ ***Solution*** We first find the number n_1 of 7-permutations of the 9-element set S which do contain 56 or 65. Consider 7 places as follows :

a b c d e f g

There are 6 pairs of consecutive places, namely ab, bc, ..., fg in which we can place 56 or 65. Thus each of 56 and 65 can be placed in 6 ways. In each case there are ${}^7P_5$ ways of filling the remaining 5 places. Hence $n_1 = 2 \times 6 \times {}^5P_7$. Also, the number of 7-permutations of S is $n_2 = {}^7P_9$. So the number of required permutations is $n_2 - n_1 = 151200$.

Practice Problems [H]

1. A shelf contains 20 books, of which of 4 are single volumes and the others are books of 8, 5 and 3 volumes respectively. In how many ways can the books be arranged on the shelf so that volumes of the same work are not separated ?
2. In how many ways can 16 rupees and 12 paise coins be arranged in a line so that no two paise coins may occupy consecutive positions ?
3. There are m intermediate stations on a railway line between two places A and B. In how many ways a train can stop at 3 of these intermediate stations no two of which are consecutive.
4. How many quadrilaterals can be made by joining the vertices of a convex polygon of 20 sides so that the quadrilaterals do not have any side common with the sides of the polygon ?
5. Let P_n denotes the number of ways in which three people can be selected out of 'n' people sitting in a row, if no two of them are consecutive.
   If, $P_{n+1} - P_n = 15$ then find the value of 'n'.

4.11 COMBINATION UNDER RESTRICTIONS

If a problem of selection of r object out of n given objects involves some condition or restriction then we can't apply the formula ${}^nC_r$ directly.

Some important situations of restricted combination are as follows :

1. Number of ways of choosing r things out of n different things if p particular things must be excluded.

Consider the objects $A_1, A_2, A_3, \ldots\ldots, A_p, A_{p+1}, \ldots\ldots, A_n$

If the p objects $A_1, A_2, \ldots, A_p$ are to be excluded then

www.pdfworld.net

www.pdfworld.net

we will have to select r objects from the remaining n – p objects $A_{p+1}, A_{p+2}, \ldots A_n$
Hence the required number of ways = $^{(n-p)}C_r$

2. Number of ways of choosing r things out of n different things if p particular things must be included ($p \le r$).

Consider the things $A_1, A_2, A_3, \ldots, A_p, A_{p+1}, \ldots, A_n$
If the p particular things $A_1, A_2, \ldots A_p$ (say) must be included in the selection then to complete the selection, we must select (r – p) more things to complete the selection. Since, these things are to selected from the remaining n – p things
$(A_{p+1}, A_{p+2}, \ldots, A_n)$
Hence the required number of ways = $^{n-p}C_{r-p}$

➤ **Example 1.** How many different selections of 6 books can be made from 11 different books, if
(i) Two particular books are always selected ;
(ii) Two particular books are never selected ?

➤ ***Solution*** (i) Since two particular books are always selected, it means that 6 – 2 = 4 books are selected out of the remaining 11 – 2 = 9 books.

$\therefore$ The required number of ways = $^9C_4 = \dfrac{9.8.7.6}{1.2.3.4}$

$= 126$

(ii) Since two particular books are never selected. It means that 6 books are selected out of the remaining 11 – 2 = 9 books.

$\therefore$ The required number of ways = $^9C_6 = {}^9C_3$

$= \dfrac{9.8.7}{1.2.3} = 84.$

➤ **Example 2.** An urn contains 3 red pens, 4 green pens and 6 yellow pens. Find number of ways of drawing 4 pens from the urn if atleast one red pen is to be included in the draw (All the pens are different from each other)

➤ ***Solution*** The required number of ways
= Total number ways choosing 4 pens – number of ways of choosing 4 non-red pens
$= {}^{3+4+6}C_4 - {}^{4+6}C_4 = {}^{13}C_4 - {}^{10}C_4$

$= \dfrac{13.12.11.10}{4.3.2.1} - \dfrac{10.4.8.7}{4.3.2.1} = 715 - 210 = 505$

➤ **Example 3.** There are m white and n black balls in an urn. In how many ways can r balls be selected from the urn of which k balls will be white ? (It is assumed that the balls of each colour are distinct)

➤ ***Solution*** There are ^mC_k ways of selecting k white from m white balls and $^nC_{r-m}$ ways of selecting the remaining r – k black balls from the group of n black balls. Every way of selecting the k white balls is associated with the $^nC_{r-k}$ different ways of selecting the black balls. Consequently, the total number of different samples is equal to the product $^mC_k \, {}^nC_{r-k}$.

3. The total number of combinations of n different things taken one or more at a time = $2^n - 1$.

The first thing may or may not be selected.
$\therefore$ The number of ways for first thing = 2
Similarly, number of ways for second thing = 2
The number of ways for the third thing = 2
.....
.....
The number of ways for nth thing = 2
Hence by multiplication rule the total number of ways = $2 \times 2 \times 2 \times \ldots$ n times = 2^n
But there is one way in which nothing is selected.
$\therefore$ Total number of ways of selecting one or more of n different things = $2^n - 1$.

Second Method :

The number of ways of selecting one thing out of n different things = $^nC_1$
The number of ways of selecting two things out of n different things = $^nC_2$
The number of ways of selecting three things out of n different things = $^nC_3$
.....
.....
The number of ways of selecting n things out of n different things = ^nC_n
Hence total number of ways of selecting one or more things out of n different things
$= {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$
$= ({}^nC_0 + {}^nC_1 + \ldots + {}^nC_n) - {}^nC_0$
$= 2^n - 1$

➤ **Example 4.** A class has n students. We have to form a team of the students including atleast two students and also excluding atleast two students. Find the number of ways of forming the team.

➤ ***Solution*** The required number of ways

www.pdfworld.net

www.pdfworld.net

$= {}^nC_2 + {}^nC_3 + ... + {}^nC_{n-3} + {}^nC_{n-2}$
$= 2^n - ({}^nC_0 + {}^nC_1 + {}^nC_{n-1} + {}^nC_n$
$= 2^n - (1 + n + n + 1)$
$= 2^n - 2(n + 1)$

➤ **Example 5.** A man has 6 friends. In how many ways he may invite one or more of them to dinner ?

➤ ***Solution*** He has to select some or all of his 6 friends, and therefore the number of selections

$= {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$
$= 6 + 15 + 20 + 15 + 6 + 1 = 63.$

Alternatively, the required number of ways = $2^6 - 1 = 63$.

➤ **Example 6.** A question paper consists of two sections having respectively 3 and 5 questions. It is not necessary to attempt all the questions. One question from each section is compulsory. In how many ways can a candidate select the questions ?

➤ ***Solution*** Here we have two sections A and B (say), the section A has 3 questions and section B has 5 questions and one question from each section is compulsory.

$\therefore$ The number of ways selecting one or more questions from section A is $2^3 - 1 = 7$.

and the number of ways selecting one or more questions from section B is

$2^5 - 1 = 31$

Hence, by multiplication rule, the required number of ways in which a candidate can select the questions

$= 7 \times 31 = 217.$

➤ **Example 7.** Given 5 different green dyes, 4 different blue dyes and 3 different red dyes. How many combinations of dyes can be chosen, taking atleast one green and one blue dyes.

➤ ***Solution*** One or more green dyes can be selected out of 5 different green dyes in

$= 2^5 - 1 = 31$ ways.

One or more blue dyes can be selected out of 4 different blue dyes in

$= 2^4 - 1 = 15$ ways.

Zero or more red dyes can be selected out 3 different red dyes in $2^3 - 1 = 8$ ways.

$\therefore$ The required number $= 31 \times 15 \times 8 = 3720.$

➤ **Example 8.** A student is allowed to select atmost n books from a collection of (2n + 1) books. If the total number of ways in which he can select atleast one book is 63, find the value of n.

➤ ***Solution*** The student can select one book or two book or three book or ..... or n books.

Hence ${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n = 63$

But we know that

${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_{2n+1} = 2^{2n+1}$

Now ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$

then $2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n)$
$+ ({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + {}^{2n+1}C_{n+3} + ... + {}^{2n+1}C_{2n}) = 2^{2n+1}$

$\Rightarrow \quad 2 + 2\,({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n) = 2^{2n+1} \quad (\because {}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r})$

$\Rightarrow \quad 2 + 2.63 = 2^{2n+1} \quad \Rightarrow \quad 1 + 63 = 2^{2n}$

$\Rightarrow \quad 64 = 2^{2n} \quad \Rightarrow \quad 2^6 = 2^{2n}$

$\therefore \quad n = 3$

Further examples

➤ **Example 9.** How many words of 4 different letters can be formed out of 7 capital letters, 3 vowels and 5 consonants if each word starts with a capital letter and contains atleast one vowel.

➤ ***Solution*** The first letter of each word must be a capital letter.

Now following cases are possible.

No. of capital letters	No. of vowels	No. of consonants	No. of words
1	1	2	${}^7C_1 \times {}^3C_1 \times {}^5C_2 \times 3! = 1260$
1	2	1	${}^7C_1 \times {}^3C_2 \times {}^5C_1 \times 3! = 630$
1	3	0	${}^7C_1 \times {}^3C_3 \times {}^5C_0 \times 3! = 42$

➤ **Example 10.** In how many ways can we form a committee of three from a set of 10 men and 8 women, such that our committee consists of at least one woman ?

➤ ***Solution*** Whenever the requirement is that there should be at least something happening, it is a piece of strategy to calculate the number associated with the complementary happening. Here, for example, we want at least one woman member in our

www.pdfworld.net

www.pdfworld.net

committee. What if we calculated the number of ways in which we could from a committee with no woman member in it ? This means all the 3 members have to come from the set of 10 men only. Out of 10 men we can select three in

$^{10}C_3 = 120$ ways

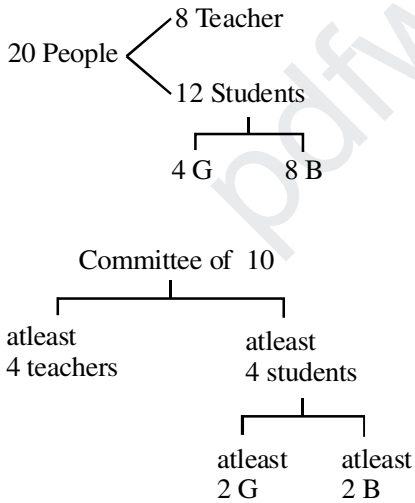
ways. So this is the complementary number. This has to be subtracted from the total number of ways in which a committee of three can be formed, without any man-woman restriction. There are 18 people. We want to select three. This can be done in

$^{18}C_3 = 816$ ways

ways. Thus the required number is 816 – 120 = 696. This is the number of ways in which a 3-member committee can be formed from 10 men and 8 women, with at least one woman member in the committee.

➤ **Example 11.** A committee of 10 is to be formed from 8 teachers and 12 students of whom 4 are girls . In how many ways this can be done so that the committee contains atleast four of either groups (teachers and students) and atleast 2 girls and atleast 2 boys are in the committee.

➤ ***Solution***



We can have

(i) 4 teachers + 6 students ⇒ 4 T and [(2 G, 4 B) + (3 G + 3 B) + (4 G + 2 B)]

(ii) 5 teachers + 5 students ⇒ 5 T and [(2 G, 3 B) + (3 G + 2 B)]

(iii) 6 teachers + 4 students ⇒ 6 T and (2 G, 2 B)

Now (i) + (ii) + (iii)

$= {}^8C_4\,[{}^4C_2 \cdot {}^8C_4 + {}^4C_3 \cdot {}^8C_3 + {}^4C_4 \cdot {}^8C_2] + {}^8C_5\,[{}^4C_2 \cdot {}^8C_3 + {}^4C_3 \cdot {}^8C_2] + {}^8C_2\,[{}^4C_2 \cdot {}^8C_2]$

$= 47040 + 25088 + 4704 = 76832$

➤ **Example 12.** In how many ways, we can choose two teams of mixed double for a tennis tournament from four couples such that if any couple participates, then it is in the same team.

➤ ***Solution*** **Case I :** No couple is chosen

we can choose 2 men in $^4C_2$ ways and hence two team can be formed in $2 \times {}^4C_2$ ways.

Case II : When only 1 couple is chosen.

A couple can be chosen in $^4C_1$ ways and the other team can be chosen in $^3C_1 \times {}^2C_1$ ways.

Hence two teams can be formed in $^4C_1 \times {}^3C_1 \times {}^2C_1$ ways.

Case III : When two couples are chosen then team can be chosen in $^4C_2$ ways.

Hence total teams

$= 2 \times {}^4C_2 + {}^3C_1 \times {}^3C_1 \times {}^2C_1 \times {}^4C_1 + {}^4C_2 = 42.$

➤ **Example 13.** The members of a chess club took part in a round robin competition in which each plays every one else once. All members scored the same number of points, except four juniors whose total score were 17.5 . How many members were there in the club? Assume that for each win a player scores 1 point, for draw $\frac{1}{2}$ point and zero for losing.

➤ ***Solution*** Let the number of members are n

Total number of points $= {}^nC_2$.

$\therefore\ {}^nC_2 - 17.5 = (n-4)\,x$ (where x is the number of point scored by each player)

$n(n-1) - 35 = 2(n-4)\,x$

$$2x = \frac{n(n-1) - 35}{n-4}$$

(where x takes the values 0.5, 1, 1.5 etc)

www.pdfworld.net

www.pdfworld.net

$$= \frac{n^2 - n - 35}{n - 4} \text{ (must be an integer)}$$

$$= \frac{n(n-4) + 3(n-4) - 23}{n-4} = (n+3) - \frac{23}{n-4}$$

$\Rightarrow \quad \frac{23}{n-4}$ must be an integer

$\Rightarrow \quad n = 27$ is the only possibility.

➤ **Example 14.** In how many ways can a team of 6 horses be selected out of a stud of 16, so that there shall always be 3 out of $A\,B\,C\,A'\,B'\,C'$, but never $A\,A'$, $B\,B'$ or $C\,C'$ together.

➤ ***Solution*** Stud means establishment or a farm in which horses are kept by an owner.

16 horses $<$ $A\,B\,C\,A'\,B'\,C'$ / 10 others

Number of ways $= {}^{10}C_3 \times$ number of ways of choosing 3 out of $A\,B\,C\,A'\,B'\,C'$ so that $A\,A'$, $B\,B'$ or $C\,C'$ are not together

$= {}^{10}C_3\,[\,{}^6C_3 - {}^3C_1 \,.\, {}^4C_1\,]$

$= {}^{10}C_3\,[\,20 - 12\,] = {}^{10}C_3 \,.\, 8$

$= \frac{10\,.\,9\,.\,8}{1\,.\,2\,.\,3} \,.\, 8 = 960\,.$

Practice Problems [I]

1. Mohan has 8 friends. In how many ways he can invite one or more of them to dinner ?
2. Set A consists of 3 elements while set B consists of 5 elements. A set S is to be formed by choosing elements from set A and set B. How many such sets can be found if each set must consist of atleast one element each from set A and set B.
3. A question paper on mathematics consists of twelve questions divided into three parts A, B and C, each containing four questions. In how many ways can an examinee answer five questions, selecting atleast one from each part.
4. A woman has 11 colleagues in her office, of whom 8 are men. She would like to have some of her colleagues to dinner. Find the number of her choices, if she decides to invite
   (i) atleast 9 of them (ii) all her women colleagues and sufficient men colleagues to make the number of women and men equal.
5. There are 5 pair of different shoes. In how many ways can each of 5 persons be distributed a pair so that atleast two persons get a complete pair.
6. In how many ways can a football team of 11 players be selected from 16 players? How many of these will (i) include 2 particular players? (ii) exclude 2 particular players?
7. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.
8. A student has to answer 10 questions, choosing atleast 4 from each of part A and part B. If there are 6 questions in part A and 7 in part B, in how many ways can the student choose 10 questions?
9. For an examination a candidate has to select 7 subjects from 3 different groups A, B, C which contain 4, 5, 6 subjects respectively. In how many different ways can a candidate make his selection if he has to select atleast 2 subjects from each group?
10. A bag contains 2 white and 3 red cubes, all of different sizes. In how many ways can 3 cubes be selected from the bag if (a) atleast one cube is white, (b) atleast one cube is red?
11. In how many ways can a ballot be validly marked if a citizen is to choose one of three candidates for mayor, one of four for city councilman, and one of three for district attorney. A citizen is not required to vote for all three positions, but he is expected to vote for atleast one.

www.pdfworld.net

www.pdfworld.net

12. A city council is composed of 5 liberals and 4 conservatives. Three members are to be selected randomly as delegates to a convention.
(a) How many delegations are possible?
(b) How many delegations could have all liberals ?
(c) How many delegations could have 2 liberals and 1 conservative ?
(d) If 1 member of the council serves as mayor, how many delegations are possible that include the mayor ?

13. A double-decker bus has 5 empty seats in the upstairs and 5 empty seats in the down stair. 10 people board the bus of which 2 are old people and 3 are children. The children refuse to take seats down stair while old people insist to say down stair. In how many different arrangements can the 10 people take seats in the bus?

14. There are 5 professors (P_1, P_2,....., P_5) and 10 students (S_1, S_2, ..., S_{10}) out of whom a committee of 2 professors and three students is to be formed. Find the number of ways in which this can be achieved ? Further find the number of committees in which P_1 and P_2 can not serve together and S_1 will serve only if S_2 is present in the committee.

4.12 COMBINATION OF ALIKE OBJECTS

1. The number of combinations of n identical things taking ($r \le n$) at a time is 1.

How many selections of 5 balls can be made from 12 red balls.
Here balls are identical it means that all balls are same colour then the number of selection of 5 balls out of 12 is 1.

2. The number of ways of selecting r things out of n identical things is n + 1.

How many combinations are there of 7 identical books ?
The books can be selected as 1 book or 2 books or 3 books or ..... or 7 books.
Hence number of combinations
$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$.

3. To find the total number of selections of one or more things from p identical things of one type, q identical things of another type, r identical things of the third type and n different things :
Since number of ways of selecting r things out of n identical things = 1 for all $r \le n$.
Hence the number of ways of selecting one thing out of p identical things = 1
the number of ways of selecting two things out of p identical things = 1
.....
.....
the number of ways of selecting p things out of p identical things = 1
the number of ways of selecting zero thing out of p identical things = 1
hence number of ways of selecting zero or more things out of p identical things = p + 1
similarly, number of ways of selecting zero or more things out of q things = r + 1
Also number of ways of selecting zero or more things out of n different things = $2 \times 2 \times 2 \times ....$ n times = 2^n
Therefore, number of ways of selecting zero or more things out of given things
$= (p + 1)(q + 1)(r + 1)2^n$
But number of ways of selecting zero thing out of given things = 1
Thus total number of ways of selecting one or more things out of given things
$= (p + 1)(q + 1)(r + 1)\, 2^n - 1$.

If out of (p + q + r + s) things, p are alike one kind, q are alike of second kind, r are alike of third kind and s are different, then the total number of combinations is
$$(p + 1)(q + 1)(r + 1)\, 2^s - 1$$

The list of alike things can be extended further.

➤ **Example 1.** Find the number of ways in which one or more letters can be selected from the collection
AAAAA BBBB CCC DD EFG.

➤ ***Solution*** Here 5A's are alike, 4B's are alike, 3C's are alike, 2D's are alike and E, F, G are different,
The selection from (AAAAA) can be made in 6 ways: include no A, include one A, include two A, include three A, include four A, include five A. Similarly selections from (BBBB) can be made in 5 ways, selections from (CCC) can be made in 4 ways,

www.pdfworld.net

www.pdfworld.net

selections from (DD) can be made in 3 ways, selection from E, F, G can be made in $2 \times 2 \times 2$ ways.
Then the total number of combinations
$= (5 + 1)(4 + 1)(3 + 1)(2 + 1)\,2^3 - 1$
$= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 8 - 1$ (excluding the case when no letter is selected) $= 2879$

➤ **Example 2.** There are 4 oranges, 5 apples and 6 mangoes in a fruit basket. In how many ways can a person select fruits from among the fruits in the basket ?
➤ ***Solution*** Here we consider all fruits of the same type as identical.
Zero or more oranges can be selected out of 4 identical oranges in $4 + 1 = 5$ ways.
Zero or more apples can be selected out of 5 identical apples in $5 + 1 = 6$ ways.
Zero or more mangoes can be selected out of 6 identical mangoes in $6 + 1 = 7$ ways
$\therefore$ The total number of selections when all the three types of fruits are selected
(the number of any type of fruit may be zero)
$= 5 \times 6 \times 7 = 210.$
But in one of these selections number of each type of fruit is zero and hence there is no selection, this must be excluded.
$\therefore$ The required number $= 210 - 1 = 209.$

CAUTION When all fruits of same type are different, the number of selections

$$= ({}^4C_0 + {}^4C_1 + ... + {}^4C_4)({}^5C_0 + {}^5C_1 + ... + {}^5C_5)$$
$$({}^6C_0 + ... + {}^6C_6) - 1$$
$$= 2^4 \cdot 2^5 \cdot 2^6 - 1 = 2^{15} - 1$$

➤ **Example 3.** Find the number of ways in which one or more letter be selected from the letters AAAAEE I II OU.
➤ ***Solution*** The number of ways of
selecting A $= 4 + 1 = 5$
selecting E $= 2 + 1 = 3$
selecting I $= 3 + 1 = 4$
selecting O $= 1 + 1 = 2$
selecting U $= 1 + 1 = 2$
Hence, total ways of selection $= 5 \times 3 \times 4 \times 2^2 = 240$.
But the above count also includes the one count in which no letter is selected. Since, we are supposed to find the number of ways in which atleast one letter is selected, therefore the required number of ways $= 240 - 1 = 239.$

➤ **Example 4.** Find the number of combinations that can be formed with 5 oranges, 4 mangoes and 3 bananas when (i) atleast one fruit is taken (ii) atleast one fruit of each kind is taken.
➤ ***Solution*** (i) The required number of combinations (when atleast one fruit)
$= (5 + 1)(4 + 1)(3 + 1) - 1$
$= 120 - 1 = 119$
(ii) The required number of combinations (when atleast one fruit of each kind)
$= 5 \times 4 \times 3 = 60.$

4. The number of ways in which r things can be selected from a group of n things of which p are identical, is

$$\sum_{0}^{r} {}^{n-p}C_r \text{ if } r \le p \quad \text{and} \quad \sum_{r=p}^{r} {}^{n-p}C_r \text{ if } r > p.$$

➤ **Example 5.** A bag contains 15 balls of which 6 are identical. Find the number of ways of selecting 8 balls.
➤ ***Solution*** We have to select 8 balls from a set of 6 identical balls and 9 different balls.
The balls can be selected in the following ways
6 identical and 2 different in $1 \times {}^9C_2$ ways
or 5 identical and 3 different in $1 \times {}^9C_3$ ways
or 4 identical and 5 different in $1 \times {}^9C_4$ ways
:
or 0 identical and 8 different in $1 \times {}^9C_8$ ways
Hence, the required number of ways
$= {}^9C_2 + {}^9C_3 + {}^9C_4 + {}^9C_5 + {}^9C_6 + {}^9C_7 + {}^9C_8$
$= 2^9 - [\,{}^9C_0 + {}^9C_1 + {}^9C_9\,]$
$= 512 - 11 = 501.$

➤ **Example 6.** Find the number of ways in which (i) a selection, (ii) an arrangement, of four letters can be made from the letters of the word PROPORTION.
➤ ***Solution*** There are 10 letters of six different types, namely O, O, O, ; P, P, ; R, R; T; I; N.
In finding groups of four these may be classified as follows :
(i) Three alike, one different. The selection can be made in 5 ways: for each of the five letters, P, R,

www.pdfworld.net

www.pdfworld.net

T, I, N, can be taken with the single group of the three like letters O.

(ii) Two alike, two others alike. The selection can be made in ${}^3C_2$ ways; for we have to choose two out of the three pairs O, O; P, P; R, R. This gives 3 selections.

(iii) Two alike, the other two different. The selection can be made in 3×10 ways; for we select one of the 3 pairs and then two from the remaining 5 letters. This gives 30 selections.

(iv) All four different. This selection can be made in ${}^6C_4$ ways as we have to take 4 different letters to choose from the six O, P, R, T, I, N. This gives 15 selections. Thus the total number of selections is 5 + 3 + 30 + 15, that is 53.

In finding the different arrangements of 4 letters we have to permute in all possible ways each of the foregoing groups.

(i) gives rise to $5 \times \frac{4!}{3!}$, or 20 arrangements.

(ii) gives rise to $3 \times \frac{4!}{2!2!}$, or 18 arrangements

(iii) gives rise to $30 \times \frac{4!}{2!}$, or 360 arrangements

(iv) gives rise to $15 \times 4!$, or 360 arrangements.

Thus the total number of arrangements is 20 + 18 + 360 + 360, that is 758.

➤ **Example 7.** How many four digit numbers are there whose decimal notation contains not more than two distinct digits ?

➤ ***Solution*** Evidently any number so formed of four digits contains

(i) only one digit (like 1111, 2222,....) and there are 9 numbers.

(ii) two digits

(a) if zero is one of the two, then the one more can be anyone of the nine, and these two digits can be arranged in
${}^9C_1\,[{}^3C_1 + {}^3C_2 + {}^3C_3] = 63$

(b) if zero is not one of them, then two of the digits have to be selected from 9, and these two can be arranged in
${}^9C_2\,[{}^4C_1 + {}^4C_2 + {}^4C_3] = 504$

Hence the total number of required numbers = 576.

Practice Problems [J]

1. Three passes are allotted to five students. In how many ways can they be distributed among the students if (a) all the passes are different (b) all the passes are identical ?

2. How many different sums of money can be formed with coins, one rupee coin, one 50 paise coin, one 25 paise coin and one 10 paise coin ?

3. A fruit basket contains 4 oranges, 5 apples and 6 mangoes. In how many ways can a person make a selection from the basket?

4. From 3 mangoes, 4 apples and 2 oranges, how many selections of fruit can be made, taking atleast one of each kind?

5. There are 5 mangoes and 4 apples. In how many different ways can a selection of fruits be made if

(i) fruits of the same kind are different

(ii) fruits of the same kind are identical?

6. Messages are conveyed by arranging 4 white, 1 blue and 3 red flags on a pole. Flags of the same colour are alike. If a message is transmitted by the order in which the colours are arranged then find the total number of messages that can be transmitted if exactly 6 flags are used.

7. Find the number of selections and of arrangements that can be made by taking 4 letters from word EXPRESSION.

4.13 DIVISION INTO GROUPS

1. The number of ways in which $(m + n)$ different things can be divided into two unequal groups containing m and n things respectively is

$$\frac{(m+n)!}{m!\,n!}.$$

www.pdfworld.net

www.pdfworld.net

If m = n, the groups are equal and in this case the number of division is $\frac{(2n)!}{n!n!2!}$; for in any one way it is possible to interchange the two groups without obtaining a new distribution.

2. However, if 2n things are to be divided equally between two persons then the number of ways

$$= \frac{(2n)!}{n!n!2!}2! = \frac{(2n)!}{n!n!} .$$

3. The number of ways in which (m + n + p) different things can be divided into three unequal groups containing m , n and p things respectively is $\frac{(m+n+p)!}{m!\,n!\,p!}$, $m \neq n \neq p$.

If $m = n = p$ then the number of groups $= \frac{(3n)!}{n!n!n!3!}$.

However , if 3n things are to be divided equally among three persons then the number of ways

$$= \frac{(3n)!}{n!n!n!3!}3! = \frac{(3n)!}{(n!)^3} .$$

For example the number of ways in which 15 recruits can be divided into three equal groups is $\frac{15!}{5!5!5!3!}$; and the number of ways in which they can be drafted into three different regiments, five into each, is $\frac{15!}{5!5!5!}$

4. The number of ways in which mn different things can be divided equally into **m groups if order of groups is not important** is

$$\frac{mn!}{(n!)^m m!}$$

5. The number of ways in which mn different things can be divided equally into **m groups if order of groups is important** is

$$\frac{mn!}{(n!)^m m!} \times m! = \frac{(mn)!}{(n!)^m}$$

➤ **Example 1.** In how many ways can a pack of 52 cards be equally among 4 players ?

➤ ***Solution*** Here order of group is important, then the number of ways in which 52 different cards can be divided equally into 4 players is

$$\left(\frac{52!}{4!(13!)^4}\right) \times 4! = \frac{52!}{(13!)^4}$$

➤ **Example 2.** In how many ways can 12 balls be divided between 2 boys, one receiving 5 and the other 7 balls ?

➤ ***Solution*** Since the order is important, the number of ways in which 12 different balls can be divided between two boys which contains 5 and 7 balls respectively, is

$$\frac{12!}{5!7!} \times 2! = \frac{12.11.10.9.8.7!}{(5.4.3.2.1)7!} .2 = 1584$$

Alternative

The first boy can be given 5 balls out of 12 balls in ${}^{12}C_5$. The second boy can be given the remaining 7 balls in one way. But the order is important (the boys can interchange 2 ways).

Thus the required number of ways

$$= {}^{12}C_5 \times 1 \times 2! = \frac{12!}{5!7!} \times 2$$

$$= \frac{12.11.10.9.8.7!.2}{5.4.3.2.1.7!} = 1584.$$

➤ **Example 3.** In how many ways can a pack of 52 cards be

(a) divided into four sets of 7, 15, 15 and 15 cards.

(b) distributed among four players having 10, 12, 14 and 16 cards.

➤ ***Solution*** (a) 7 cards chosen out of 52 cards in ${}^{52}C_7$ ways. 15 cards chosen out of remaining 52 – 7 = 45 cards in ${}^{45}C_{15}$ ways. 15 cards chosen out of remaining 45 – 15 = 30 cards in ${}^{30}C_{15}$ ways. and 15 cards chosen out of remaining 30 – 15 = 15 cards in ${}^{15}C_{15}$ ways.

www.pdfworld.net

www.pdfworld.net

Since the last three groups are identical and can be interchanged amongst themselves in 3! ways, hence the required number of ways

$$= {}^{52}C_7 \times {}^{45}C_{15} \times {}^{30}C_{15} \times {}^{15}C_{15} \times \frac{1}{3!}$$

$$= \frac{52!}{7!(15!)^3} \times \frac{1}{3!}$$

(b) 10 cards chosen out of 52 cards in ${}^{52}C_{10}$ ways.
12 cards chosen out of remaining 52 – 10 = 42 cards in ${}^{52}C_{12}$ ways.
14 cards chosen out of remaining 42 – 12 = 30 cards in ${}^{30}C_{14}$ ways.
and 16 cards chosen out of remaining 30 – 14 = 16 cards in ${}^{16}C_{16}$ ways.
The above chosen groups can now be distributed among four different players in 4! ways. Thus, the required number of ways

$$= {}^{52}C_{10} \times {}^{42}C_{12} \times {}^{30}C_{14} \times {}^{16}C_{16} \times 4!$$

$$= \frac{52!}{10!12!14!16!} \times 4!$$

Alternative

The above solution can be written in a better way :

(a) In this case order is not important, hence

$$\text{required number of ways} = \frac{52!}{7!(15!)^3} \times \frac{1}{3!}$$

(b) In this case order is important, hence required

$$\text{number of ways} = \frac{52!}{10!12!14!16!} \times 4!$$

➤ **Example 4.** Find the number of ways in which 9 different toys be distributed among 4 children belonging to different age groups in such a way that distribution among the 3 elder children is even and the youngest one is to receive one toy more.

➤ ***Solution*** The distribution should be 2, 2, 2 and 3 to the youngest. Now 3 toys for the youngest can be selected in ${}^{9}C_3$ ways, remaining 6 toys can be divided into three equal groups in

$$\frac{6!}{(2!)^3 . 3!}$$ ways and can be distributed in 3 ! ways.

Thus the required number of ways = $${}^{9}C_3 . \frac{6!}{(2!)^3}$$

$$= \frac{9!}{3!\,(2!)^3}$$

➤ **Example 5.** In an eleven storeyed building 9 people enter a lift cabin. It is known that they will leave the lift in groups of 2, 3 and 4 at different residential storeys. Find the number of ways in which they can get down.

➤ ***Solution*** Note that in an eleven storeyed building there will be 10 floors.
9 people can be divided into groups of 2, 3 and 4 in $\frac{9!}{2!\,3!\,4!}$ ways.

Now each group can be distributed in ${}^{10}C_3 . 3!$ ways

Thus the required number of ways = $\frac{9!}{2!\,3!\,4!}$ 10. 9. 8

$$= \frac{10!}{4}$$

➤ **Example 6.** How many ways are there to invite 1 of 3 friends for dinner on 6 successive nights such that no friend is invited more than 3 times ?

➤ ***Solution*** Let x, y, z be the friends and let (a, b, c) denote the case where x is invited a times, y b times and z c times. For example, one possible arrangement corresponding to the triplet (3, 2, 1) is

x, x, y, x, y, z

Then we have the following possibilities :

(i) (a, b, c) = (1, 2, 3); (1, 3,2); (2, 3, 1); (2, 1, 3); (3, 1, 2); (3, 2,1).
(ii) (a, b, c) = (3,3,0); (3,0,3); (0, 3,3).
(iii) (a, b, c) = (2, 2, 2). So the total number of ways is $6 \times 6! / 1!2!3! + 3 \times 6! / 3!3! + 6!/2!2!2!$.

➤ **Example 7.** Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty, if

(i) balls and boxes are all different
(ii) balls are identical but boxes are different
(iii) balls are different but boxes are identical

www.pdfworld.net

www.pdfworld.net

(iv) balls as well as boxes are identical

(v) balls as well as boxes are identical but boxes are kept in a row ?

➤ ***Solution*** As no box is to remain empty, boxes can have balls in the following numbers :

Cases : 1, 1, 3 or 1, 2, 2,

(i) The number of ways to distribute the balls in groups of 1, 2, 3

$= {}^5C_1 \times {}^4C_1 \times {}^3C_3$.

But the boxes can interchange their content, no exchange giving a new way when boxes containing balls in equal numbers interchange.

$\therefore$ The number of ways to distribute 1, 1, 3 balls to the boxes $= {}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!}$

Similarly, the number of ways to distribute 1, 2, 2 balls to the boxes

$= {}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!}$

$\therefore$ The total required number of ways

$= {}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} + {}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!}$

$= 5 \times 4 \times 3 + 5 \times 6 \times 3 = 60 + 90 = 150.$

$\therefore$ the required number of ways $= 60 + 90 = 150.$

(ii) When the balls are identical but boxes are different the number of combinations will be 1 in each case. $\therefore$ The required number of ways

$= 1 \times \frac{3!}{2!} + 1 \times \frac{3!}{2!} = 3 + 3 = 6.$

(iii) The balls should be divided into two groups as

$\frac{5!}{1!1!3!2!} + \frac{5!}{1!2!2!2!} = 10 + 15 = 25.$

(iv) When balls as well as boxes are identical, the number of combinations and arrangements will be 1 each in both cases.

$\therefore$ the required number of ways $= 1 + 1 = 2.$

(v) When boxes are kept in a row, they will be treated as different. So, in this case the number of ways will be the same as in (ii).

➤ **Example 8.** The Indian cricket team with eleven players, the team manager, the physiotherapist and two umpires are to travel from the hotel where they are staying to the stadium where the test match is to be played . Four of them residing in the same town own cars, each a four seater which they will drive themselves . The bus which was to pick them up failed to arrive in time after leaving the opposite team at the stadium . In how many ways can they be seated in the cars ? In how many ways can they travel by these cars so as to reach in time, if the seating arrangement in each car is immaterial and all the cars reach the stadium by the same route.

➤ ***Solution*** There are 15 members : 11 Players, 1 manager, 1 Doctor, 2 Umpires.

4 people drive their own cars . There are 12 seats for 11 people .

Hence, the total number of ways $= {}^{12}C_{11} \cdot 11! = (12)!$

For the second part, 11 people can sit in 4 cars in group of 3, 3, 3 & 2 only.

which can be done in $\frac{(11)!}{(3!)^3 \cdot 2! \cdot 3!}$ and can be distributed in four cars in $4!$ ways .

Hence, the total number of ways $= \frac{(11)!\ 4!}{2!\ (3!)^4}$

$= \frac{(11)!}{108} = 369600.$

➤ **Example 9.** (a) In how many ways can five people be divided into three groups.

(b) In how many ways can five people be distributed in three different rooms if no room must be empty.

(c) In how many ways can five people be arranged in three different rooms if no room must be empty.

➤ ***Solution*** (a) Five people can be divided into three groups in the following ways :

1, 1, 3 or 1, 2,2

Hence total number of ways $= \frac{5!}{3!} \times \frac{1}{2!} + \frac{5!}{(2!)^2} \times \frac{1}{2!}$

$= 10 + 15 = 25.$

(b) The three different rooms can be filled in the following ways :

www.pdfworld.net

www.pdfworld.net

Room	I	II	III
People	1	1	3

or

Room	I	II	III
People	1	2	2

Number of ways = ways in which such groups can be formed × no of ways in which the groups be arranged in the three different rooms

$$= \left(\frac{5!}{3!}\times\frac{1}{2!}\right)3! + \left(\frac{5!}{(2!)^2}\times\frac{1}{2!}\right)3! = 60 + 90 = 150.$$

(c) In this case, the positioning of the people amongst themselves is also to be taken into account.

Case I Number of ways = $60 \times 3! = 360$

Case II Number of ways = $90 \times 2! \times 2! = 360$

Hence, the required number of ways = 360 + 360 = 720.

➤ **Example 10.** In how many ways 13 non-distinguishable books can be distributed among 7 persons so that every person get at least one book and atleast one person gets 4 books but not more.

➤ ***Solution*** When atleast one person to be given 4 books category

(i) 4 4 1 1 1 1 1
(ii) 4 3 2 1 1 1 1
(iii) 4 2 2 2 1 1 1

Case(i) | B B B B | | B B B B | B B B B B

$$\text{number of ways} = \frac{7!}{2!\cdot 5!} = 21$$

Case-(ii) | B B B B | | B B B | | B B | B B B B

$$\text{number of ways} = \frac{7!}{4!} = 210$$

Case-(iii) | B B B B | | B B |

| B B | | B B | B B B

$$\text{number of ways} = \frac{7!}{3!\cdot 3!} = 140$$

Total number of ways = 21 + 210 + 140 = 371.

➤ **Example 11.** Show that $\sum_{a+b+c=n} \frac{n!}{a!b!c!} = 3^n$

➤ ***Solution*** Consider the distribution of n distinct objects in 3 distinct boxes which can be done in 3^n ways. This can also be done by finding distribution of n objects with respect to any 3 ordered partition of n and then adding. For example 2, 3, n – 5 is a 3–partition of n.

The number of distributions corresponding to this partition is $\frac{n!}{2!\,3!(n-5)!}$.

Thus $\sum_{a+b+c=n} \frac{n!}{a!\,b!\,c!} = 3^n$

➤ **Example 12.** Show that $(6!)^{5!}$ is a divisor of (6!)!

➤ ***Solution*** Consider (6!) objects which are grouped into 5! groups, each group containing 6 objects. Note the $6 \times 5! = 6!$. Let each group be considered a separate kind, but the members within group as identical kind. Then if we look at permutations of the 6! objects, since it is divided into 5! groups of different kinds, each containing 6 objects of the same kind the total number of permutations is $\frac{(6!)!}{6!\,6!.....6!}$ there being 5! factors in the denominator, each equal to 6!. This number is therefore $\frac{(6!)!}{(6!)^{5!}}$

Since this counts the number of permutations of the 6! objects, it is an integer. This incidentally means $(6!)^{5!}$ is divisor of (6!)!.

Practice Problems [K]

1. There are 3 copies each of 4 different books. In how many ways can they be arranged on a shelf?
2. In how many ways 50 different things can be divided in 5 sets three of them having 12 things each and two of them having 7 things each.

www.pdfworld.net

www.pdfworld.net

3. In how many ways 50 different things can be distributed among 5 persons so that, three of them get 12 things each and two of them get 7 things each.

4. If $n \in N$, show that $\frac{n^2!}{(n!)^{n+1}}$ is an integer.

5. Show that $(kn)!$ is divisible by $(n!)^k$.

4.14 INCLUSION–EXCLUSION PRINCIPLE

Consider the following examples:

(i) How many integers between 1 and 6300 inclusive are not divisible by 5?

Since precisely every fifth number is divisible by 5, we see that of the 6300 numbers under consideration, exactly 6300/5 or 1260 are divisible by 5. Hence the answer to the questions is 6300 – 1260 = 5040.

(ii) How many integers between 1 and 6300 inclusive are divisible by neither 5 nor 3?

To answer this we could begin by paralleling the argument in (i) and say that the number of integers under consideration that are divisible by 5 in 1260, and the number divisible by 3 is 6300 / 3 or 2100. But 6300 – 2100 – 1260

is not the correct answer to the problem, because too many integers have been subtracted from the 6300. Numbers like 15, 30, 45, ... which are divisible by both 3 and 5 have been removed twice from the 6300 integers under consideration. So we see that we must add back the number of integers divisible by both 3 and 5, that is, divisible by 15. There are 6300/15 or 420 of those. Thus we get the answer

$$6300 - 2100 - 1260 + 420 = 3360$$

The rule of sum can be applied to express the number of elements in the union of two non-disjoint sets A and B in terms of n(A), n(B) and $n(A \cap B)$ and $n(A \cap B)$.

We proceed as follows:

Since we can write A as the union of the pairwise disjoint sets $A \cap B$ and $A \cap B'$, therefore

$$n(A) = n(A \cap B) + n(A \cap B'). \quad ...(1)$$

Again, since we can write B as the union of the pairwise disjoint sets $A \cap B$ and $A' \cap B$, therefore

$$n(B) = n(A \cap B) + n(A' \cap B). \quad ...(2)$$

Also, since the set $A \cup B$ can be expressed as the union of pairwise disjoint sets $A \cap B$, $A \cap B'$, and $A' \cap B$, therefore

$$n(A \cap B) = n(A \cap B) + n(A \cap B') + n(A' \cap B) \quad ...(3)$$

Adding corresponding sides of (1) and (2), and using (3), we have

$$n(A) + n(B) = n(A \cup B) + n(A \cap B),$$

$$n(A \cup B) = n(A) + n(B) - (A \cap B).$$

➤ **Example 1.** 105 students take an examination out of whom 80 students pass in English, 75 students pass in Mathematics and 60 students pass in both subjects. How many students fail in both subjects ?

➤ ***Solution*** Let X = the set of students who take the examination,

A = the set of students who pass in English

B = the set of students who pass in Mathematics

We are given that n(X) = 105, n(A) = 80, n(B) = 75, $n(A \cap B) = 60$

Since $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, therefore

$$n(A \cup B) = 80 + 75 - 60 = 95.$$

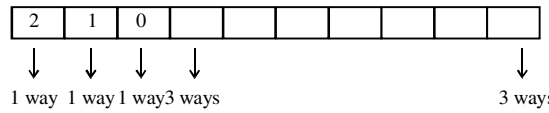
The required number = $n(X) - n(A \cup B)$

$$= 105 - 95 = 10.$$

Thus 10 students fail in both subjects.

➤ **Example 2.** How many ternary sequences of length 9 are there which either begin with 210 or end with 210.

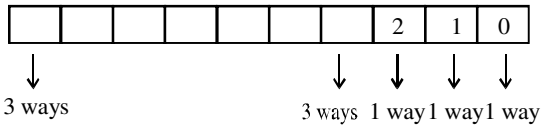
➤ ***Solution*** In a ternary sequence each term may be 0 or 1 or 2



The number of ternary sequences of length 9 which begin with 210 = $3^6 = 729$

www.pdfworld.net

www.pdfworld.net



The number of ternary sequence which begin with 210 and end with 210 = 3^3 = 27

The number of ternary sequences of length 9 which begin with 210 = 3^6 = 729

$\therefore$ The required number = 729 + 729 – 27

= 1458 – 27 = 1431

The number of objects having none of the properties α, β, γ .... is

$$N - N(\alpha) - N(\beta) - N(\gamma) - .... + N(\alpha,\beta) + N(\alpha,\gamma) + N(\beta,\gamma) + ... - N(\alpha,\beta,\gamma) - .....$$

.....

This is the general **inclusion-exclusion principle.**

➤ **Example 3.** Find the number of ways of dealing a five card hand from a regular 52 card deck such that the hand contains atleast one card in each suit.

➤ ***Solution*** Number of all 5 card hands is $^{52}C_5$

Let P_1 be the property of the hand not having any spade ;

Let P_2 be the property of the hand not having any club;

Let P_3 be the property of the hand not having any diamond ;

Let P_4 be the property of the hand not having any heart;

We want of calculate n(0).

n(1) is calculated by removing one suit from the deck and dealing the rest. Hence

$n(1) = 4 \times {}^{39}C_5$

similarly,

$n(2) = {}^4C_2 \times {}^{26}C_5 = 6 \times {}^{26}C_5$

$n(3) = {}^4C_3 \times {}^{13}C_5 = 4 \times (135)$

and $n(4) = {}^4C_4 \times 0 = 0$

Hence $n(0) = {}^{52}C_5 - 4({}^{39}C_5) + 6({}^{26}C_5) - 4({}^{13}C_5)$.

➤ **Example 4.** How many six digit numbers contain exactly three different digits ?

➤ ***Solution*** The three different digits are either

0, x, y (x, y $\in$ {1, 2, 3,....., 9})

or x, y, z (x, y, z $\in$ {1, 2, 3, ..., 9}).

If the number of the two types are α and β respectively then the required answer must be

$^9C_2 \cdot \alpha + {}^9C_2 \cdot \beta$

We will first find β : For the sake of the simplicity let the three non zero digits be 1, 2, 3, and

let A : 1 is not there B : 2 is not there C : 3 is not there

then $n(A \cup B \cup C)$ = Number of ways if which some or more digits are absent

$= n(A) + n(B) + n(C) - n(A \cap B) + ... + n(A \cap B \cap C)$

$= 2^6 + 2^6 + 2^6 - 1^6 - 1^6 - 1^6 + 0 = 96$

Therefore $\beta = 3^6 - n(A \cap B \cap C) = 540$

Again to find α let the three digits be 0, 1 and 2 then

$\alpha = 2 \times 3^5 - (2^6 + 2^5 + 2^5) + (1 + 1 + 0) - 0 = 360$

Thus the required answer is $^9C_2 \cdot 360 + {}^9C_3 \cdot 540 = 58320$.

➤ **Example 5.** Find the number of permutations of the 8 letters AABBCCDD, taken all at a time, such that no two adjacent letters are alike.

➤ ***Solution*** First disregard the restriction that no two adjacent letters be alike.

The total number of permutations is then

$$N = \frac{8!}{2!2!2!2!} = 2250$$

Now apply the inclusion exclusion principle, where a permutation has property α in case the A's are adjacent, property β in case the B's are adjacent, etc. It can be calculated that

$$N(\alpha) = \frac{7!}{2!2!2!} = 630, \qquad N(\alpha,\beta) = \frac{6!}{2!2!} = 180$$

$$N(\alpha,\beta,\gamma) = 60, \qquad N(\alpha,\beta,\gamma,\delta) = 24.$$

Hence the answer is

$$N - 4N(\alpha) + 6N(\alpha,\beta) - 4N(\alpha,\beta,\gamma) + N(\alpha,\beta,\gamma,\delta) = 864.$$

4.15 CIRCULAR PERMUTATION

We have seen that the number of permutations of n different things taken all at a time is n!, where each permutation is an arrangement of the n things in a row or a straight line. These are called linear permutations. When we arrange objects round a circle, we are said to make circular permutations. In a circular permutation, we cannot speak of first or last place of an object. Hence the rules of linear permutations are not applicable here.

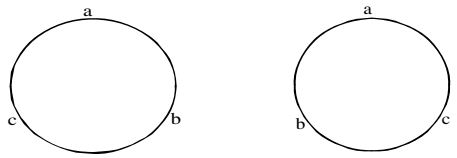
Let us examine a simple case in which three different letters, say, a, b, c are arranged round a circle.

www.pdfworld.net

www.pdfworld.net

In the circular permutations, abc, bca, cab the relative positions of the letters involved are exactly the same. A cyclic change of the letters of a circular arrangement does not alter it.

The three circular permutations above are merely the repetitions of each other. But viewed linearly, they are all different. Thus one circular permutation of three objects yields three different linear permutations.



If there be N circular permutations, then we have 3N linear permutations, Now the three different letters a, b, c can be linearly arranged among themselves in 3! ways.

$\therefore \quad 3N = 3! \qquad \therefore N = \frac{3!}{3} = 2! = 2$

In general, the number of circular permutations of n different things among themselves is

$$\frac{n!}{n} = (n-1)!$$

The number of circular permutations of n different things taken r at a time = ${}^{n}C_{r}.\ (r-1)!$.

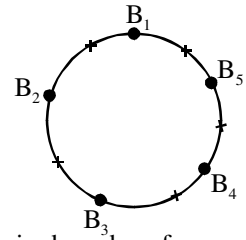
➤ **Example 1.** In how many ways can 24 persons be seated round a table, if there are 13 seats ?

➤ ***Solution*** In case of circular table the clockwise and anticlockwise orders are different Thus the required number of circular permutations = ${}^{24}C_{13}$.

$(13-1)! = \frac{24!}{13 \times 11!}$.

➤ **Example 2.** In how many different ways can five boys and five girls form a circle such that the boys and girls alternate ?

➤ ***Solution*** After fixing up one boy on the table the remaining can be arranged in 4 ! ways, but boys and girls are to alternate. There will be 5 places, one place each between two boys. These five places can be filled by 5 girls in 5 ! ways.



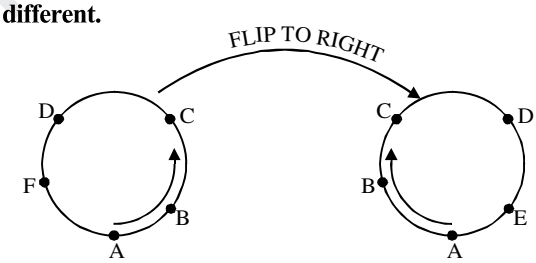
Hence, the required number of ways = 4 ! × 5! = 2880.

Arrangements of beads or flowers (all different) around a necklace or garland :

Consider five beads A, B, C, D, E in a necklace or five flowers A, B, C, D, E in a garland.

If the necklace or garland on the left is turned over we obtain the arrangement on the right, i.e., anticlockwise and clockwise order of arrangements are not different we will get arrangements as follows : We see that arrangements in the above figures are not different.

Then, the number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken as not different.



If clockwise and anticlockwise orders are not taken as different, then the number of circular permutations $= \frac{1}{2}\ {}^{n}C_{r}.\ (r-1)!$.

➤ **Example 3.** Find the number of ways in which 12 different beads can be arranged to form a necklace.

➤ ***Solution*** 12 different beads can be arranged among themselves in a circular order in (12 – 1)! = 11! ways. Now in the case of necklace there is no distinction between clockwise and anticlockwise arrangements. So the required number of arrangements = $\frac{1}{2}(11\ !)$.

www.pdfworld.net

www.pdfworld.net

➤ **Example 4.** In how many ways can we arrange 6 different flowers in a circle ? In how many ways we can form a garland using these flowers?

➤ ***Solution*** The number of circular arrangements of 6 different flowers = (6 – 1)! = 120

When we form a garland, clockwise and anticlockwise arrangements are similar. Therefore, the number of ways of forming garland = $\frac{1}{2}$ (6 – 1)! = 60.

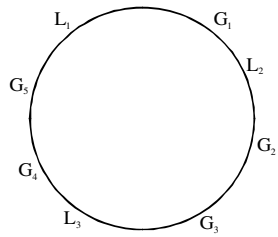
➤ **Example 5.** How many necklaces of 12 beads each can be made from 18 beads of various colours ?

➤ ***Solution*** In the case of necklace there is no distinction between the clockwise and anticlockwise arrangements, then the required number of circular permutations = $\frac{1}{2}\ {}^{18}C_{12}.\ (12-1)!$

$$= \frac{18\times17\times16\times15\times14\times13!}{6\times5\times4\times3\times2\times1\times24} = \frac{119\times13!}{2}$$

➤ **Example 6.** In how many ways can 3 ladies and 5 gentlemen arrange themselves about a round table so that (i) every gentleman may have atleast one lady by his side (ii) if no two ladies are to be together.

➤ ***Solution*** (i) The five gentleman can arrange themselves about the round table in 4! ways . If every gentleman is to have atleast one lady by his side, then two ladies must be in two of the adjacent gaps and the third lady must occupy only one gap for every choice of the first two ladies (see the figure). Thus for every arrangement of gentlemen, ladies can be arranged in 5.3! ways. Two adjacent gaps may be chosen in 5 ways and the 3 ladies each time may be arranged in 3! ways.



∴ The required number of ways in this case is 4! × 5 3! i.e. 720.

(ii) If no two ladies are to be together, then the number of ways is ${}^5P_3$. 4! i.e. 1440.

In the 5 gaps amongst the gentlemen, ladies can be seated in ${}^5P_3$ ways.

➤ **Example 7.** In how many ways 5 Indians and 4 Englishmen can be seated at a round table if

(i) there is no restriction ?

(ii) all the four Englishmen sit together ?

(iii) all the four Englishmen do not sit together ?

(iv) no two Englishmen sit together ?

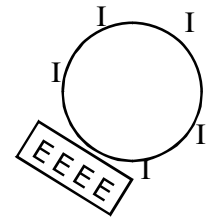
➤ ***Solution*** (i) Total number of persons = 5 + 4 = 9. These 9 persons can be seated at the round table in 8! ways.

∴ The required number = 8!

(ii) Regarding the 4 Englishmen as one unit we have only 5 + 1 i.e. 6 persons.

These 6 persons can be seated at the round table in 5! ways. But the b4 Englishmen can be arranged among themselves in 4! ways.

∴ The required number = 5! 4!



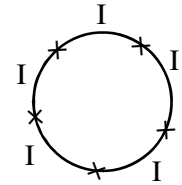
(iii) Total number of arrangements when there is no restriction = 8!;

The number of arrangements when all the four Englishmen sit together = 5! 4!

∴ The number of arrangements when all the four Englishmen do not sit together = 8! – 5! 4!

(iv) Since there is no restriction on Indians therefore, first of all we arrange the 5 Indians.

Now the 5 Indians can be seated around a table in 4! ways



If the Englishmen sit at the places indicated by '×' then no two Englishmen will sit together.

Now there are 5 places for 4 Englishmen therefore, 4 Englishmen can be seated in ${}^5P_4$ ways.

www.pdfworld.net

www.pdfworld.net

$\therefore$ The required number $= 4! \times {}^5P_4 = 4! \times 5!$

➤ **Example 8.** In how many ways can 20 persons sit around a circle such that there is exactly one person between A and B.

➤ ***Solution*** Let the remaining 18 persons be P_1, P_2, ..., P_{18}. Take one of the r persons, say P_1 and make him sit between A and B. Then there are 18 objects in a circular order namely P_2,....., P_{18} and (A, P_1, B). The number of these ways must be 17!. But every such case will be doubled because of B P_1 A. Again there are 18 ways of choosing one person between A and B. Therefore, the final answer = $17! \times 2! \times 18 = 18! \times 2!$

➤ **Example 9.** How many necklaces can be made using 7 beads of which 5 are identical red beads and 2 are identical blue beads ?

➤ ***Solution*** The different necklaces are determined by the number of red beads between the two blue beads, taking both arcs into consideration. So there are exactly 3 distinct necklaces :

b b r r r r r, b r b r r r r, b r r b r r r.

➤ **Example 10.** Six persons A, B, C, D, E and F are to be seated at a circular table. Find the number of ways this can be done if A must have either B or C on his right and B must have either C or D on his right.

➤ ***Solution*** When A has B or C to his right we have AB or AC

When B has C or D to his right we have BC or BD

Thus, we must have ABC or ABD or AC and BD

for ABC D, E, F on a circle number of ways = 3 ! = 6

for ABD C, E, F on a circle number of ways = 3 ! = 6

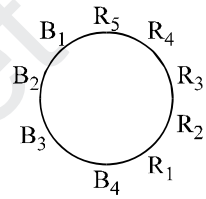
for AC, BD E, F the number of ways = 3 ! = 6

Hence, the required number of ways= 18

➤ **Example 11.** Suppose that there are 5 red points and 4 blue points on a circle. Find the number of convex polygons whose vertices are among the 9 points and having at least one blue vertex.

➤ ***Solution***

Sr.No.		n(A)
1	Triangle	${}^9C_3 - {}^5C_3$
2	Quadrilateral	${}^9C_4 - {}^5C_4$
3	Pentagon	${}^9C_5 - {}^5C_5$
4	Hexagon	${}^9C_6$
5	Heptagon	${}^9C_7$
6	Octagon	${}^9C_8$
7	Nonagon	${}^9C_9$



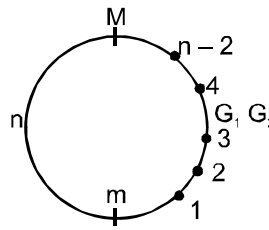
Remarks: When 6 or more vertices are taken then at least one blue has to be taken.

Hence, the required number of ways

$= n(A) = (84 - 10) + (126 - 5) + (126 - 1) + 84 + 36 + 3 + 1 = 74 + 121 + 125 + 130 = 450.$

➤ **Example 12.** There are 2n guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must not be placed next to one another . Show that the number of ways in which the company can be placed is $(2n - 2)! \cdot (4n^2 - 6n + 4)$.

➤ ***Solution*** Excluding the two specified guests , 2n persons can be divided into two groups one containing n and the other $(n - 2)$ in $\dfrac{(2n-2)!}{n!\ (n-2)!}$ and can sit on either side of master &



www.pdfworld.net

www.pdfworld.net

mistress in 2 ! ways and can arrange themselves in n ! (n − 2) !

Now the two specified guests where (n − 2) guests are seated will have (n − 1) gaps and can arrange themselves in 2 ! ways . Number of ways when G_1 G_2 will always be together

$$= \frac{(2n-2)\,!}{n\,!\;(n-2)\,!}\; 2\,!\,n\,!\;(n-2)\,!\;(n-1)\,.\,2\,!$$

$$= (2\,n-2)\,!\,.\,4\,(n-1)$$

Hence number of ways when G_1G_2 are never together

$$= \frac{2\,!}{n\,!\;n\,!\;2\,!}\; 2\,!\,.\,n\,!\,.\,n\,! \;-\; 4\,(n-1)\,.\,(2n-2)\,!$$

$$= (2\,n-2)\,!\;\left[2n\,(2n-1) \;-\; 4\,(n-1)\right]$$

$$= (2\,n-2)\,!\;\left[4\,n^2 - 6\,n + 4\right]$$

➤ **Example 13.** How many differently coloured blocks of a fixed cubical shape can be made if six colors are available, and a block is to have a different colour on each of its six faces ?

➤ ***Solution*** Let one of the colours be white. Then since one face must be white, let it be the bottom of the block. The top can be coloured in any of the 5 remaining colours. That done, the vertical faces are to be coloured with the 4 remaining colors. This now amounts to problem in circular permutations, because the cubical block can now be rotated about a vertical axis through the center of the block without altering the colours of the top and bottom. Hence there are 3! ways of colouring the vertical faces, and this is multiplied by 5 to get the answer 30.

Practice Problems [L]

1. How many integers from 1 to 33,000 inclusive are divisible by none of 3, 5, 11 ?
2. Seven girls are to dance in a circle. In how many different ways can they stand on the circumference of the circle ?
3. In a class of students there are 6 boys and 4 girls. In how many ways can they be seated around a table so that all the 4 girls sit together.
4. A round table conference is to be held between 20 delegates of 20 countries. In how many ways can they and the host be seated if two particular delegates are always to sit one either side of the host.
5. In how many ways can 7 persons sit around a table so that all shall not have same neighbours in any two arrangements.
6. In how many ways can four men and four ladies be seated at a round table, if no two men are to be in adjacent seats ?
7. In the preceding question, suppose the persons are four married couples. What would be the answer to the question if no husband and wife, as well as no two men are to be in adjacent seats ?

4.16 LINEAR EQUATIONS WITH UNIT COEFFICIENTS

Consider the solutions of the equation x + y + z + w = 12 in positive integers x, y, z, w.

The number of solutions of the given equation is easily determined by viewing the problem in this way: If 12 units (denoted by 12 u's) separated by 11 spaces (denoted by 11s's) are lined up,

u s u s u s u s u s u s u s u s u s u s u,

and if we choose any 3 of the s's and let the others disappear, for example,

u u s u u u u s u u u u u s u,

then the remaining s's separate the units into four batches. The number of units in these batches can be used as the values of x, y, z, w. In the example the values are x = 2, y = 4, z = 5, w = 1. Thus any selection of three of the s's gives a solution of x + y + z + w = 12 in positive integers and any solution corresponds to such a selection. It follows that the number of solutions of this equation in positive integers is the same as the number of ways of choosing 3 things out of 11, which is

www.pdfworld.net

www.pdfworld.net

$$^{11}C_3 = \frac{11.10.9}{3.2.1} = 165.$$

For a more general result let us replace 12 by m and x + y + z + w by a sum of k variables, and ask for the number of solutions in positive integers of the equation

$x_1 + x_2 + x_3 + ... + x_k = m.$

In the special case just studied, m = 12 and k = 4, so the equation could have been written

$x_1 + x_2 + x_3 + x_4 = 12$; using $x_1\ x_2\ x_3\ x_4$ in place of x, y, z, w makes no difference in the number of solutions, of course.

The number of solutions of above equation in positive integers $x_1, x_2, ..., x_k$ is $^{m-1}C_{k-1}$.

Let us turn to the question of the number of solutions of an equation in non-negative integers, the difference here being that zero values for the variables are now allowed. To begin with a special case, we ask for the number of solutions of the equation

$x + y + z + w = 12$

in non-negative integers. Let us take 12 u's and 3s's lined up in a row, for example :

u s u u u u u s u u s u u u

If we look upon the s's as separators of the 12u's into four batches, we see that the illustration gives batches of 1, 5, 2, 4 corresponding to the solution x = 1, y = 5, z = 2, w = 4 of the equation x + y + z + w = 12 under discussion.

We claim that every arrangement of 12 u's and 3s's gives a solution. To find the value of x, we locate the first s and count the number of u's to its left. If the arrangement begins with an s, x = 0 because no u's are to its left. To find the value of y, we count the number of u's between the first and second s. If the first and second s are adjacent, then y = 0. Similarly, the value of z is the number of u's between the second and third s, and the value of w is the number of u's to the right of the third s.

Conversely, if we start with any solution, such as x = 0, y = 1 z = 2, w = 9 we can write 12 u's and 3 s's in a row

s u s u u s u u u u u u u u u corresponding to this solution.

The solution x = 9, y = 0, z = 1, w = 2 corresponds to the arrangement

u u u u u u u u u s s u s u u

Thus the number of solutions of x + y + z + w = 12 in non-negative integers is the same as the number of ways of writing 12 u's and 3 s's in a row. This number is $^{15}C_3$. This is called as **method of fictitious partition.**

We generalize this result to the equation

$x_1 + x_2 + x_3 + ... + x_k = m$

in k variables whose sum must always be m, by an argument similar to that given in the special case. The number of solutions of equation in non-negative integers is the same as the number of arrangements in a row of m u's and k – 1 s's and there are

$^{m+k-1}C_{k-1}$ such arrangements.

Coefficient Method

(i) The number of non-negative integral solutions of equation $x_1 + x_2 + ... + x_r = n$

= Number of ways of distributing n identical things among r persons

when each person can get zero, one or more things

= coeff. of x^n in $[(1 + x + x^2 + ... + x^n)(1 + x + x^2 + ... + x^n)(1 + x + x^2 + ... + x^n).....$upto r factors$]$

= coeff. of x^n in $(1 + x + x^2 + ... + x^n)^r$

$$= \text{coeff. of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^r$$

= coeff. of x^n in $(1 - x^{n+1})^r(1 - x)^{-r}$

= coeff. of x^n in $(1 - x)^{-r}$

[leaving terms containing powers of x greater than n]

$= {}^{n+r-1}C_{r-1}$

If n is a positive integer, then

$$(1-x)^{-n} = 1 + \frac{(-n)}{1!}(x) + \frac{(-n)(-n-1)}{2!}(-x)^2 + \frac{(-n)(-n-1)(-n-2)}{3!}(-x)^3 + ...\text{to } \infty$$

$$= 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ... \text{ to } \infty$$

$= 1 + {}^nC_1x + {}^{n+1}C_2x^2 + {}^2C_3x^3 + ... \text{ to } \infty$

coeff. of x^r in $(1 - x)^{-n} = {}^{n+r-1}C_r$

www.pdfworld.net

www.pdfworld.net

$\Rightarrow$ coeff. of x^n in $(1-x)^{-r} = {}^{n+r-1}C_n = {}^{n+r-1}C_{r-1}$

(ii) The number of positive integral solutions of equation $x_1 + x_2 + ... + x_r = n$

= Number of ways of distributing n identical things among r persons when each person can get atleast one thing

= coeff. of x^n in $[(x + x^2 + ... + x^n)(x + x^2 + ... + x^n)(x + x^2 + ... + x^n)$.....upto r factors]

= coeff. of x^n in $(x + x^2 + ... + x^n)^r$

$$= \text{coeff. of } x^n \text{ in } x^r\left(\frac{1-x^n}{1-x}\right)^r$$

= coeff. of x^{n-r} in $(1 - x^n)^r(1 - x)^{-r}$

= coeff. of x^{n-r} in $(1 - x)^{-r}$

[leaving terms containing powers of x greater than n–r]

$= {}^{n-r+r-1}C_{r-1} = {}^{n-1}C_{r-1}$.

➤ **Example 1.** Find the positive number of solutions of $x + y + z + w = 20$ under the following conditions : (i) zero values of x, y, z, w are included. (ii) zero values are excluded.

➤ ***Solution*** (i) Since $x + y + z + w = 20$ $x \geq 0, y \geq 0, z \geq 0, w \geq 0$

The number of solutions of the given equation in this case is same as the number of ways of distributing 20 things among 4 persons.

Hence total number of solutions $= {}^{20+4-1}C_{4-1}$

$= {}^{23}C_3 = 1771$

(ii) Since $x + y + z + w = 20$ ...(1)

Here $x \geq 1, y \geq 1, z \geq 1, w \geq 1$

or $x_1 = x - 1 \Rightarrow x = x_1 + 1$

Let $y_1 = y - 1 \Rightarrow y = y_1 + 1$

$z_1 = z - 1 \Rightarrow z = z_1 + 1$

$w_1 = w - 1 \Rightarrow w = w_1 + 1$

Then from (1),

$x_1 + 1 + y_1 + 1 + z_1 + 1 + w_1 + 1 = 20$

$\Rightarrow$ $x_1 + y_1 + z_1 + w_1 = 16$

and $x_1 \geq 0, y_1 \geq 0, z_1 \geq 0, w_1 \geq 0$

Hence total number of solutions $= {}^{16+4-1}C_{4-1} = {}^{19}C_3 = 969$.

Alternative (ii)

$\because$ $x + y + z + w = 20$

$x \geq 1, y \geq 1, z \geq 1, w \geq 1$

Hence total no. of solutions $= {}^{20-1}C_{4-1} = {}^{19}C_3 = 969$.

➤ **Example 2.** How many integral solutions are there to $x + y + z + w = 29$, when $x \geq 1, y \geq 1, z \geq 3$ and $w \geq 0$?

➤ ***Solution*** $x + y + z + w = 29$ ...(1)

$x \geq 1, y \geq 2, z \geq 3, w \geq 0$

$\Rightarrow$ $x - 1 \geq 0, y - 2 \geq 0, z - 3 \geq 0, w \geq 0$

let $x_1 = x - 1, x_2 = y - 2, x_3 = z - 3$

or $x = x_1 + 1, y = x_2 + 2, z = x_3 + 3$ and then $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, w \geq 0$

From (1), $x_1 + 1 + x_2 + 2 + x_3 + 3 + w = 29$

$\Rightarrow$ $x_1 + x_2 + x_3 + w = 23$

Hence total number of solutions $= {}^{23+4-1}C_{4-1}$

$= {}^{26}C_3 = 2600$

➤ **Example 3.** How many integral solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 = 15$ when $x_k \geq 0$?

➤ ***Solution*** We have $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ ...(1)

and $x_1 + x_2 = 15$ ...(2)

then from (1) and (2) we get two equations

$x_3 + x_4 + x_5 = 5$ ...(3)

$x_1 + x_2 = 15$ ...(4)

with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$ and $x_5 \geq 0$

The number of solutions of equation (3) $= {}^{5+3-1}C_{3-1} = {}^7C_2 = 21$ and number of solutions of equation (4). Hence total number of solutions of the given system of equations $= 21 \times 16 = 336$

➤ **Example 4.** Find the number of non-negative integral solutions of $3x + y + z = 24$.

➤ ***Solution*** We have

$3x + y + z = 24$, $x \geq 0, y \geq 0, z \geq 0$

let $x = k$ $\therefore$ $y + z = 24 - 3k$ ...(1)

Here $0 \leq 24 - 3k \leq 24$

Hence $0 \leq k \leq 8$

The total number of integral solutions of (1) is

$${}^{24-3k+2-1}C_{2-1} = {}^{25-3k}C_1 = 25 - 3k$$

Hence, the total number of solutions of the original equation

$$= \sum_{k=0}^{8} (25 - 3k) = 25 \sum_{k=0}^{8} 1 - 3 \sum_{k=0}^{8} k$$

$$= 25 \cdot 9 - 3 \cdot \frac{8 \cdot 9}{2} = 225 - 108 = 117.$$

➤ **Example 5.** Find the number of solutions of the equation $x + y + z = 6$, where $x, y, z \in W$.

➤ ***Solution*** Number of solutions = coefficient of x^6 in $(1 + x + x^2 + ...x^6)^3$

= coefficient of x^6 in $(1 - x^7)^3 (1 - x)^{-3}$

= coefficient of x^6 in $(1 - x)^{-3}$

$= {}^{6+3-1}C_{3-1} = {}^8C_2 = 280$.

www.pdfworld.net

www.pdfworld.net

➤ **Example 6.** Find the number of integral solutions of equation

$x + y + z + t = 29, x > 0, y > 1, z > 2$ and $t \geq 0$

➤ ***Solution*** Given, $x + y + z + t = 29$,

where $x \geq 1, y \geq 2, z \geq 3, t \geq 0$

Let $p = x - 1, q = y - 2, r = z - 3, s = t$

Then $p + q + r + s = x + y + z + t - 6 = 29 - 6 = 23$, where , $p, q, r, s \geq 0$

$\therefore$ $p + q + r + s = 23, p, q, r, s \geq 0$

$\therefore$ The required number = number of ways in which 23 identical things can be distributed among 4 persons when each person can get any number of things.

$$= {}^{4+23-1}C_{4-1} = {}^{26}C_3 = \frac{26 \times 25 \times 24}{3!} = 2600$$

➤ **Example 7.** Find the number of non-negative integral solutions of $x + y + z + 4t = 20$.

➤ ***Solution*** Case I : When $4t = 0$, i.e. $t = 0$.

In this case $x + y + z = 20$

The number of non-negative integral solutions

$= {}^{3+20-1}C_{3-1} = {}^{22}C_2$

Case II : When $t = 1$, number of non-negative integral solutions $= {}^{3+16-1}C_2 = {}^{18}C_2$

Case III : When $t = 2$, number of non-negative integral solutions $= {}^{14}C_2$

Case IV : When $t = 3$, number of non-negative integral solutions $= {}^{10}C_2$

Case V : When $t = 4$, number of non-negative integral solutions $= {}^{6}C_2$

Case VI : When $t = 5$, number of non-negative integral solutions $= {}^{2}C_2$

$\therefore$ The required number $= {}^{22}C_2 + {}^{18}C_2 + {}^{14}C_2 + {}^{10}C_2 + {}^{8}C_2 + {}^{2}C_2 = 536$

Alternative

The required number

$= \text{coefficient of } x^{20} \text{ in } (1-x)^{-1}(1-x)^{-1}(1-x)^{-1}(1-x^4)^{-1}$

$= \text{coefficient of } x^{20} \text{ in } [(1-x)^{-3}(1-x^4)^{-1}]$

$= \text{coefficient of } x^{20} \text{ in } (1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_3 x^3 + \ldots + {}^{22}C_{20} x^{20} + \ldots.)[1 + x^4 + x^8 + x^{12} + x^{16} + x^{20} \ldots..]$

$= 1 + {}^6C_4 + {}^{10}C_8 + {}^{14}C_{12} + {}^{18}C_{16} + {}^{22}C_{20}$

$= 1 + {}^6C_2 + {}^{10}C_2 + {}^{14}C_2 + {}^{18}C_2 + {}^{22}C_2$

$= 1 + 15 + 45 + 91 + 153 + 231 = 536$

➤ **Example 8.** How many integers between 1 and 1000000 have the sum of the digits 18 ?

➤ ***Solution*** Any number between 1 and 1000000 must be of less than seven digits. Therefore, it must be of the form $a_1\, a_2\, a_3\, a_4\, a_5\, a_6$

where $a_1, a_2, a_3, a_4, a_5, a_6 \in \{0, 1, 2, \ldots.., 9\}$

Thus $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 18$

where $0 \leq a_i \leq 9, i = 1,2,3, \ldots., 9$

The required number

$= \text{coeff. of } x^{18} \text{ in } (1 + x + x^2 + \ldots + x^9)^6$

$$= \text{coeff. of } x^{18} \text{ in } \left(\frac{1-x^{10}}{1-x}\right)^6$$

$= \text{coeff. of } x^{18} \text{ in } [(1-x^{10})^6 (1-x)^{-6}]$

$= \text{coeff. of } x^{18} \text{ in } [(1 - {}^6C_1 x^{10} \ldots.)(1-x)^{-6}]$

[leaving terms containing powers of x greater than 18]

$= \text{coeff. of } x^{18} \text{ in } (1-x)^{-6} - {}^6C_1 \text{ . coeff. of } x^8 \text{ in } (1-x)^{-6}$

$= {}^{6+18-1}C_5 - 6 \,.\, {}^{6+8-1}C_5 = {}^{23}C_5 - 6.{}^{13}C_5$

$$= \frac{23.22.21.20.19}{120} - 6.\frac{13.12.11.10.9}{120}$$

$= 33649 - 7722 = 25927$

➤ **Example 9.** Find the number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 \leq n$ (where n is a positive integer).

➤ ***Solution*** Let x_5 be such that $x_1 + x_2 + x_3 + x_4 + x_5 = n$.

The problem now reduces to finding the integral solutions to the equation

$x_1 + x_2 + x_3 + x_4 + x_5 = n$, where $x_1, x_2, x_3, x_4, x_5 \geq 0$

The number of required solutions

$= \text{coeff. of } x^n \text{ in } (1 + x + x^2 + x^3 + \ldots....)^5$

$= \text{coeff. of } x^n \text{ in } (1-x)^{-5} = {}^{n+4}C_4$

➤ **Example 10.** In how many ways can a dice be thrown thrice by a person to make a sum of 12.

➤ ***Solution*** The minimum number that can appear on a throw is 1 and the maximum number is 6. Thus, the problem is equivalent to finding the number of integral solutions to the equation

$x_1 + x_2 + x_3 = 12$, where $1 \leq x_1, x_2, x_3 \leq 6$.

www.pdfworld.net

www.pdfworld.net

Hence, the required number of ways

$= \text{coeff. of } x^{12} \text{ in } (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^3$

$= \text{coeff. of } x^{12} \text{ in } x^3(1 + x + x^2 + \ldots\ldots + x^5)^3$

$= \text{coeff. of } x^9 \text{ in } (1 - x^6)^3 (1 - x)^{-3}$

$= \text{coeff. of } x^9 \text{ in } (1 - 3x^6 + 3x^{12} - x^{18})(1 - x)^{-3}$

$= \text{coeff. of } x^9 \text{ in } (1 - x)^{-3}$

$- [3 \times \text{coeff. of } x^3 \text{ in } (1 - x)^{-3}]$

$= {}^{9+3-1}C_9 - 3 \times {}^{3+3-1}C_3$

$= {}^{10}C_1 - 3 \times {}^5C_2 = 55 - 30 = 25.$

➤ **Example 11.** In a bakery four types of biscuits are available. In how many ways a person can buy 10 biscuits if he decide to take atleast one biscuit of each variety.

➤ ***Solution*** Let x be the number of biscuits the person select from first variety, y from the second, z from the third and w from the fourth variety.

Then the number of ways

= number of solutions of the equation $x + y + z + w = 10$.

where $x, y, z, w \in \{1, 2, \ldots\ldots\ldots, 7\}$

This is equal to coefficient of x^{10} in $(x + x^2 + \ldots. + x^7)^4$

$= \text{coefficient of } x^6 \text{ in } (x + x + \ldots. + x^6)^4$

$= \text{coefficient of } x^6 \text{ in } (1 - x^7)^4 (1 - x)^{-4}$

$= \text{coefficient } x^6 \text{ in } (1 - x)^{-4}$

$= {}^{6+4-1}C_{4-1} = 84.$

➤ **Example 12.** An examination consists of four papers, each paper has a maximum of m marks. Show that the number of ways in which a student can get 2m marks is $\frac{1}{3}(m + 1)(2m^2 + 4m + 3)$.

➤ ***Solution*** Let the student get x_i marks in the its paper. Then $0 \le x_i \le m$ and $x_1 + x_2 + x_3 + x_4 = 2m$. The number of ways of getting 2m marks equals the number of non-negative integral solutions of this equation, i.e., the coefficient of t^{2m} in

$(1 + t + t^2 + \ldots. + t^m)^4 = (1 - t^{m+1})^4 (1 - t)^{-4}$

This is the coefficient of t^{2m} in

$(1 - 4t^{m+1})(1 + {}^4C_1 t + {}^5C_2 t^2 + {}^6C_3 t^3 + \ldots..)$,

which is ${}^{2m+3}C_{2m} - 4({}^{m+2}C_{m-1})$

$= {}^{2m+3}C_3 - 4({}^{m+2}C_3)$

$= \frac{(2m + 3)(2m + 2)(2m + 1)}{6} - 4\left[\frac{(m + 2)(m + 1)m}{6}\right]$

$\frac{2}{6}(m + 1)[(2m + 3)(2m + 1) - 2(m + 2)m]$

$= \frac{1}{3}(m + 1)(2m^2 + 4m + 3).$

Practice Problems [M]

1. How many solutions are there of $x + y + z + w = 50$ (a) in positive integers ? (b) in non-negative integers ?
2. How many integers between 1 and 1,000,000 inclusive have sum of digits (a) equal to 6; (b) less than 6 ?
3. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ in positive integers (a) with $x_5 > 12$; (b) with $x_5 > 12$ and $x_4 > 7$.
4. Find the number of solutions of $x + y + z + w = 1$ in integers greater than –4, i.e., in integers selected from –3, –2, –1, 0 , 1, 2, 3, 4, 5,.....
5. Find the number of solutions of $x + y + z + w = 20$ in non-negative integers (a) with $x \ge 6$; (b) with $x \ge 6$ and $y \ge 6$
6. Show that the equations $x_1 + \ldots. + x_7 = 13$ and $x_1 + \ldots. + x_{14} = 6$ have the same number of non-negative integer solutions.
7. Show that the equations $x_1 + x_2 + x_3 = 12$ and $x_1 + \ldots. + x_{10} = 12$ have the same number of positive integers solutions.
8. Find the number of non negative integral solutions of $2x + 2y + z = 10$.

4.17 APPLICATIONS OF COEFFICIENT METHOD

Combinations with Repetitions

1. The symbol ${}^rC_n$ denotes the number of combinations of r different things, taken n at a time. Suppose now that there are several identical copies of each of the r things, as for example identical copies of the books in a bookstore. Thus we can ask the question : Given r distinct categories of things, each category containing an unlimited supply, how many

www.pdfworld.net

www.pdfworld.net

different combinations of n things are there ? (A combination may include several indistinguishable objects from the same category). This question amounts to asking for the number of solutions of

$$x_1 + x_2 + x_3 + ... + x_r = n$$

in non-negative integers, because we can take x_1 of the first thing, x_2 of the second thing, ..., x_r of the rth thing. Thus we can use the formula in the previous section to get the following conclusion.

The number of combinations, n at a time, of r different things each of which is available in unlimited supply is $^{n+r-1}C_{r-1}$.

For instance, consider twenty different books each of which is available in unlimited supply. How many selections of ten book can be made, (a) if repetitions are not allowed, (b) if repetitions are allowed ?

Part (a) is just a matter of choosing ten things out of twenty, so the answer is $^{20}C_{10}$.

Part (b) is a question of the number of combinations, ten at a time, of twenty different things each of which is available in multiple copies. So the answer is given by the above formula with n = 10 and r = 20, namely

$$^{10+20-1}C_{20-1} = {}^{29}C_{19} = \frac{29!}{10!19!}$$

2. The number of combinations of r things out of n things of which p are of one type, q of second type, s of third type and so on.

$$= \text{coeff. of } x^r \text{ in } [(1 + x + x^2 + ... + x^p)(1 + x + x^2 + ... + x^q)(1 + x + x^2 + ... + x^s).....]$$

Proof : Let the different things be denoted by the letters a, b, c,....

Now we consider the continued product

$(1 + ax + a^2x^2 + ... + a^px^p)\ (1 + bx + b^2x^2 + ... + b^qx^q)\ (1 + cx + c^2x^2 + ... + c^sx^s)....$

It is clear that all the terms in the product is of the same degree in the letters a, b, c as in x. Also the coefficient of x^r in the product is the number of ways of taking r of the letters a, b, c, ... with the restriction that maximum number of a's is p, maximum number of b's is q, maximum number of c's is s and so on.

But the coefficient of x^r in the product will not change if we take a = b = c = ... = 1.

3. The number of selections of r things out of n things of which p are of one type, q of second type, s of third type and so on, when each thing is taken atleast once

$$= \text{Coeff. of } x^r \text{ in } \{(x + x^2 + x^3 + ... + x^p)\ (x + x^2 + x^3 + ... + x^q)\ (x + x^2 + x^3 + ... + x^s).....\}$$

➤ **Example 1.** In how many ways can we choose 6 candies from 8 brands that are available ?

➤ ***Solution*** This is a problem of combinations with repetitions allowed. We choose 6 brands from 8 brands, but now repetitions are allowed. So the answer is not $^8C_6$. For example you can choose all the 6 candidates from the same brand. Let us call the brands

$$B_1\ B_2\ B_3\ B_4\ B_5\ B_6\ B_7\ B_8$$

so now it is a question of how many we choose of Brand B_1, how many of brand B_2 and so on. Let us represent this symbolically as

$$x\ x\ ....\ x\ \ |\ x\ ....\ x\ |\ x\ ...\ x\ |\ \ ...\ \ |\ x\ ...\ x$$
$$(1) \qquad (2) \qquad (3) \qquad (8)$$

Here the eight brands are denoted as 8 boxes. If for instance, our choice is

$$B_2\ B_2\ B_4\ B_7\ B_7\ B_7$$

we write this as

$$|\ x\ x|\ \ |\ x\ |\ |\ |\ x\ x\ x\ |\qquad (*)$$

Each vertical separator stands for the separation between one brand and the next brand. The no. of x's says how many we have taken of that brand. An arrangement of 6 x's and 7 vertical separators as in (*) says precisely what our choice is. In this case it says that B_1 is not chosen, B_2 is chosen twice, B_3 is not chosen, B_4 is chosen once, B_5, B_6 are chosen, B_7 is chosen three times and B_8 is not chosen. So it is $B_2\ B_2\ B_4\ B_7\ B_7\ B_7$. So now it is up to us to count in how many ways we can have 6 x's and 7 separators arranged in a line. This means we have 13 spaces in which we can put the seven vertical separators anywhere we like and fill up the rest of the spaces with x's. This can be done in $^{13}C_7$ ways.

The answer is therefore $^{13}C_6$, which is the same as $^{13}C_7$. Generalizing this to r-combinations of n things with repetitions, we have spaces for the r things chosen and the n – 1 vertical separators between the n

www.pdfworld.net

www.pdfworld.net

objects. Thus we have n – 1 + r entities. We have to select n –1 entities as spaces for the separators.
So the answer is ${}^{n+r-1}C_{n-1}$ which is the same as ${}^{n+r-1}C_{r}$. This is the same as the number of non-negative integer solutions of $x_1 + x_2 + ... + x_n = r$.

➤ **Example 2.** Show that a selection of 10 balls can be made from an unlimited number of red, white, blue and green balls in 286 different ways and that 84 of these contain balls of all four colours.

➤ ***Solution*** First part : Let the number of red, white, blue and green balls be x, y, z and w respectively, then $x + y + z + w = 10$, where $x, y, z, w \geq 0$
The required number = number of ways of distributing 10 identical things among 4 persons when each gets zero or more things.

$$= {}^{4+10-1}C_{4-1} = {}^{13}C_3 = \frac{13.12.11}{6} = 286$$

Second part : $x + y + z + w = 10$, where $x, y, z, w \geq 1$
Let $a = x - 1, b = y - 1, c = z - 1, d = w - 1$
Then $a + b + c + d = x + y + z + w - 4 = 10 - 4$
or $a + b + c + d = 6$
where $a, b, c, c, d \geq 0$
$\therefore$ The required number $= {}^{4+6-1}C_{4-1} = {}^{9}C_3 = 84$

Alternative

First part : The required number
$= $ coeff. of x^{10} in $(x^0 + x^1 + x^2 + ... + x^{10})^4$
$= $ coeff. of x^{10} in $\left(\frac{1-x^{11}}{1-x}\right)^4 =$ coeff. of x^{10} in $(1 - x^{11})^4(1 - x)^{-4}$
$= $ coeff. of x^{10} in $(1 - x)^{-4}$ [leaving terms containing powers of x greater than 10]
$= {}^{4+10-1}C_{4-1} = {}^{13}C_3 = 286$

Second part : Here atleast one ball of each colour is to be selected, therefore, required number
$= $ coeff. of x^{10} in $(x^1 + x^2 + ... + x^7)^4$
$= $ coeff. of x^{10} in $x^4 (x^1 + x + ... + x^6)^4$
$= $ coeff. of x^6 in $(1 + x + ... + x^6)^4$
$= $ coeff. of x^6 in $\left(\frac{1-x^7}{1-x}\right)^4$
$= $ coeff. of x^4 in $[(1 - x^7)^4(1 - x)^{-4}]$
$= $ coeff. of x^6 in $(1 - x)^{-4}$ [leaving terms containing powers of x greater than 6]
$= {}^{9}C_3 = 84$

Arrangement in Groups

1. The number of ways in which n different things can be arranged into r different groups is
$n\,!\,.\,{}^{n+r-1}C_{r-1}$ if blank groups are allowed.
Proof : Assume that the n things are identical. Let first, second, ..., rth group be allotted $x_1, x_2, ..., x_r$ things respectively. The number of solutions of $x_1 + x_2 + x_3 + ... + x_r = n$ in non-negative integers is ${}^{n+r-1}C_{r-1}$. Now the n things can be permuted thoroughly in n! ways, considering them to be different, among the r groups. Hence, the number of ways is $n\,!\,.\,{}^{n+r-1}C_{r-1}$
Alternatively,
Let n letters a, b, ... k be written in a row in any order. All the arrangements of the letters in r groups, blank groups being admissible, can be obtained thus ; place among the letters r – 1 marks of partition, and arrange the n + r – 1 things (consisting of letters and marks) in all possible orders. Since r – 1 of the things are alike, the number of different arrangements is $(n + r - 1)!/(r - 1)! = n\,!\,.\,{}^{n+r-1}C_{r-1}$.

2. The number of ways in which n different things can be arranged into r different groups is
$n\,!\,.\,{}^{n-1}C_{r-1}$ if blank groups are not allowed.
Proof : As before assume that the n things to be identical. Let first, second, ..., rth group be allotted $x_1, x_2, ..., x_r$ things respectively. The number of solutions of $x_1 + x_2 + x_3 + ... + x_r = n$ in positive integers is ${}^{n-1}C_{r-1}$. Now the n things can be permuted thoroughly in n! ways, considering them to be different, among the r groups. Hence, the number of ways is $n\,!\,.\,{}^{n-1}C_{r-1}$
Alternatively,
All the arrangements of the letters in r groups, none of the groups being blank, can be obtained as follows : (a) Arrange the letters in all possible orders. This can be done in n! ways. (b) In every such arrangement, place (r – 1) marks of partition in (r – 1) out of the (n – 1) spaces between the letters. This can be done in ${}^{r-1}C_{n-1}$ ways.
Hence the required number is $n!\,{}^{r-1}C_{n-1}$

www.pdfworld.net

www.pdfworld.net

➤ **Example 3.** In how many ways 5 different balls can be arranged into 3 different boxes so that no box remains empty ?

➤ ***Solution*** The required number of ways
$= 5! \,.\, {}^{5-1}C_{3-1} = 5! \,.\, {}^{4}C_{2} = 720.$

Alternative

Since no box remains empty, boxes can have balls in the following ways : 1, 1, 3 or 1, 2, 2

All 5 balls can be arranged by 5 ! ways and boxes can be arranged in $\frac{3!}{2!}$ ways.

$\therefore$ Hence the required number of ways

$$5! \times \frac{3!}{2!} + 5! \times \frac{3!}{2!}$$
$$= 120 \times 3 + 120 \times 3 = 720$$

➤ **Example 4.** A conference attended by 200 delegates is held in a hall. The hall has 7 doors, marked A, B, ..., G. At each door, an entry book is kept and the delegates entering through that door sign it in the order in which they enter. If each delegate is free to enter any time and through any door he likes, how many different sets of seven lists would arise in all ? (assume that every person signs only at his first entry).

➤ ***Solution*** Let a, b, c, d, e, f, g, be the number of persons entering through the doors A, B, C, D, E, F, G respectively.

Then $a + b + c + d + e + f + g = 200$

The number of ways of dividing 200 persons in 7 groups $= {}^{n+r-1}C_{r-1} = {}^{7+200-1}C_{7-1} = {}^{206}C_{6}$

The number of ways of arranging 200 persons in these groups containing a, b, c, d, e, f, g persons thoroughly is 200!.

$\therefore$ The required number $= {}^{206}C_{6} \,.\, 200! \,.$

Distribution in Groups

1. The number of ways in which n identical things can be distributed into r different groups is ${}^{n+r-1}C_{r-1}$ or ${}^{n-1}C_{r-1}$, according as blank groups are or are not admissible.

Proof : If there are no blank lots, any such distribution can be effected as follows: place the n things in a row and put marks of partition in a selection of r – 1 out of the n – 1 spaces between them. This can be done in ${}^{n-1}C_{r-1}$ ways, which is therefore the number of such distributions.

If blank lots are allowed, the number of distributions is the same as that of n + r identical things into r groups with no blank groups. For such a distribution can be effected thus: put one of the n + r things into each of the r groups, and distribute the remaining n things into r groups, blank groups being allowed. Hence the number in question is ${}^{n+r-1}C_{r-1}$.

Alternatively,

The number of ways in which n identical things can be distributed into r different groups when blank groups are admissible.

$$= \text{coeff. of } x^n \text{ in } (1 + x + x^2 + ... + x^n)^r$$
$$= {}^{n+r-1}C_{r-1}$$

The number of ways in which n identical things can be distributed into r different groups when blank groups are not admissible.

$$= \text{coeff. of } x^n \text{ in } (x + x^2 + ... + x^n)^r$$
$$= {}^{n-1}C_{r-1}$$

The number of terms in the expansion of $(a_1 + a_2 + a_3 + ... + a_n)^r$
= number of ways of distributing n identical things among r persons when each person can get zero, one or more things $= {}^{n+r-1}C_{r-1}$

➤ **Example 5.** In how many ways can 20 oranges can given to four children if each child should get atleast one orange.

➤ ***Solution*** The maximum number of oranges that a child can get = 20 – 3 = 17.

Hence, the required number of ways

$$= \text{coeff. of } x^{20} \text{ in } (x + x^2 + x^3 + ... + x^{17})^4$$
$$= \text{coeff. of } x^{16} \text{ in } (1 + x + x^2 + ... + x^{16})^4$$
$$= \text{coeff. of } x^{16} \text{ in } (1 - x^{17})^4 (1 - x)^{-4}$$
$$= \text{coeff. of } x^{16} \text{ in } (1 - x)^{-4}$$
$$= {}^{16+4-1}C_{16}$$
$$= {}^{19}C_{3} = 969$$

➤ **Example 6.** Four boys picked up 30 mangoes. In how many ways can they divide them, if all mangoes are identical ?

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Clearly, 30 mangoes can be distributed among 4 boys such that each boy can receive any number of mangoes.

Hence the total number of ways

$= {}^{30+4-1}C_{4-1} \qquad = {}^{33}C_3 \quad = 5456.$

➤ **Example 7.** In how many ways 5 identical balls can be distributed into 3 different boxes so that no box remains empty ?

➤ ***Solution*** The required number of ways

$= {}^{5-1}C_{3-1} = {}^4C_2 = 6$

Alternative :

Each box must contain atleast one ball since no box remains empty. Boxes can have balls in the following ways : 1, 1, 3 or 1, 2, 2

Here balls are identical but boxes are different the number of combinations will be 1 in each way.

$\therefore$ The required number of ways

$$= 1 \times \frac{3!}{2!} + 1 \times \frac{3!}{2!} = 6.$$

➤ **Example 8.** Find the number of ways of distributing 4 identical things among two persons when each person gets atleast one thing.

➤ ***Solution*** Let power of x denote the number of things given to persons. Here each person will get a minimum of one thing and maximum of 3 things.

The required number $=$ coeff. of x^4 in $(x^1 + x^2 + x^3)^2$

$$= {}^{4-1}C_{3-1} = 3$$

2. The number of ways in which n identical things can be distributed into r groups so that no group contains less than ℓ things and more than m things (ℓ < m) is coefficient of x^n in the expansion of $(x^\ell + x^{\ell+1} + x^{\ell+2} + ... + x^m)^r$

➤ **Example 9.** In how many ways 16 identical things can be distributed among 4 persons if each person gets atleast 3 things.

➤ ***Solution*** The required number

$=$ coeff. of x^{16} in $(x^3 + x^4 + ... + x^7)^4$

$=$ coeff. of x^{16} in $x^{12}\,(1 + x + ... + x^4)^4$

$$= \text{coeff. of } x^4 \text{ in } \left(\frac{1-x^5}{1-x}\right)^4$$

$=$ coeff. of x^4 in $[(1 - x^5)^4(1 - x)^{-4}]$

$=$ coeff. of x_4 in $[(1 - 4x^5 + 6x^{10} - ....)(1 - x)^{-4}]$

$=$ coeff. of x^4 in $(1 - x)^{-4}$ [leaving terms containing powers of x greater than 4]

$$= {}^7C_3 = \frac{4.5.6.7}{4!} = 35$$

Alternative

Let the four persons be given a, b, c, d things respectively, then $a+b+c+d=16$, where $a, b, c, d \geq 3$

Let $x = a - 3, \quad y = b - 3, \; z = c - 3, \quad t = d - 3$

Then $a + b + c + d = 16$

or $x + y + z + t + 12 = 16$

or $x + y + z + t = 4, \; x, y, z, t \geq 0$

The required number = number of ways of distributing 4 identical things among 4 persons when each person can get zero or more things

$= {}^{4+4-1}C_{4-1} = {}^7C_3 = 35.$

3. The number of ways in which n different things can be distributed into different groups, blank groups being admissible is r^n. For each of the n things can be assigned to any one of the r groups. Alternatively, the number of ways

$$= \text{coeff. of } x^n \text{ in } n!\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} .....\right)^r$$

$$= \text{coeff. of } x^n \text{ in } n!.\, e^{rx} = = \frac{r^n}{n!}$$

Coefficient of x^n in e^{rx}

$$= \left(1 + \frac{rx}{1!} + \frac{r^2x^2}{2!} + \frac{r^3x^3}{3!} .....\right) = \frac{r^n}{n!}$$

4. The number of ways in which different things can be distributed into r different groups, there being no blank group, is

$$= \text{coeff. of } x^n \text{ in } n!\left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} .....\right)^r$$

$=$ coeff. of x^n in $n!.\,(e^x - 1)^r$

$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - ... + (-1)^{r-1}\,{}^rC_{r-1}$

www.pdfworld.net

www.pdfworld.net

➤ **Example 10.** In how many ways 5 different balls can be distributed into 3 boxes so that no box remains empty ?

➤ ***Solution*** The required number of ways

$= 3^5 - {}^3C_1 (3-1)^5 + {}^3C_2 (3-2)^5 - {}^3C_3 (3-3)^5$

$= 243 - 96 + 3 - 0 \quad = 150$

OR

Coefficient of x^5 in $5!\,(e^x - 1)^3$

= coeff. of x^5 in $5!\,(e^{3x} - 3e^{2x} + 3e^x - 1)$

$$= 5!\left(\frac{3^5}{5!} - 3\frac{2^5}{5!} + 3\frac{1}{5!}\right)$$

$= 3^5 - 3.2^5 + 3 = 243 - 96 + 3 = 150$

Alternative

Each box must contain atleast one ball since no box remains empty. Boxes can have balls in the following ways : 1, 1, 3 or 1, 2, 2

The number of ways to divide the balls in the first case $= {}^5C_1 \times {}^4C_1 \times {}^3C_3$

∴ The total number of ways to distribute 1, 1, 3 balls to the boxes

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!}$$

$= 5 \times 4 \times 1 \times 3 = 60$

and the number of ways to distribute the balls in the second case

$$= {}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!}$$

$= 5 \times 6 \times 1 \times 3 = 90$

∴ The required number of ways = 60 + 90 = 150

Permutations with repetitions

The number of permutations of r things out of n things of which p are of one kind, q are of second kind and so on = coeff of x^r in

$$r!\left[\left(1+\frac{x}{1!}+\frac{x^2}{2!}+....+\frac{x^p}{p!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+....+\frac{x^q}{q!}\right)....\right]$$

Let the different things be denoted by a, b, c, ... Consider the product

$$\left(1+\frac{ax}{1!}+\frac{a^2x^2}{2!}+.....+\frac{a^px^p}{p!}\right)$$

$$\left(1+\frac{bx}{1!}+\frac{b^2x^2}{2!}+.....+\frac{b^qx^q}{q!}\right) \quad ...(1)$$

The required number of permutations

= sum of all possible terms of the form

$$\frac{r!}{p!q....!}a^pb^q.....,\quad \text{where } p+q+....=r$$

= coeff. of x^r in

$$r!\left[\left(1+\frac{x}{1!}+.....+\frac{x^p}{p!}\right)\left(1+\frac{x}{1!}+.....+\frac{x^q}{q!}\right).....\right]$$

➤ **Example 11.** Find the number of combinations and the number of permutations of the letters of the word PARALLEL, taken four at a time.

➤ ***Solution*** First Part : Here L occurs three times, A occurs two times and there are three other different letters P, R, E, from which we have to select 4 letters.

∴ The required number

$= \text{coeff. of } x^4 \text{ in } [(1+x+x^2+x^3)(1+x+x^2)(1+x)^3]$

$= \text{coeff. of } x^4 \text{ in } (1-x^4)(1-x^3)(1-x^2)^3\,(1-x)^{-5}$

$= \text{coeff. of } x^4 \text{ in } (1-x^3-x^4+x^7)\,(1-3x^2+3x^4 + x^6)(1-x)^{-5}$

$= \text{coeff of } x^4 \text{ in } (1-x^3+2x^4-3x^2....)(1-x)^{-5}$

$= \text{coeff. of } x^4 \text{ in } (1-x)^{-5} - \text{coeff. of x in } (1-x)^{-5}$

$+ 2\,\text{coeff. of } x^0 \text{ in } (1-x)^{-5} - 3\,\text{coeff. of } x^2 \text{ in } (1-x)^{-5}$

$$= \frac{5.6.7.8}{4!} - 5 - 2 - 3\,.\,\frac{6.5}{2!} = 22$$

Second Part :

The required number = coeff . x^4 in

$$4!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)\left(1+\frac{x}{1!}\right)^3$$

$= 4!$. coeff. of x^4 in

$$[1+\frac{2x}{1!}+2\frac{x^2}{2!}+\frac{x^2}{1!}+2\frac{x^3}{2!}+\frac{x^4}{2!\,2!}+\frac{x^3}{3!}\frac{x^2}{2!}]$$

$.(1 + 3x + 3x^2 + x^3)$

$= 4!.$ coeff. of x^4 in $(1 + 2x + 2x^2 + \frac{7}{6}x^3 + \frac{5}{12}x^4)$

$$.(1 + 3x + 3x^2 + x^3) \;= 24\left(2+6+\frac{7}{2}+\frac{5}{12}\right) = 286$$

Alternative

Total number of letters = 8,

L occurs thrice and A occurs twice

The different letters are A, L, P, R, E, (five)

www.pdfworld.net

www.pdfworld.net

Case I : When 3 letters are identical and remaining one different :

The letters used will be three L's and one out of A, P, R, E

The number of arrangements in this case

$$= 1 \times {}^4C_1 \times \frac{4!}{3!} = 16$$

Case II : When 2 letters are identical and remaining 2 are identical :

The letters used will be two L's and two A's

$$\text{The number of arrangements} = 1 \times 1 \times \frac{4!}{2!2!} = 6$$

Case III : When 2 letters are identical and remaining 2 are different :

The letters used will be (i) two L's and two out of P, R, A, E :

$$\text{The number of arrangements} = 1 \times {}^4C_2 \times \frac{4!}{2!} = 72$$

(ii) two A's and two out of P, L, R, E :

$$\text{The number of arrangements} = 1 \times {}^4C_2 \times \frac{4!}{2!} = 72$$

Case IV : When all the 4 letters are different :

The number of arrangements $= {}^5P_4 = 120$

$\therefore$ The required number $= 16 + 6 + (72 + 72) + 120$
$= 286$

➤ **Example 12.** Find the number of combinations and permutations of four alphabets taken from the word MATHEMATICS.

➤ ***Solution*** The word MATHEMATICS consists of
2M's, 2A's, 2T's, 1H, 1E, 1I, 1C, 1S

Hence, the required number of combinations of four alphabets, is

$$= \text{coeff. of } x^4 \text{ in } (1 + x + x^2)^3 (1 + x)^5$$
$$= \text{coeff of } x^4 \text{ in } [(1 + x)^3 + 3(1 + x)^2 x^2 + 3(1 + x)x^4 + x^6]. (1 + x)^5$$
$$= \text{coeff. of } x^4 \text{ in } [(1 + x)^8 + 3x^2(1 + x)^7 + 3x^4(1 + x)^6 + x^6 (1 + x)^5]$$
$$= {}^8C_4 + 3 \times {}^7C_2 + 3 \times {}^6C_0 + 0$$
$$= 70 + 63 + 3 = 136$$

and number of permutations of 4 alphabets, is

$$= \text{coeff. of } x^4 \text{ in } 4! \left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)^3 \left(1 + \frac{x}{1!}\right)^5$$
$$= \text{coeff. of } x^4 \text{ in } 4! \left(1 + x + \frac{x^2}{2}\right)^3 (1 + x)^5$$
$$= \text{coeff. of } x^4 \text{ in } 4! \left[(1+x)^3 + \frac{3}{2}(1+x)^2 x^2 + \frac{3}{4}x^4(1+x) + \frac{1}{8}x^6\right](1+x)^5$$
$$= \text{coeff. of } x^4 \text{ in } 4! \left[(1+x)^8 + \frac{3}{2}(1+x)^2(1+x)^7 + \frac{3}{4}x^4(1+x) + \frac{1}{8}x^6(1+x)^5\right]$$
$$= 4! \left({}^8C_4 + \frac{3}{2} \times {}^7C_2 + \frac{3}{4} \times {}^6C_0 + 0\right)$$
$$= 24\left(70 + \frac{3}{2} \times 63 + \frac{3}{4}\right)$$
$$= 1680 + 756 + 18 = 2454$$

➤ **Example 13.** Find the number of combinations and permutations of 4 letters taken from the word EXAMINATION.

➤ ***Solution*** There are 11 letters A, A; I, I; N, N; E, X, M, T, O.

The number of combinations = coefficient of x^4 in
$(1 + x + x^2)^3 (1 + x)^5$

($\because$ 2A's, 2R's, 2N's, 1E, 1X, 1M, 1T and 1O)

$$= \text{coeff. of } x^4 \text{ in } \{(1 + x)^3 + x^6 + 3(1 + x)^2 x^2 + 3 (1 + x) x^4\} (1 + x)^5$$
$$= \text{coeff. of } x^4 \text{ in } \{(1 + x)^8 + x^6(1 + x)^5 + 3x^2 (1 + x)^7 + 3x^4 (1 + x)^6\}$$
$$= {}^8C_4 + 0 + 3 \,.\, {}^7C_2 + 3$$
$$= \frac{8.7.6.5}{1.2.3.4} + 3.\frac{7.6}{1.2} + 3$$
$$= 70 + 63 + 3 = 136$$

and the number of permutations

$$= \text{coefficient of } x^4 \text{ in } 4 ! \left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)^3 \left(1 + \frac{x}{1!}\right)^5$$
$$= \text{coefficient of } x^4 \text{ in } 4 ! \left(1 + x + \frac{x^2}{2}\right)^3 (1 + x)^5$$
$$= \text{coefficient of } x^4 \text{ in } 4 ! \left\{(1+x)^3 + \frac{x^6}{8} + \frac{3}{2}(1+x)^2 x^2 + \frac{3}{4}x^4(1+x)\right\}(1 + x)^5$$
$$= \text{coefficient of } x^4 \text{ in } 4 ! \left\{(1+x)^8 + \frac{x^6}{8}(1+x)^5 + \frac{3}{2}x^2(1+x)^7 + \frac{3}{4}x^4(1+x)^6\right\}$$

www.pdfworld.net

www.pdfworld.net

$$= 4\,!\left\{{}^{8}C_4 + 0 + \frac{3}{2}.{}^{7}C_2 + \frac{3}{4}\right\}$$

$$= 24\left\{\frac{8.7.6.5}{1.2.3.4} + \frac{3}{2} + \frac{7.6}{1.2} + \frac{3}{4}\right\}$$

$$= 8\,.\,7\,.\,6\,.\,5 + 6(3\,.\,7\,.\,6) + 6\,.\,3$$

$$= 1680 + 756 + 18 = 2454$$

Alternative

A, A; I, I; N, N; E, X, M, T, O.

The following cases arise :

Case I : All letters different : The required number of choosing 4 different letters from 8 different (A, I, N, E, X, M, T, O) types of the letters

The number of combinations $= {}^{8}C_4 = \frac{8.7.6.5}{1.2.3.4} = 70$

and the number of permutations $= {}^{8}P_4 = 8.7.6.5$
$= 1680$

Case II : Two alike of one type & Two alike of another type : This must be 2A's 2I's, or 2I's, 2N's, 2A's.

$\therefore$ The number of combinations $= {}^{3}C_2 = 3$

and the number of permutations $= 3\,.\,\frac{4\,!}{2\,!\,2\,!} = 18$

Case III : Two alike and Two different :

This must be 2A's or 2I's or 2N's

and for each case 7 different letters.

e.g. for 2A, 7 differents' are I, N, E, X, M, T, O.

$\therefore$ The number of combinations $= {}^{3}C_1 + {}^{7}C_2$

$$= 3 \times \frac{7 \times 6}{1 \times 2} = 63$$

and the number of permutations $= 63.\,\frac{4\,!}{2\,!} = 756$

From Cases I, II and III,

The number of combinations $= 70 + 3 + 63 = 136.$

and the number of permutations $= 1680 + 18 + 756$
$= 2454.$

STUDY TIP

(i) m different white balls and n different red balls are to be arranged in a line such that the balls of the same colour are always together $= (m\,!\,n\,!\,2\,!)$

(ii) m different white balls and n different red balls are to be arranged in a line such that all the red balls are together $= ((m+1)!\,n!)$

(iii) m different white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \geq n - 1) = ({}^{m+1}C_n\, m!\, n!)$

(iv) m different white balls and m different red balls are to be arranged in a line such that colour of the balls is alternating $= (2 \times (m\,!)^2)$

(v) m identical white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \geq n - 1) = ({}^{m+1}C_n)$

(vi) m identical white balls and n different red balls are to be arranged in a line such that no two red balls are together $(m \geq n - 1) = ({}^{m+1}C_n\, n!)$

(vii) If n things are arranged in a line the number of selections of r things $(n \geq 2r - 1)$ such that no two things are adjacent is same number of way of arranging $n - r$ identical white balls and r identical red ball in a line such that no two balls are together $= ({}^{n-r+1}C_r)$. e.g. suppose there are n stations on trains's route and a train has to stop at r stations such that no two stations are adjacent. The number of ways must be ${}^{n-r+1}C_r$.

(viii) Suppose there are N seats in a particular row of a theatre. The number of ways of making n people sit $(N \geq 2n - 1)$ such that no two people sit side by side is same as number of ways of arranging $N - n$ identical white balls (empty seats) and n different red balls (n people) such that no two red balls are together. The required number of ways are ${}^{N-n+1}C_n \times n!$.

➤ **Example 14.** Find the number of integers which lie between the numbers 786 and 999786, if the sum of their digits is 25.

➤ ***Solution*** The required number of integers

(Note that the numbers will be of 3, 4, 5 and 6 digits)

$$= \text{co-efficient of } x^{25} \text{ in } (1 + x + x^2 + \ldots\ldots + x^9)^6$$

$$= \text{coeff. of } x^{25} \text{ in } (1 - x^{10})^6 (1 - x)^{-6}$$

$$= \text{coeff. of } x^{25} \text{ in } (1 - 6x^{10} + 15x^{20} + \ldots.)\,(1 - x)^{-6}$$

$$= {}^{30}C_5 - 6 \times {}^{15}C_5 + 15 \times {}^{10}C_5$$

$$= 142506 - 96024 + 3780$$

$$= 53262$$

www.pdfworld.net

www.pdfworld.net

➤ **Example 15.** How many integers between 10^2 and 10^5 have the sum their digits equal to 14.

➤ ***Solution*** Let us find all five digit numbers lying between 0 and 10^5 whose digits add up to a sum of 14. The problem is equivalent to finding the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 14$$

where $0 \le x_1, x_2, x_3, x_4, x_5 \le 9$,

which is = coeff. of x^{14} in $(1 + x + x^2 + ... + x^9)^5$

= coeff. of x^{14} in $(1 - x^{10})^5(1 - x)^{-5}$

= coeff. of x^{14} in $(1 - 5x^{10} + 10x^{20} + ...)$

$.(1 + {}^5C_1x + {}^6C_2x^2 + ...)$

$= {}^{18}C_{14} - 5 \times {}^8C_4$

$$= \frac{18\times17\times16\times15}{1\times2\times3\times4} - \frac{5\times8\times7\times6\times5}{1\times2\times3\times4} = 3410$$

The numbers lying between 0 and 10^2 whose digits add upto 14 are 59, 68, 77, 86, 95.

Hence, the required number of such numbers lying between 10^2 and 10^5 whose digits add up to 14, is

$= 3410 - 5 = 3405$

➤ **Example 16.** 100 different objects are arranged in a row. In how many ways can 3 objects be selected so that no two objects are consecutive.

➤ ***Solution*** Let $a_1, a_2, a_3, ..., \times a_{97}$ denote the objects which are not selected.

$$\times a_1 \times a_2 \times a_3 \times .....\times a_{96} \times a_{97}$$

then the three objects which are to be selected must be at any of the places indicated by cross. number of cross marks in 98.

$\therefore$ The required number = ${}^{98}C_3 = 152096$

Alternative

Let b_1, b_2, b_3 denote the three objects which are selected $\boxed{x}\ b_1\ \boxed{y}\ b_2\ \boxed{z}\ b_3\ \boxed{t}$

Let x, y, z, t be the number of points on the left of b_1 between b_1 and b_2, between b_2 and b_3 and on the right on b_3 respectively.

Then $x + y + z + t = 97$

where $x \ge 0, y \ge 1, z \ge 1, t \ge 0$ ...(1)

Let $a = x, b = y - 1, c = z - 1, d = t$

Then $a + b + c + d = x + y + z + t - 2 = 97 - 2 = 95$

or, $a + b + c + d = 95$, where $a, b, c, d \ge 0$ ...(2)

$\therefore$ The required number = number of non-negative integral solutions of equation (2)

$= {}^{n+r-1}C_{r-1} = {}^{95+4-1}C_{4-1} = {}^{98}C_3 = 152096$

➤ **Example 17.** There are 12 intermediate stations betwen two places A and B. In how many ways can a train be made to stop at 4 of these 12 intermediate stations no two of which are consecutive.

➤ ***Solution*** Let $S_1, S_2, ..., S_8$ denote the stations where the train does not stop. The four stations where the train stops should be at any four of the nine places indicated by cross.

$\therefore$ The required number = ${}^9C_4 = \frac{9.8.7.6}{4!} = 126$

Alternative:

Let S_1, S_2, S_3, S_4 be the four intermediate stations where the train stops.

$\boxed{a}\ S_1\ \boxed{b}\ S_2\ \boxed{c}\ S_3\ \boxed{d}\ S_4\ \boxed{e}$

Let a, b, c, d, e, be the number of stations between A and S_1, S_1 and S_2, S_2 and S_3, S_3 and S_4 and S_4 and B respectively.

Then $a + b + c + d + e = 8$ ...(1)

where $a \ge 0, b \ge 1, c \ge 1, d \ge 1, e \ge 0$

Let $x = a, y = b - 1, z = c - 1, t = d - 1, w = e$

Now, $x + y + z + t + w = a + b + c + d + e - 3 = 8 - 3 = 5$

or, $x + y + z + t + w = 5$, where $x, y, z, t, w \ge 0$ ...(2)

$\therefore$ The required number

= number of non-negative integral solutions of (2)

$= {}^{n+r-1}C_{r-1} = {}^{5+5-1}C_{5-1} = 126$

➤ **Example 18.** 12 seats are to be occupied by 4 people. Find the number of possible arrangements if

(i) no two persons sit side by side.

(ii) there should be atleast two empty seats between any two persons.

(iii) each person has exactly one neighbour.

➤ ***Solution*** (i) Let A, B, C, D represent the 4 persons and x_1, x_2, x_3, x_4, x_5 denote the number of vacant seats on either side of the persons as shown below

$$x_1(A)\ x_2\ (B)\ x_3\ (C)\ x_4\ (D)\ x_5$$

We have

$$x_1 + x_2 + x_3 + x_4 + x_5 = 12 - 4 = 8$$

where $x_1, x_5 \ge 0$ and $x_2, x_3, x_4 \ge 1$.

The number of integral solutions to the above equation

www.pdfworld.net

www.pdfworld.net

$= \text{coeff. of } x^8 \text{ in } (1+x+x^2+....)^2 (x+x^2+x^3+.....)^3$
$= \text{coeff of } x^{8-3} \text{ in } (1+x+x_2+.....)^5$
$= \text{coeff. of } x^5 \text{ in } (1-x)^{-5}$
$= {}^{5+5-1}C_{5-1} = {}^9C_4.$

Now, the four persons can be arranged amongst themselves in 4! ways. Hence, the required number of seating arrangements $= 4! \times {}^9C_4 = {}^9C_4$

(ii) In this case, we have

$x_1 + x_2 + x_3 + x_4 + x_5 = 8$

where $x_1, x_5 \geq 0$ and $x_2, x_3, x_4 \geq 2$.

The number of integral solutions to the above equation

$= \text{coeff. of } x^8 \text{ in } (1+x+x^2+.....)^2 (x^2+x^3+x^4+.....)^3$
$= \text{coeff. of } x^{8-6} \text{ in } (1+x+x^2+.....)^5$
$= \text{coeff. of } x^2 \text{ in } (1-x)^{-5}$
$= {}^{5+2-1}C_{5-1} = {}^6C_4$

Now, the four persons can be arranged amongst themselves in 4! ways. Hence, the required number of seating arrangements $= 4! \times {}^6C_4 = {}^6P_4$.

(iii) In this case it is required that the persons sit in pairs as shown below

$x_1(AB)\, x_2\, (CD)\, x_3$

We have

$x_1 + x_2 + x_3 = 12 - 4 = 8$

where $x_1, x_3 \geq 0$ and $x_2 \geq 1$.

The number of integer solutions to the above equation

$= \text{coeff. of } x^8 \text{ in } (1+x+x^2+.....)^2 (x+x^2+x^3+.....)^1$
$= \text{coeff. of } x^{8-1} \text{ in } (1+x+x^2+.....)^3$
$= \text{coeff. of } x^7 \text{ in } (1-x)^{-3}$
$= {}^{7+3-1}C_{3-1} = {}^9C_2.$

Now the four persons can be arranged amongst themselves in 4! ways. Hence, the required number of seating arrangements $= 4! \times {}^9C_2$.

➤ **Example 19.** Find the number of ways in which two different numbers between 1 and 100 can be chosen such that the difference between them is atmost 10.

➤ ***Solution*** Let P, Q denote the chosen numbers and x_1, x_2, x_3 denote the number of numbers on either side of the chosen numbers as shown below x_1 (P) x_2 (Q) x_3 We have $x_1 + x_2 + x_3 = 98$ where $0 \leq x_2 \leq 9$ and $0 \leq x_1, x_3 \leq 98$.

Hence, the required number of ways

$= \text{coeff. of } x^{98} \text{ in } (1+x+x^2+....+x^{98})^2 .(1+x+x^2+.....+x^9)$
$= \text{coeff. of } x^{98} \text{ in } \left(\frac{1-x^{99}}{1-x}\right)\left(\frac{1-x^{10}}{1-x}\right)$
$= \text{coeff of } x^{98} \text{ in } (1-x^{99})^2 (1-x^{10})(1-x)^{-3}$
$= \text{coeff. of } x^{98} \text{ in } (1-x^{10})(1-x)^{-3}$
$= \text{coeff. of } x^{98} \text{ in } (1-x)^{-3} - \text{coeff. of } x^{88} \text{ in } (1-x)^{-3}$
$= {}^{100}C_2 - {}^{90}C_2.$

➤ **Example 20.** From the integers 1, 2, ..., 11, two groups (not necessarily disjoint) are selected; the first group contains 5 integers and the second group contains 2 integers. In how many ways, allowing repetitions, can the selection be made if

(i) there are no further conditions ?

(ii) each group contains either all odd integers or all even integers ?

➤ ***Solution*** (i) The first group G_1 of 5 integers from the 11 integers can be chosen in ${}^{11+5-1}C_{11-1} = {}^{15}C_{10} = {}^{15}C_5$ ways. Similarly, the second group G_2 of 2 integers from the 11 integers can be chosen in ${}^{11+2-1}C_{11-1} = {}^{12}C_{10} = {}^{12}C_2$ ways. Therefore the two groups G_1 and G_2 can be chosen in ${}^{15}C_5 \times {}^{12}C_2$ ways.

(ii) We have 4 mutually exclusive cases :

(a) G_1 all even, G_2 all even
(b) G_1 all odd, G_2 all odd
(c) G_1 all even, G_2 all odd
(d) G_1 all even, G_2 all even.

Hence the total number of ways is

${}^{10}C_5 \times {}^6C_2 + {}^{10}C_5 \times {}^7C_2 + {}^9C_5 \times {}^7C_2 + {}^9C_5 \times {}^6C_2$

➤ **Example 21.** Show that ${}^{n+5}C_5$ distinct throws are possible with a throw of n dices which are indistinguishable among themselves. (Say that two outcomes are the same if they contain the same number ones, the same number of twos, ..., and the same number of sixes.)

➤ ***Solution*** The dices are indistinguishable. Let us understand the problem with n = 2. The different possible throws are :

www.pdfworld.net

www.pdfworld.net

1, 1					
1, 2	2, 2				
1, 3	2, 3	3, 3			
1, 4	2, 4	3, 4	4, 4		
1, 5	2, 5	3, 5	4, 5	5, 5	
1, 6	2, 6	3, 6	4, 6	5, 6	6, 6

The total number is 21 which is equal to ${}^{2+6-1}C_{6-1} = {}^{7}C_{5}$.

The fact that the dices are indistinguishable express itself thus. It is not a question of which dice shows up what number. It is only a question of what numbers are shown up on top. The numbers on the dices are 1, 2, 3, 4, 5, 6. These numbers show up in the n dice in various combinations with repetitions. So we are now looking at combinations of 6 things taken n at a time with repetitions permitted, up to a maximum of n times. This is the number of solutions of $x_1 + x_2 + ... + x_6 = n$ is non-negative integers, where x_1 is the number of the dices showing ones, x_2 the number showing twos, and so on.

The answer is ${}^{n+6-1}C_{6-1} = {}^{n+5}C_{5}$.

➤ **Example 22.** Find the number of ways in which n distinct objects can be put into two identical boxes so that no box remains empty.

➤ ***Solution*** For every object, we have two options, putting them in either of the two boxes. Thus, the n objects can be put in 2^n ways.

Since, the boxes which can be arranged amongst themselves in 2! ways are identical, therefore,

$$\text{the number of ways} = \frac{1}{2!} \times 2^n = 2^{n-1}$$

But the above count also includes the one case in which all the objects are put in one box. Hence, excluding that case, the required number of ways $= 2^{n-1} - 1$.

➤ **Example 23.** Find the number of ways in which n distinct objects can be put into three identical boxes so that no box remains empty.

➤ ***Solution*** For every object, we have three options, putting them in either of the three boxes. Thus, the objects can put in 3^n ways.

But this includes ${}^{3}C_{1}$ cases in which all the objects are put in one box.

It also includes ${}^{3}C_{2}$ $(2^n - 2)$ cases in which all the objects are put in two boxes only.

Since the boxes which can be arranged amongst themselves in 3! ways, are identical, therefore, the

$$\text{number of ways} = \frac{1}{3!} \times \text{Total number of ways}$$

Hence, required number of ways is

$$= \frac{1}{3!}\left(3^n - {}^{3}C_{2}\,(2^n - 2) - {}^{3}C_{1}\right).$$

Practice Problems [N]

1. If balls come in three colors, red, white and blue, how many different combinations of ten balls are there ?
2. How many different collections of cakes, eight cakes in each collection, can be formed from four kinds of cake ?
3. Three boys picked 5 apples. In how many ways can these 5 apples be distributed among the 3 boys so that each can have any number of apples.
4. Show that a selection of 10 balls can be made from an unlimited number of red, white, blue and green balls in 286 different ways, and that 84 of these contain balls of all four colours.
5. Show that the number of different selections of 5 letters which can be made from five a's, four b's three c's, two d's and one e is 71.
6. Show that a selection of (n – 1) things can be made from 3n things, of which n are alike of one kind, n are alike of another kind and n are alike of a third kind, in $n.2^{n-1}$ different ways.
7. In the expansion of $(a_1 + a_2 + ... + a_p)^n$ where n is any positive integer not greater than p, prove that the coefficient of any term in which none of the numbers $a_1, a_2, ... a_p$ occurs more than once is n!.

www.pdfworld.net

www.pdfworld.net

8. In how many ways can a white balls, b black balls and c red balls be put in m different bags, if one or more of the bags may remain empty ?

9. In how many ways can 5 rings be worn on the 4 fingers of one hand ?

10. Show that the number of combinations of (p + q) things, r at a time, p things being of one sort and q of another is $(p + q - r + 1)$ where $r > p > q$.

11. Six cards are drawn one by one from a set of unlimited number of cards, each card is marked with numbers -1, 0 or 1. The number of different ways in which they can be drawn if the sum of the numbers shown by them vanishes.

12. n objects are arranged in a row . A subset of these objects is called unfriendly, if no two of its elements are consecutive. Show that the number of unfriendly subsets of a k−element set is ${}^{n-k+1}C_k$.

13. There are 7 balls of different colours which are to be placed in 3 different boxes. Each box can hold all the 7 balls. Find the number of ways in which the balls can be placed so that no box remains empty.

4.18 FACTORS OF NATURAL NUMBER

Let $N = p^a . q^b . r^c$ ..... where p , q , r ..... are distinct primes & a , b , c .... are natural numbers then :

(i) The total numbers of divisors of N including 1and N is = (a + 1)(b + 1) (c + 1).....

(ii) The sum of these divisors is
$= (p^0 + p^1 + p^2 + ... + p^a)\ (q^0 + q^1 + q^2 + ... + q^b)\ (r^0 + r^1 + r^2 + ... + r^c)$ ....

(iii) Number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a+1)(b+1)(c+1)\ .... & \text{if N is not a perfect square} \\ \frac{1}{2}[(a+1)(b+1)(c+1)\ .... + 1] & \text{if N is a perfect square} \end{cases}$$

(iv) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

1. To find the sum of all the divisors of a composite number.

Let N denote the number, and suppose $N = a^p b^q c^r ....$, where a, b, c, ..., are different prime numbers and p, q, r, ... are positive integers. Then it is clear that each term of the product

$$(1 + a + a^2 + ... + a^p)(1 + b + b^2 + ... + b^q) .(1 + c + c^2 + ... c^r).....$$

is a divisor of the given number, and that no other number is a divisor. This includes as divisors, both unity and the number itself.

Hence the sum of all the divisors of N including 1 and N itself is

$$(1 + a + a^2 + ... + a^p)(1 + b + b^2 + ... + b^q) .(1 + c + c^2 + ... c^r).....$$

$$= \left(\frac{a^{p+1} - 1}{a - 1}\right)\left(\frac{b^{q+1} - 1}{b - 1}\right)\left(\frac{c^{r+1} - 1}{c - 1}\right).....$$

2. To find the number of divisors of a composite number.

Let $N = a^p b^q c^r ....$

Since each term of the product $(1 + a + a^2 + ... + a^p)(1 + b + b^2 + ... + b^q)(1 + c + c^2 + ... c^r).....$ is a divisor of the given number hence the number of divisors of N including 1 and N itself is the number of terms in the product, that is, $(p + 1)(q + 1)(r + 1)....$

Note that the number of proper divisors is $(p + 1)(q + 1)(r + 1).... - 2$, which excludes 1 and N.

3. To find the number of ways in which a composite number can be resolved into two factors.

Let $N = a^p b^q c^r ....$

The number of divisors of N is $(p + 1)(q + 1)(r + 1)....$. Among the divisors we can pair two divisors whose product is N. A pair of such divisors corresponds to

www.pdfworld.net

www.pdfworld.net

one way of resolving N into two factors. Hence, the number of ways of resolving N into two factors is half the total number of divisors.

Thus, the required number is $\frac{1}{2}(p+1)(q+1)(r+1)....$

But, this assumes that N is not a perfect square (atleast one of the quantities p, q, r.... is an odd number). If N is a perfect square, of resolution into factors is $\sqrt{N}\times\sqrt{N}$, and to this way there corresponds only one divisor $\sqrt{N}$.

If we exclude this, the number of ways of resolution is $\frac{1}{2}\{(p+1)(q+1)(r+1)...-1\}$

and to this we must add the one way $\sqrt{N}\times\sqrt{N}$.

Thus the required number is

$\frac{1}{2}\{(p+1)(q+1)(r+1)...+1\}$.

The number of ways of resolving N into two factors is $\frac{1}{2}(p+1)(q+1)(r+1)...$, when N is not a perfect square $\frac{1}{2}\{(p+1)(q+1)(r+1)...+1\}$, when N is a perfect square.

4. To find the product of all the divisors of a composite number.

Let $N = a^p b^q c^r....$

Assume that N is not a perfect square. Each way of resolving N into two factors implies that a pair of divisors can be multiplied to obtain N.

The number of such pairs is

$\frac{1}{2}(p+1)(q+1)(r+1)....$ as found above. Hence the product of all the divisors of N is $N^{\frac{1}{2}(p+1)(q+1)(r+1)...}$

If N is a perfect square, one way of resolution into factors is $\sqrt{N}\times\sqrt{N}$, and to this way there corresponds only one divisor $\sqrt{N}$. So in the product $N^{\frac{1}{2}\{(p+1)(q+1)(r+1)...+1\}}$ we must divide $\sqrt{N}$.

Hence, again the product of all the divisors of N is $N^{\frac{1}{2}(p+1)(q+1)(r+1)...}$.

The product of all the divisors of N is $N^{\frac{1}{2}(p+1)(q+1)(r+1)...}$ in both cases.

5. To find the number of ways in which a composite number can be resolved into two factors which are co-prime.

Let $N = a^p b^q c^r....$

Of the two factors one must contain a^p, for otherwise there would be some power of a in one factor and some power of a in the other factor, and thus the two factors would not be prime to each other. Similarly b^q must occur in one of the factors only and so on. Hence, the required number is equal to the number of ways in which the product abc... can be resolved into two factors; that is, the number of ways is

$\frac{1}{2}(1+1)(1+1)(1+1)...$ or 2^{n-1}, where n is the number of different prime factors in N.

The number of ways in which N can be resolved into two factors which are co-prime is 2^{n-1}, where n is the number of different prime factors in N.

6. To find the number of ordered pairs (x, y) such that L.C.M. of x and y is a composite number.

Let $N = a^p b^q c^r....$

Of the two numbers x and y one must contain a^p, for otherwise the L.C.M. would have smaller power of a. If x contains a^p then y can have one of the factors 1, a, a^2, a^3,....., a^p. And if y contains a^p then x can have one of the factors 1, a, a^2, a^3,....., a^{p-1}. Hence the number of ways to distribute powers of a to x and y to ensure that the L.C.M. contains a^p is (2p + 1).

Thus, the total number of ways is $(2p+1)(2q+1)(2r+1).....$

➤ **Example 1.** Find the number of divisors and sum of the divisors of 2520.

➤ ***Solution*** $2520 = 2^3 \times 3^2 \times 5 \times 7$

Each term of the product $(1+2+2^2+2^3)(1+3+3^2)(1+5)(1+7)$ is a divisor of 2520.

$\therefore$ Total number of divisors of 2520 = number of terms in the product = $4 \times 3 \times 2 \times 2 = 48$

The sum of divisors of 2520

www.pdfworld.net

www.pdfworld.net

$= (1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2)(1 + 5)(1 + 7)$
$= 15 \times 13 \times 6 \times 8 = 9360$

➤ **Example 2.** How many positive integers are factors of 30030 ?

➤ ***Solution*** Here $m = 30030 = 2 \times 3 \times 5 \times 7 \times 11 \times 13$ and each prime factor occurs only once. Hence every factor of m corresponds to the product of elements of a particular subset of the 6 set $S = \{2, 3, 5, 7, 11, 13\}$. Note that the empty subset corresponds to the factor 1. Hence the number of factors of m is equal to the number of subsets of S, namely 2^6.

➤ **Example 3.** Find the number of factors of the number 38808 (excluding 1 and the number itself). Find also the sum of these divisors.

➤ ***Solution*** $38808 = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11$

Hence the total number of divisors (excluding 1 and itself)
$= (3 + 1)(2 + 1)(2 + 1)(1 + 1) - 2 = 70$

The sum of these divisors
$= (2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2)$
$.(11^0 + 11^1)$
$= (15)(13)(57)(12) - 1 - 38808 = 94571$

➤ **Example 4.** Find the sum of all the odd divisors of 360.

➤ ***Solution*** $360 = 2^3 \cdot 3^2 \cdot 5$. Any odd divisor of 360 is of the form $3^a \cdot 5^b$

Hence the sum is $= (3^0 + 3 + 3^2)(5^0 + 5^1) = 78$

➤ **Example 5.** If p, q, r, s, t are prime numbers. Find the number of ways in which the product, $p q^2 r^3 s t$ can be expressed as product of two factors, excluding 1 as a factor .

➤ ***Solution*** Total factors $= 2 \times 3 \times 4 \times 2 \times 2 = 96$

Hence the total ways $= \frac{96}{2} = 48$, but this includes $(1, p q^2 r^3 s t)$ also. Hence required number of ways $= 48 - 1 = 47$

➤ **Example 6.** Find the number of even divisors of 1350. Also find the sum of all divisors whose unit digit is 0.

➤ ***Solution*** $1350 = 2 \times 3^3 \times 5^2$

For even divisors, 2 must be always included.

$\therefore$ Number of even divisors $= (1)(3 + 1)(2 + 1) = 12$

A divisor has unit digit 0 if it includes 2 as well as 5 as factors.

$\therefore$ The sum of such divisors $= (2)(1 + 3 + 3^2 + 3^3)(5 + 5^2) = 2400$.

➤ **Example 7.** In how many ways 8100 can be resolved into product of two factors.

➤ ***Solution*** $8100 = 2^2 \times 3^4 \times 5^2$ Here 8100 is a perfect square. $\therefore$ Number of ways

$$= \frac{1}{2}\{(2 + 1)(4 + 1)(2 + 1) + 1\} = 23$$

➤ **Example 8.** In how many ways the number 10800 can be resolved as a product of two factors.

➤ ***Solution*** Let $N = 10800 = 2^4 \cdot 3^3 \cdot 5^2$

Here N is not a perfect square ($\because$ power of 3 is odd)

Hence the number of ways $= \frac{1}{2}(4 + 1)(3 + 1)(2 + 1)$
$= 30$

➤ **Example 9.** In how many ways the number 18900 can be split in two factors which are relative prime.

➤ ***Solution*** Let $N = 18900 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7^1$

$\therefore$ $n = 4$ (number of different factors in N)

Hence the number of ways in which a composite number N can be resolved into two factors which are relative prime $= 2^{4-1} = 8$.

➤ **Example 10.** Find the number of ordered pairs (x, y) such that L.C.M. of x and y is 2520.

➤ ***Solution*** We have $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$

The various ways to distribute 2^3 to (x, y) are $(2^3, 1)$, $(2^3, 2)$, $(2^3, 2^2)$, $(2^3, 2^3)$, $(1, 2^3)$, $(2, 2^3)$, $(2^2, 2^3)$, which count to be $2 \times 3 + 1 = 7$. Similarly, other factors can be distributed to x and y. Thus the total number of ways is $(2 \times 3 + 1)(2 \times 2 + 1)(2 \times 1 + 1)(2 \times 1 + 1) = 315$.

➤ **Example 11.** Find the number of integral solutions of the equation $x_1 x_2 x_3 x_4 = 210$.

➤ ***Solution*** 210 can be factorised as 210
$= 2 \times 3 \times 5 \times 7$

All the factors are prime. Clearly any number can be assigned to x_i's in 4 ways. Hence total ways of assigning the factors to x_i's are 4^4. Now for a particular positive integral solution, we have ${}^4C_2$ solutions when any two any two x_i's become negative, and all the four solutions can be negative in ${}^4C_4$ ways. Hence, total number of integral solutions $= 4^4(1 + {}^4C_2 + {}^4C_4)\ 8.4^4$.

www.pdfworld.net

www.pdfworld.net

➤ **Example 12.** Find the number of positive integers which divide 10^{999} by not 10^{998}.

➤ ***Solution*** All the positive divisors of p^r (p a positive prime, $r \in N$) are 1, p, p^2, p^3,..., p^r. That is, the number of positive divisors of p^r is r + 1. Similarly, the number of the positive divisors of q^r (q a positive prime, $s \in N$) is s + 1. So the number of positive divisors of $p^r q^s$ is (r + 1)(s + 1).

Now, $10^{999} = 2^{999} \times 5^{999}$, where 2 and 5 are positive primes. So the number of positive divisors of 10^{999} is $(999 + 1)(999 + 1) = 1000^2$. For similar reasons, the number of positive divisors of 10^{998} is 999^2. So the number of positive numbers which divide 10^{999} but not 10^{998} is

$= 1000^2 - 999^2$

$= (1000 + 999)(1000 - 999)$

$= 1999 \times 1 = 1999$

➤ **Example 13.** Find the number of positive integer solutions of $x_1x_2x_3 = 30$.

➤ ***Solution*** $x_1x_2x_3 = 2 \times 3 \times 5$. If we treat 2, 3, 5 as objects and x_1, x_2, x_3 as distinct boxes then finding the number of positive integer solutions is same as finding the number of ways of distributing 3 distinct objects in 3 distinct boxes. Thus the required number of solutions is $3^5 = 27$

(For example if all the objects are held by x_1 the corresponding solution is $x_1 = 30, x_2 = 1, x_3 = 1$, if 2 and 3 are held by x_1 and 5 by x_3 then $x_1 = 6, x_2 = 1, x_3 = 5$ etc}

4.19 DERANGEMENTS

Any change in the order of things arranged in a row is called a derangement. Derangement of n different objects occupying n places in a row is the number of ways in which the objects in the row can be relocated such that none of the objects occupies its original place. For instance, a derangement of

1	2	3	4	5

is

2	3	1	5	4

Let there be n letters and n addressed envelopes. In how many ways could all the letters be put into the wrong envelopes ?

The number of ways to put 1 letter in 1 addressed envelope so that one letter is in wrong envelope = 0 ...(1)

The number of ways to put 2 letters in 2 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope

$= 2! - 1 - 0$ from (1)

$= 2 - 1 = 1$ ...(2)

The number of ways to put 3 letters in 3 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope – The number of ways in which 2 letters are in correct envelopes

$= 3! - 1 - {}^3C_1 \times 1 - 0$ from (1), (2) = 2 ...(3)

($^3C_1$ to select one envelope to put the letter correctly)

The number of ways to put 4 letters in 4 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope – The number of ways in which 2 letter are in correct envelopes – The number of ways in which 3 letters are in correct envelopes

$= 4! - 1 - {}^4C_1 \times 2 - {}^4C_2 \times 1 - 0$ from (1), (2) & (3)

$= 24 - 1 - 8 - 6 - 0 = 9$ ...(4)

The number of ways to put 5 letters in 5 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in correct envelope – The number of ways in which 2 letters are in correct envelopes – The number of ways in which 3 letters are in correct envelopes – The number of ways in which 4 letters are in correct envelopes

$= 5! - 1 - {}^5C_1 \times 9 - {}^5C_2 \times 2 - {}^5C_3 \times 1 - 0$

from (1), (2), (3), (4)

$= 120 - 1 - 45 - 20 - 10 - 0 = 44$

The number of ways to put 6 letters in 6 addressed envelopes so that ll are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the

www.pdfworld.net

www.pdfworld.net

correct envelope – The number of ways in which 2 letters are in correct envelopes – The number of ways in which 3 letters are in correct envelopes – The number of ways in which 4 letters are in correct envelopes – The number of ways in which 5 letters are in correct envelopes.

$= 6! - 1 - {}^6C_1 \times 44 - {}^6C_2 \times 9 - {}^6C_3 \times 2 - {}^6C_4 \times 1 - 0$

from (1), (2), (3), (4) & (5)

$= 720 - 1 - 264 - 135 - 40 - 15$

$= 720 - 455 = 265.$

Let us now develop a formula for derangement of n objects at n places.

Suppose that there are n letters and n addressed envelopes. In how many ways could all the letters be put into the wrong envelopes ?

Let the letters be denoted by the letters a, b, c,... and the corresponding envelopes by a′, b′, c′,.....

Let D(n) be the required number of ways.

Then a can be put into any one of the n – 1 envelopes b′, c′,...

Suppose a is put into k′; then k may be put into a′, in which case there will be D(n – 2) ways of putting all the other wrong. Also if a is put into k′, the number of ways of disposing of the letters so that k is not put in a′, b not in b′ etc., is D(n – 1).

Hence the number of ways of satisfying the conditions when a is put into k′ is D(n – 1) + D(n – 2). The same is true when a is put into any other of the envelopes b′, c′,.....

Hence we have

$$D(n) = (n-1)\{D(n-1) + D(n-2)\}$$

$$\therefore \quad D(n) - nD(n-1) = -\{D(n-1) - (n-1)D(n-2)\}$$

Similarly $D(n-1) - (n-1)D(n-2)$

$$= -\{D(n-2) - (n-2)D(n-3)\}$$

.....=.....

$$D(3) - 3D(2) = -\{D(2) - 2D(1)\}.$$

But obviously $D(2) = 1$ and $D(1) = 0$;

$$\therefore \quad D(n) - nD(n-1) = (-1)^n.$$

Hence $$\frac{D(n)}{\lfloor n} - \frac{D(n-1)}{\lfloor n-1} = (-1)^n \cdot \frac{1}{\lfloor n}.$$

Similarly $$\frac{D(n-1)}{\lfloor n-1} - \frac{D(n-2)}{\lfloor n-2} = (-1)^{n-1} \cdot \frac{1}{\lfloor n-1}.$$

and $$\frac{D(2)}{\lfloor 2} - \frac{D(1)}{\lfloor 1} = (-1)^2 \cdot \frac{1}{\lfloor 2}.$$

Hence, by addition, D(n)

$$= \lfloor n \left\{ \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} ..... + \frac{(-1)^n}{\lfloor n} \right\}.$$

If out of n things, r things go to wrong places and (n – r) things go to their original places then the required number= ${}^nC_{n-r} D_r$

where D_r = No. of ways, if r things go to wrong places

$$= r! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^r \frac{1}{r!}\right).$$

➤ **Example 1.** In how many ways we can put 5 differently coloured balls into 5 boxes of colours same as that of the balls so that no ball goes to the box of the same colour.

➤ ***Solution*** The problem is the number of derangements of 5 objects.

This is equal to $5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44.$

➤ **Example 2.** Four slip of papers with the numbers 1, 2, 3, 4 written on them are put in a box. They are drawn one by one (without replacement) at random. In how many ways it can happen that the ordinal number of atleast one slip coincide with its own number.

➤ ***Solution*** Total number of ways = 4 ! = 24.

The number of ways in which ordinal number of any slip does not coincide with its own number is the number of derangements of 4 objects

$$= 4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$$

Thus the required number of ways = 24 – 9 = 15.

➤ **Example 3.** Six persons go to a birthday party. They leave their coats in the lounge and pick them while returning back. In how many ways can they pick up the coats, so that

(i) exactly one person picks up his own coat;

(ii) exactly two persons pick up their own coats.

➤ ***Solution*** (i) Since any one of the six persons can be the one to pick up his own coat, and the remaining five persons all pick up the wrong coats, therefore the desired number of ways = $6.D_5 = 6 \times 44 = 264$

www.pdfworld.net

www.pdfworld.net

(ii) Two persons (who pick up their own coats) can be selected out of 6 persons in ${}^6C_2$ ways. For each such selection the remaining four persons pick up wrong coats in D_4 ways. Therefore , the desired number of ways = ${}^6C_2 \times D_4 = 15 \times 9 = 135$.

➤ **Example 4.** In how many ways four pairs of gloves (a pair consist of a right-handed and a left-handed glove) can be distributed among four persons such that every person gets an odd pair ?

➤ ***Solution*** Let the pairs be (G_1, G'_1); (G_2, G'_2); (G_3, G'_3) and (G_4, G'_4).

First of all distribute all right-hand gloves (G'_1, G'_2, G'_3, G'_4) among four person say P_1, P_2, P_3 and P_4. This can be done in 4! ways and for each distribution of right-hand gloves the left-handed gloves can be deranged in $D_4 = 9$ ways.

Thus the required number of ways = 4!.9 = 216.

➤ **Example 5.** Find the number of derangements of the integers from 1 to 10 inclusive, satisfying the condition that the set of elements in the first five places is (a) 1, 2, 3, 4, 5, in some order ; (b) 6, 7, 8, 9, 10 in some order.

➤ ***Solution*** (a) The integers 1, 2, 3, 4, 5 can be put into the first five places in D_5 ways, because there are D_5 derangements of five things; the remaining integers from 6 to 10 can be put into the last five places in D_5 ways, so the answer is $D_5 . D_5 = 1936$.

(b) Any arrangement of 6, 7, 8, 9, 10 in the first five places is a derangement, so there are 5! possibilities; the same is true for the integers 1, 2, 3, 4, 5 in the last five places. Hence the answer is $(5!)^2 = 14400$.

➤ **Example 6.** A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) atleast two of them are in the wrong envelopes. (ii) all the letters are in the wrong envelopes.

➤ ***Solution*** (i) The number of ways in which atleast two of them in the wrong envelopes = $\sum_{r=2}^{6} {}^6C_{6-r} D_r$

$= {}^6C_{6-2} D_2 + {}^6C_{6-3} D_3 + {}^6C_{6-4} D_4 + {}^6C_{6-5} D_5 + {}^6C_{6-6} D_6$

$= {}^6C_4 . 2! \left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 . 3! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$

$+ {}^6C_2 . 4! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$

$+ {}^6C_1 . 5! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$

$+ {}^6C_0 \, 6! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$

$= 15 + 40 + 135 + 264 + 265 = 719$

(ii) The number of ways is which all letters be placed in wrong envelopes

$= 6! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$

$= 720 \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$

$= 360 - 120 + 30 - 6 + 1 = 265$

Alternative

(i) The number of all the possible ways of putting 6 letters into 6 envelopes is 6 !.

The number of ways to place all letters correctly into corresponding envelopes =1

It is not possible that only one letter goes in the wrong envelope.

If 5 letters go in the right envelopes, then the remaining one letter also goes in the right envelope.

Hence the number of ways to place atleast two letters in the wrong envelopes = 6 ! – 0 – 1 = 719

Pigeon Hole Principle (PHP)

Consider the following statements :

(a) If 6 balls are distributed among 5 children, then one of them must get more than one ball.

(b) There are two people in New Delhi who were born in the same year.

If more than n objects are distributed in n boxes, then atleast one box has more than one object in it. The above fact goes by the name of Pigeon Hole Principle (PHP).

Another (amusing) way of expressing PHP is to say that if m pigeons are placed in n pigeon holes, where

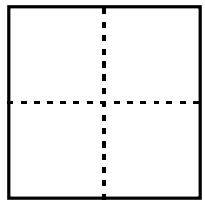
www.pdfworld.net

www.pdfworld.net

m > n. Then there will be more than one pigeon in at least one pigeon hole.

➤ **Example 7.** Five points lie within a square, each of whose sides has length 2 units. Show that there exist two of them the distance between which is atmost $\sqrt{2}$.

➤ ***Solution***



Divide the square into four equal squares. By Pigeon Hole Principle, atleast two of them (say x and y) must lie in one of the four small squares. The distance between x and y cannot exceed the length of the diagonal of the square i.e. $\sqrt{2}$

Practice Problems [O]

1. Find the number of divisors of 8064.
2. In how many ways can the number 7056 be resolved into two factors ?
3. Find the number of positive integer solutions of $x_1x_2x_3 = 500$.
4. How many permutations of the integers from 1 to 9 inclusive have exactly three of the numbers in their natural positions, and the other six not?
5. There are 5 letters and 5 directed envelopes, (i) In how many ways can all the letters be put into the wrong envelopes ? (ii) In how many ways can 2 letters be rightly placed and 3 letters wrongly placed ?
6. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{x_1, x_2, x_3, x_4, x_5\}$. How many one-one mapping $f : A \to B$ can be defined so that $f(i) \neq x_i$ for all i.
7. In how many ways can the letters M, A, D, I, S, O, N be written down so that the word spelled completely disagrees with MADISON ?

4.20 NUMBER OF SUBSETS

➤ **Example 1.** A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is chosen. Find the number of ways of selecting P and Q so that P and Q are non-intersecting.

➤ ***Solution*** Each of the elements is available for selection in making P as well as Q. But P and Q must not have any common element.

So each element can be taken in 3 ways (element $\in$ P, $\notin$ Q ; $\notin$ P, $\in$ Q ; $\notin$ P, $\notin$ Q)

$\therefore$ n elements can be taken in $3^n - 1$ ways

which is equal to the number of ways to select non-intersecting subsets P and Q

(excluding the case when P, Q are ϕ)

➤ **Example 2.** A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is next chosen. Find the number of ways of choosing P and Q such that $P \cap Q$ contains exactly two elements.

➤ ***Solution*** Let $A = \{a_1, a_2, ..., a_n\}$

The two elements which must be present in subset P as well as subset Q can be chosen in ${}^nC_2$ ways.

For each of the remaining (n – 2) elements of set A we have 3 options; chosen either for P or for Q or not chosen at all.

Thus, we have 3^{n-2} ways.

Hence, the total number of ways of choosing P and Q such that they have exactly 2 elements in common is $= {}^nC_2 \times 3^{n-2}$

➤ **Example 3.** A is a set containing n elements. A subset P_1 of A is chosen. The set A is reconstructed by replacing the elements of P_1 . A subset P_2 of A is next chosen and the set A is again reconstructed by replacing the elements of P_2. In this way m (>1) subsets P_1, P_2,....., P_m of A are chosen. Find the number of ways of choosing P_1, P_2, ..., P_m so that

(i) $P_1 \cap P_2 \cap ..... \cap P_m = \phi$

(ii) $P_1 \cup P_2 \cup ..... \cup P_m = A$

➤ ***Solution*** (i) Let $A = \{a_1, a_2, ..., a_n\}$

During construction of any subset, say P_i $(1 \leq i \leq m)$, for each element of A, say a_j $(1 \leq j \leq n)$, we have two choices, either $a_j \in P_i$ or $a_j \notin P_i$.

www.pdfworld.net

www.pdfworld.net

Thus, a_j can be chosen for m subsets in 2^m ways. Excluding the case in which it is chosen for all the subsets, since that is not favourable to $P_1 \cap P_2 \cap \ldots\ldots\ldots \cap P_m = \phi$, the choice for a_j reduces to $2^m - 1$ ways.

Since, there are n elements, the total number of choices = $(2^m - 1)^n$.

(ii) a_j can be chosen for m subsets in 2^m ways. Excluding the case in which it is not chosen for any of the subsets, since that is not favourable to $P_1 \cup P_2 \cup \ldots\ldots\ldots \cup P_m = A$, the choice for a_j reduces to $2^m - 1$ ways. Since, there are n elements, the total number of choices = $(2^m - 1)^n$.

➤ **Example 4.** Consider the collection of all three–element subsets drawn from the set {1, 2, 3, 4,.....,299, 300}. Determine the number of these subsets for which the sum of the three elements is a multiple of 3.

➤ ***Solution*** For $0 \le j \le 2$, let A_j denote the set of all integers between 1 and 300 which leave remainder j when divided by 3. Then $|A_j| = 100$ for $0 \le j \le 2$. If a, b, c is a 3–element subset of the given set S = 1, 2,...,300, then 3 divides a + b + c if and only if

(i) all a, b, c are in A_0 or in A_1 or in A_2.

(ii) one of the a, b, c is in A_0, another in A_1, and the third one in A_2.

The number of 3–element subsets of A_j, $0 \le j \le 2$ is $^{100}C_3$. For each choice of a in A_0, b in A_1 and c in A_2, we get a 3–element subset such that 3 divides a + b + c. Thus the total number of 3–element subsets {a, b, c} such 3 divides a + b + c is equal to $3\ ^{100}C_3 + 100^3 = 1495100$.

➤ **Example 5.** How many 3–element subsets of the set {1, 2, 3, ...,19, 20} are there such that the product of the three numbers in the subset is divisible by 4?

➤ ***Solution*** We count the 3–element subsets {a, b, c} such that 4 does not divide abc. This is possible if and only if either all the three are odd numbers or any two of them are odd and the other is an even number not divisible by 4. There are 10 odd numbers in the set {1, 2, 3,....,20} and 5 even numbers not divisible by 4. Thus the number of 3–element subsets {a, b, c} such that 4 does not divide abc is equal to $^{10}C_3 + 5\ ^{10}C_3 = 345$. The number of 3–element subsets is $^{20}C_3 = 1140$.

Thus the number of 3–element subsets such that the product of these elements is divisible by 4 is equal to 1140 – 345 = 795.

4.21 NUMBER OF FUNCTIONS

➤ **Example 1.** How many functions can be defined from a set A containing 5 elements to a set B having 3 elements? How many these are onto functions?

➤ ***Solution*** Image of each element of A can be taken in 3 ways.

∴ Number of functions from A to B = $3^5 = 243$.

Number of into functions from A to B = $2^5 + 2^5 + 2^5 - 3 = 93$.

∴ Number of onto functions = 243 – 93 = 150.

➤ **Example 2.** Let A = {1, 2, 3, 4, 5} and B = {1, 2, 3} be two sets. How many functions can be defined from A to B ?

(a) In how many functions the range does not contain unity ?

(b) In how many functions the element 1 in B has exactly 2 pre-images in A ?

(c) In how many function the elements 1 has two, 2 has two and 3 has 1 pre-images ?

(d) In how many functions f(1) = 1 ?

➤ ***Solution*** The number of functions is clearly 3^5 since we are distributing 5 distinct objects in three boxes and each object can be placed in 3 ways.

(a) 2^5 since we are distributing five distinct objects in two boxes.

(b) Select any two from five to keep in box 1 which can be done in $^5C_2$ ways and distribute remaining three objects in box 2 and 3.
Thus the required number is $^5C_2 \cdot 2^3 = 80$

(c) The answer must be $^5C_2 \cdot {}^3C_2 \cdot {}^1C_1$ (It follows from here: We are distributing 5 distinct objects in 3 boxes where first box contains 2, second contains 2 and third contains 1 which is $\frac{5!}{2!2!1!}$)

(d) 3^4 since we have to distribute four distinct objects in three boxes (when f(1) = 1 remaining elements in A can still have their images as 1 in B)

www.pdfworld.net

www.pdfworld.net

➤ **Example 3.** If X is a set containing n elements and Y is a set containing m elements, then

(i) How many functions are there from X to Y?

(ii) How many of these functions are one-one?

(iii) How many of these functions are onto?

➤ ***Solution***

(i) Every element in X can be matched with elements of Y in m ways.

Therefore, the number of ways in which all the elements of X can be matched with elements of Y, is $= m^n$

which is the number of possible functions from X to Y.

(ii) **Case I** $(n > m)$

For n elements in X, we need to choose n distinct images from the m elements in y which cannot be done since $n > m$

Hence, the number of one-one functions = 0 if $n > m$.

Case II $(n < m)$ For n elements in X, we need to choose n distinct images from the m elements in y which can be done in ${}^mP_n$ ways.

Hence the number of one-one functions $= {}^mP_n$ if $n < m$

(iii) If any one element of Y(m elements) does not have a pre-image in X (n elements), then the number of functions that can be formed $= (m-1)^n$

Thus, total number of such functions in which one element of Y has no pre-image

$= {}^mC_1 \times (m-1)^n$

Similarly, number of such functions in which two elements of Y have no pre-images

$= {}^mC_2 \times (m-2)^n$

Thus, number of such functions in which atleast one element of Y has no pre-image

$$= \sum_{r=1}^{m} (-1)^{r-1}\, {}^mC_r \times (m-r)^n$$

An onto function from X to Y is function in which every element of Y has a pre-image in X.

Hence, the number of onto functions

= total number of functions – function in which atleast one element in Y has no pre-image.

$$= m^n - \sum_{r=1}^{m} (-1)^{r-1}\, {}^mC_r \times (m-r)^n = \sum_{k=1}^{m} (-1)^{m-r}\, {}^mC_r\, r^n$$

➤ **Example 4.** How many functions are there from the set $\{1, 2, ..., n\}$ to the set $\{0, 1\}$ (i) that are one-to-one? (ii) that assign 0 to both 1 and n? (iii) that assign 1 to exactly one of the positive integers $< n$?

➤ ***Solution***

(i) If n = 1, the number of one to one functions from A to B is 2 : the function which maps 1 to 0 and the function which maps 1 to 1.

If n = 2, the number of one to one functions from A to B is again 2: The function which maps 1 to 0 and 2 to 1 and the function which maps 1 to 1 and 2 to 0.

If $n \le 3$, the number of one to one functions from A to B is 0

(ii) The number of required functions is 1 if n = 1.

If $n > 1$, it is 2^{n-2}, since f(1) = 0, f(n) = 0 and each of f(2), ..., f(n – 1) can be 0 or 1.

(iii) The number of required functions is 2(n – 1), since for each number k = 1, 2, ..., (n – 1), we have the function f for which f(k) = 1 and f(i) = 0 for every $i, 0 \le i \le (n-1), i \ne k$, and f(n) = 0 or 1.

4.22 GEOMETRICAL PROBLEMS

➤ **Example 1.** There are ten points in a plane. Of these ten points four points are in a straight line and with the exception of these four points no other three points are in the same straight line. Find

(i) the number of triangles formed.

(ii) the number of straight lines formed.

(iii) the number of quadrilaterals formed, by joining these ten points

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** (i)

No. of points selected out of 4 collinear points	No. of points selected out of remaining 6 points	No. of triangles formed
0	3	$^4C_0 \cdot {}^6C_3 = 20$
1	2	$^4C_1 \cdot {}^6C_2 = 60$
2	1	$^4C_2 \cdot {}^6C_1 = 36$
3	0	0
		Total = 116

Hence required number = 116

Alternative :

Total number of selections of 4 points out of 4 collinear points = $^4C_3 = 4$

The required number = 120 – 4 = 116

(ii)

No. of points selected out of 4 collinear points	No. of points selected out of remaining 6 points	No. of str. lines formed
0	2	$^4C_0 \cdot {}^6C_2 = 15$
1	1	$^4C_1 \cdot {}^6C_1 = 24$
2	0	1
		Total = 40

Hence the required number = 40

Alternative :

Total number of selections of 2 points out of 10 points

$= {}^{10}C_2 = \dfrac{10 \times 9}{2!} = 45$

The number of selections of two points when only one straight line is formed $= {}^4C_2 = 6$

$\therefore$ The required number $= {}^{10}C_2 - {}^4C_2 + 1$

$= 45 - 6 + 1 = 40$

(iii)

No. of points selected out of 4 collinear points	No. of points selected out of remaining 6 points	No. of quadrilaterals formed
0	4	$^4C_0 \cdot {}^6C_4 = 15$
1	3	$^4C_1 \cdot {}^6C_3 = 80$
2	2	$^4C_2 \cdot {}^6C_2 = 90$
3	1	0
4	0	0
		Total = 185

Hence the required number = 185

➤ **Example 2.** Prove that if each of m points on one straight line be joined to each of n points on the other straight line terminated by the points, then excluding the points on the given two lines, number of points of intersection of these lines is $\dfrac{1}{4}$mn (m – 1)(n – 1).

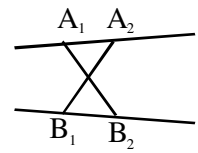
➤ ***Solution*** To get one point of intersection as desired in the question we need two points on the first line and two points on the second line. Now two points on the first line can be selected out of m points in $^mC_2$ ways. Again two points on the second line can be selected out of n points in $^nC_2$ ways.

One such selection of 4 points two on one line and two on the other gives one point of intersection as desired in the question.

$\therefore$ The required number $= {}^mC_2 \times {}^nC_2$

$$= \frac{m(m-1)}{2!} \cdot \frac{n(n-1)}{2!}$$

$$= \frac{1}{4} mn\,(m-1)(n-1).$$



➤ **Example 3.** There are m points in a plane which are joined by straight lines in all possible ways and of these no two are coincident or parallel and no three of them are concurrent except at the points. Show that the number of points of intersection, other than the given points, of the lines so formed is $\dfrac{m!}{8(m-4)!}$

➤ ***Solution*** Let the m points be $A_1, A_2, A_3, \ldots, A_m$

We consider four points A_1, A_2, A_3, A_4.

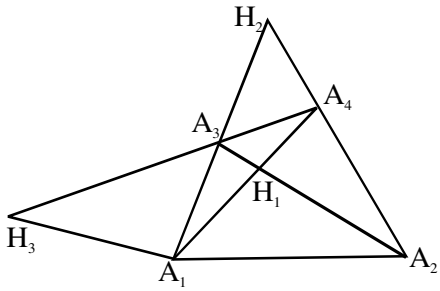
If these four points are joined in all possible ways, then we will get three points of intersection.

H_1, H_2 and H_3 other than the given points A_1, A_2, A_3, A_4

www.pdfworld.net

www.pdfworld.net

$\therefore$ The required number of points of intersection



$$= 3.\ {}^{m}C_4 = 3.\ \frac{m(m-1)(m-2)(m-3)}{4!}$$

$$= \frac{1}{8}.\frac{m!}{(m-4)!}$$

➤ **Example 4.** If m parallel lines are intersected by n other parallel lines find the number of parallelograms thus formed.

➤ ***Solution*** Two parallel lines can be selected from the first set of m parallel lines in ${}^{m}C_2$ ways.

Again two parallel lines can be selected from other set of n parallel lines in ${}^{n}C_2$ ways.

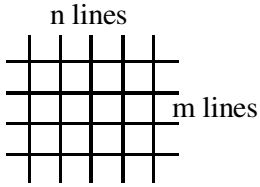
$\therefore$ The number of selections of 4 lines, 2 from first set and 2 from second set = ${}^{m}C_2 \times {}^{n}C_2$

And each such selection of 4 lines gives a parallelogram

$\therefore$ The number of parallelograms thus formed

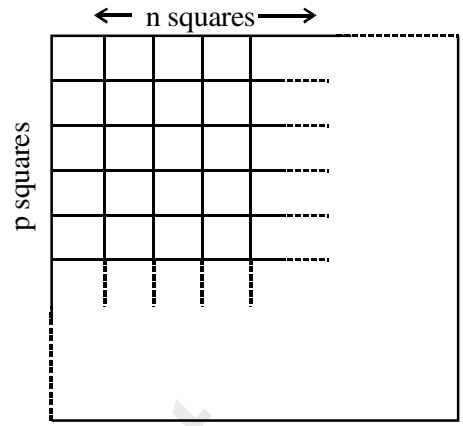
$$= {}^{m}C_2 \times {}^{n}C_2 = \frac{m!}{2!(m-2)!} \times \frac{n!}{2!(n-2)!}$$

$$= \frac{1}{4}.\ mn(m-1)(n-1)$$



Number of Rectangles and Squares

The fig. shows n squares horizontally and p squares vertically (n < p).



1. The number of squares of any size that can be formed is

= squares of the sizes 1 + 2 + 3 + ... + n units

= np + (n – 1)(p – 1) + (n – 2)(p – 2) + ... + (n – n + 1)(p – n + 1)

$$= \sum_{r=1}^{n}(n-r+1)(p-r+1) \text{ if } n < p$$

which reduces to $\sum_{r=1}^{n} n^2$ if n = p

2. The number of rectangles of any size that can be formed is

$= {}^{n+1}C_2 \times {}^{p+1}C_2$

$\because$ there are n + 1 vertical lines and p + 1 horizontal lines and to form a rectangle we need two vertical and two

$$= \frac{n(n+1)}{2} \times \frac{p(p+1)}{2} \text{ horizontal lines.}$$

which reduces to $\sum_{r=1}^{n} n^3$ if n = p.

➤ **Example 5.** Find the number of rectangles excluding squares from a rectangle of size 7 × 4

➤ ***Solution*** The required number of rectangles excluding squares

$$= \left[\frac{7(7+1)}{2} \times \frac{4(4+1)}{2}\right] - [(4\times 7) + (3\times 6) + (2\times 5) + (1\times 4)]$$

$= (28 \times 6) - (28 + 18 + 10 + 4)$

$= 168 \times 60 = 108.$

www.pdfworld.net

www.pdfworld.net

➤ **Example 6.** In how many ways we can choose two squares of same dimension from a square grid, formed by (2m + 1) horizontal and (2m + 1) vertical equi-spaced lines, such that they have either a edge or corner edge common.

➤ ***Solution*** If two squares of (r × r) have a side common

Total number of selections = $2(2m+1)-r)(3m+1-2r)$

If two squares of (r × r) have a corner common, then total number of selections

$$= 2((2m+1)-2r)^2$$

So total number of selections of two square of dimension (r × r), having either a side or corner common

$$= 2\left((2m+1)^2 - 3r(2m+1) + 2r^2 + (2m+1)^2 - 4r(2m+1) + 4r^2\right)$$

Hence the total number of such squares

$$= 2\sum_{r=1}^{n} (2(2m+1)^2 - 7(2m+1)r + 6r^2)$$

$$= 2\left[2(2m+1)^2 m - \frac{7(2m+1)m(m+1)}{2} + \frac{6m(m+1)(2m+1)}{6}\right]$$

$$= 4(2m+1)^2 m - 7(2m+1)m(m+1) + 2m(m+1)(2m+1)$$

$$= m(2m+1)(4(2m+1) - 7(m+1) + 2(m+1))$$

$$= m(2m+1)(8m+4-7m-7+2m+2)$$

$$= m(2m+1)(3m-1).$$

Path Problem

➤ **Example 7.** A man works in a building located seven blocks east and eight blocks north of his home. Thus in walking to work each day he goes fifteen blocks. All the streets in the rectangular pattern are available to him for walking. In how many different paths can he go from home to work, walking only fifteen blocks ?

➤ ***Solution*** Let us denote by E the act of walking a block east, and by N the act of walking a block north and let us interpret a string of E's and N's such as

EENNNENN

as meaning (reading from left to right) that a man walks two blocks east, then three blocks north, then one block east and finally two block north. Then any path from home to work can be identified with an appropriate pattern of seven E's and eight N's in a row. For example, the path beginning with three blocks east, then two north, then four east, and finally six north is

EEENNEEEENNNNNN

Thus to each path there corresponds a string of seven E's and eight N's properly interspersed in a row; and conversely, to any such string of E's and N's there corresponds exactly one path. We can therefore rephrase the problem as follows: in how many ways can seven Es and eight N's be written in a row ?

If we think of fifteen boxes to be filled with seven E's and eight N's, we see that the answer to this question is just the number of ways we can choose seven boxes out of fifteen to fill with E's, and this number is

$${}^{15}C_7 = \frac{15!}{7!8!} = \frac{15.14.13.12.11.10.9}{7.6.5.4.3.2} = 6435.$$

➤ **Example 8.** The streets of a city are arranged like the lines of a chessboard. There are m streets running north and south and n east and west. Find the number of ways in which a man can travel from the north-west to the south-east corner, going the shortest possible distance.

➤ ***Solution*** Let the city be represented by a rectangle whose sides are of length a and b.

For each path total distance covered in horizontal direction is a and in vertical direction is b. a is the sum of lengths of (m – 1) horizontal line segments and b is the sum of lengths of (n – 1) vertical line segments.

Each path to go from P to Q will be an arrangement of (m + n – 2) line segments of which (m – 1) are horizontal and (n – 1) are vertical.

∴ Required number = number of arrangments of (m + n – 2) different things of which (m – 1) are of one kind and (n – 1) are of another kind

$$= \frac{(m+n-2)!}{(m-1)!(n-1)!}$$

➤ **Example 9.** In the path problem, denote the north south streets by A, B, C, ..., H and the east west streets by 1st, 2nd, ..., 9th. Presume that the man lives at the corner of 1st and A, and works at the corner of 9th and H. Given the information that all streets are

www.pdfworld.net

www.pdfworld.net

available for walking with one exception, namely that E street from 5th to 6th is not cut through, in how many different paths can the man walk from home to work, walking only fifteen blocks.

➤ ***Solution*** We subtract from the total number $^{15}C_7 = 6435$ of unrestricted paths the number that includes the block from 5th to 6th one. From 1st and A to 5th and E there are $^8C_4$ paths. From 6th and E to 9th and H there are $^6C_3$ paths. Hence from 6435 we subtract $^8C_4 \cdot {}^6C_3 = 5035$.

➤ **Example 10.** As a generalization of the path problem to three dimensions, consider a three dimensional steel framework; how many different paths of length fifteen units are there from one intersection point in the framework to another that is located four units to the right, five units back, and six units up?

➤ ***Solution*** Denoting by R, B, U the motions of distance one unit to the right, back, and up, we see that the problem is the same as finding the number of permutations, all at a time, of the fifteen letters

R R R R B B B B B U U U U U U

Hence the required number of paths is 15! / (4!5!6!).

➤ **Example 11.** In two-dimensional space the point (a, b) is said to be a lattice point if a, b are integers. Two lattice points (a, b) and (c, d) are neighbours if they agree in one of the coordinates and differ in the other coordinate by 1. Every pair of neighbours is connected by a directed line with the direction coinciding with the positive direction of the coordinate axis to which it is parallel. In how many ways can one move from (0, 0) to (a, b) along the directed path ?

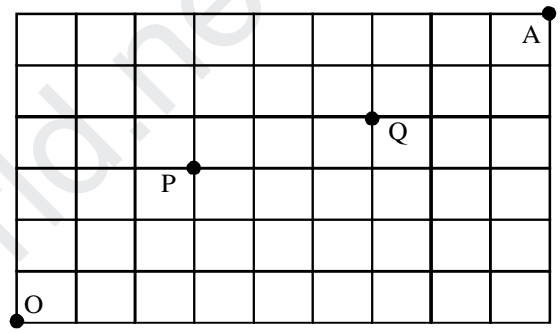
➤ ***Solution*** To move from (0, 0) to (a, b) one has to move through a steps of 1 unit each along the x-axis and b steps of 1 unit each along the y axis.

Each such move can be written as a permutation x x y x y y x ..... x x y where there are a 'x' s and b y's. Here each x represents a 1–unit move along the x–axis and y represents a 1–unit move along the y–axis. Thus there are as many moves from (0, 0) to (a, b) as there are permutations of the above kind. These permutations are nothing but permutations of (a + b) things, 'a' of which are of one kind (viz., they are x) and b of which are of a different kind (viz. they are y).

So the required number is the number of all such permutations is $\frac{(a+b)!}{a!b!}$

➤ **Example 12.** A person starts from the origin O (0,0) in the X-Y plane. He takes steps of one unit along the positive X-axis or the positive Y-axis. Travelling in this manner, find the total number of ways he can reach A(9, 6) avoiding both the points P(3, 3) and (6, 4).

➤ ***Solution*** We use the following formula : the number of paths from (0, 0) to (m, n) is $^{m+n}C_m$.



Let T(O, A) be the set of all paths from O(0,0) to A(9,6), T(O, P, A) be the set of all paths from O to A via P, etc.

Hence $|T(O, P, A)| = |T(O, P)| \times |T(P, A)|$,

where |T(P, A)| is to be evaluated by shift of origin to P. $|T(O, A)| = {}^{9+6}C_6 = 5005$,

$|T(O, P, A)| = {}^{3+3}C_3 \times {}^{6+3}C_3 = 20 \times 84 = 1680$,

$|T(O, P, Q, A)| = {}^{3+3}C_3 \times {}^{3+1}C_1 \times {}^{3+2}C_2 = 20 \times 4 \times 10 = 800$

By the inclusion-exclusion principle, the required number of paths is

$= |T(O,A)| - |T(O,P,A)| - |T(O,Q,A)| + |T(O,P,Q,A)|$

$= 5005 - 1680 - 2100 + 800 = 2025$

Number of Regions Formed by n Straight Lines

Show that n straight lines, no two of which are parallel and no three of which meet in a point, divide a plane into $\frac{1}{2}n(n+1) + 1$ parts.

www.pdfworld.net

www.pdfworld.net

Consider n lines drawn in a plane such that no two lines are parallel and no three lines are concurrent.
One line divides the plane into 2 regions.
A second line divides the two regions that it passes through into 4 regions, thereby increasing the number of regions by 2.
A third line divides the three regions that it passes through into 6 regions, thereby increasing the number of regions by 3.
Similarly, the nth line will divide the n regions that it will pass through into 2n regions, thereby increasing the number of regions by n.
Hence, the total number of regions thus formed by n lines such that no two lines are parallel and no three lines are concurrent, is

$$= 2 + 2 + 3 + 4 + ... + n = 1 + \Sigma n = \frac{1}{2} n(n+1) + 1.$$

Alternative

The nth straight line is cut by each of the other n – 1 line; and hence it is divided into n portions. Now there are two parts of the plane on the two sides of each of these portions of the nth line which would become one part if the nth line were away. Hence the plane is divided by n lines into n more parts than it is divided by n – 1 lines.
Hence, if F(x) be put for the number of parts into which the plane is divided by x straight lines, we have

$F(n) = F(n-1) + n.$
Similarly $F(n-1) = F(n-2) + (n-1)$,
..... = ... and
$F(2) = F(1) + 2.$
But obviously $F(1) = 2.$

Hence $F(n) = 2 + 2 + 3 + 4 + ... + n = 1 + \frac{1}{2} n(n+1).$

➤ **Example 13.** Six straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent, Then find the number of parts into which these lines divide the plane.

➤ ***Solution*** No. of parts of the plane

$$= 1 + \Sigma 6 = 1 + \frac{6.7}{1.2} = 22$$

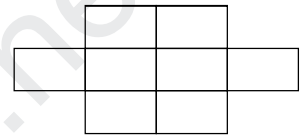
Selection of Squares

Let there be n rows, I row has α_1 squares. II row has α_2 squares, III row has α_3 squares, and so on. If we placed β Xs in the squares such that each row contain atleast one X then the number of ways

= coefficient of x^β in

$$({}^{\alpha_1}C_1 x + {}^{\alpha_1}C_2 x^2 + ... + {}^{\alpha_1}C_{\alpha_1} x^{\alpha_1})$$
$$\times ({}^{\alpha_2}C_1 x + {}^{\alpha_2}C_2 x^2 + ... + {}^{\alpha_2}C_{\alpha_3} x^{\alpha_2})$$
$$\times ({}^{\alpha_3}C_1 x + {}^{\alpha_3}C_2 x^2 + ... + {}^{\alpha_3}C_{\alpha_3} x^{\alpha_3}).....$$

➤ **Example14.** Six X's have to be placed in the squares of the figure below, such that each row contains atleast one X. In how many different ways can this be done ?



➤ ***Solution*** The required number of ways

$= $ coefficient of x^6 in $({}^2C_1 x + {}^2C_2 x^2)$
$({}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4)$ $({}^2C_1 x + {}^2C_2 x^2)$
$= $ coeff. of x^6 in $\{(1+x)^2 - 1\}\{(1+x)^4 - 1\}\{(1+x)^2 - 1\}$
$= $ coeff. of x^6 in $\{(1+x)^2 - 1\}^2 \{(1+x)^4 - 1\}$
$= $ coeff. of x^6 in $\{(1+x)^4 - 2(1+x)^2 + 1\}\{(1+x)^4 - 1\}$
$= $ coeff. of x^6 in $\{(1+x)^8 - 2(1+x)^6 .....\}$
$= {}^8C_6 - 2 = 26$

OR

$= $ coefficient of x^6 in $({}^2C_1 x + {}^2C_2 x^2)$
$({}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4)$ $({}^2C_1 x + {}^2C_2 x^2)$
$= $ coeff. of x^3 in $(2 + x)^2 (4 + 6x + 4x^2 + x^3)$
$= $ coeff. of x^3 in $(4 + 4x + x^2)(4 + 6x + 4x^2 + x^2)$
$= 4 + 16 + 6 = 26$

Alternative :

In the given figure there are 8 squares and we have to place 6X's.
This can be done in ${}^8C_6 = {}^8C_2 = 28$ ways
But these include the possibility that either first row or third row may not have any X when the six X's are placed in the other two rows. These two possibilities are to be excluded.
$\therefore$ The required number of ways = 28 – 2 = 26

www.pdfworld.net

www.pdfworld.net

Practice Problems [P]

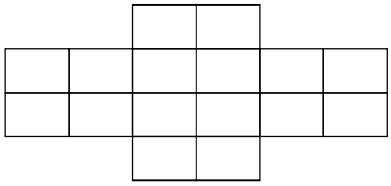
1. m points are taken on one of two given parallel lines and n points on the other. Join with line segments each of the m points on the former line to each of the n points on the latter. What is the number of points of intersection of the segments if it is known that there are no points in which three or more segments intersect.

2. n points are given on the circumference of a circle and the chords determined by them are drawn. If no three chords have a common point, how many triangles are there all of whose vertices lie inside the circle ?

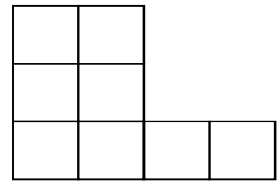
3. A rectangular city is divided by streets into squares. There are n such squares from north to south and k squares from east to west. Find the number of shortest walks from the north-eastern end of the city to the south-western end.

4. Consider the set of points S = {(x, y); x, y are non-negative integers $\leq$ n}. Find the number of squares that can be formed with vertices belonging to S and sides parallel to the axes.

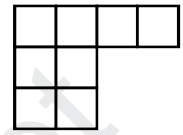
5. 14X's have to be placed in the squares of the figure given below such that each row contains atleast one X. In how many ways can this be done ?



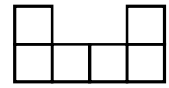
6. In how many ways the letters of the word PERSON can be placed in the squares of the adjoining figure so that no row remains empty?



7. In how many different ways can the letters of the word VECTOR be placed in the 8 boxes of the given figure so that no row remains empty?



8. In how many different ways can the six digits 1, 2, 3, 4, 5, 6 be placed in the boxes of the given figure so that only even digits occupy top two boxes and the number read by the four bottom digits is even ?



Target Problems for JEE Advanced

➤ **Problem 1.** If all the permutations of the letters of the word "UNIVERSAL" are arranged (and numbered serially) in alphabetical order as in a dictionary.

(i) What is the first word ?
(ii) What is the last word ?
(iii) How many words are there under each letter ?
(iv) What is the serial number of the word : RIVENSULA?

➤ ***Solution*** (i) AEILNRSUV is the first word

(ii) The reverse of the above i.e. VUSRNLIEA is the last word.

(iii) 8! = 40320; because we can keep one letter fixed (i.e., as the first letter of the word) and permute only the remaining 8 letters.

(iv) To calculate the serial number of the word RIVENSULA in the alphabetical order, we have to systematically exhaust the words that go before the specific word. This is done in the following table which is self explanatory.

www.pdfworld.net

www.pdfworld.net

Words beginning with	No. of such words	Count for totalling
A	8!	40320
E	8!	40320
I	8!	40320
L	8!	40320
N	8!	40320
RA	7!	5040
RE	7!	5040
RIA	6!	720
RIE	6!	720
RIL	6!	720
RIN	6!	720
RIU	6!	720
RIVA	5!	120
RIVEA	4!	24
RIVEL	4!	24
RIVENA	3!	6
RIVENL	3!	6
RIVENSA	2!	2
RIVENSL	2!	2
RIVENSUA	1!	1
		2,16,185

The next word after the above list is RIVENSULA. So the serial number required is 2,16,816.

➤ **Problem 2.** How many times is the digit 0 written when listing all numbers from 1 to 3333 ?

➤ ***Solution*** We are considering the integers t such that $1 \le t \le 3333$. Clearly, the largest number t having 0 in the units place is 3330. So there are 333 numbers t having 0 in the units place : they are 10, 20, ..., 3330. We can describe these numbers as t = x0 where x is any one of 1, 2, ... 333.

Similarly, number t = x0 y i.e. numbers having 0 in the tens place are in all 33×10 because x can be any one of 1, 2, ... 33 and y can be any one of 0, 1, 2,.... 9. Thus there are $33 \times 10 = 330$ numbers like x0y. In the same way, there are $3 \times 10^2 = 300$ numbers with 0 in the hundreds place (i.e. x0yz where $1 \le x \le 3$, $0 \le y, z \le 10$), hence the total number of times 0 is written is $333 + 330 + 300 = 963$.

(We assume in this solution that the number such as 11 is written as 11 and not as 0011.)

➤ **Problem 3.** How many eight digit numbers formed from 1, 3, 4, 5, 6, 7, 8, 9 will be multiples of 275?

➤ ***Solution*** We have $275 = 25 \times 11$.

Thus, the required number must be divisible by 25 as well as by 11. For divisibility by 25, the last two digits must be divisible by 25. There is only one such choice, i.e. 75

Let the eight digit number be denoted by

a	b	c	d	e	f	7	5

For divisibility by 11, we have

$(a + c + e + 7) - (b + d + f + 5) = 11$

i.e. $(a + c + e) - (b + d + f) = 9$

where a, b, ..., f are to be chosen from 1, 3, 4, 6, 8, 9.

The only choice is (3, 8, 9) for (a, c, e) and (1, 4, 6) for (b, d, f) which can be arranged amongst themselves in 3! ways each.

Hence, the required number of multiples of 275, is

$= 3! \times 3! = 36.$

➤ **Problem 4.** Find the number of ways of choosing m coupons out of an unlimited number of coupons bearing the letters A, B and C so that they cannot be used to spell the word BAC.

➤ ***Solution*** The word BAC cannot be spelt if the m selected coupons do not contain atleast one of A, B and C.

The number of ways of selecting m coupons which are A or B = 2^m.

This also includes the case when all the m coupons are A or all are B.

The number of ways of selecting m coupons when all are A = 1.

Hence the number of ways of selecting m coupons with atleast one A and one B = $2^m - 2$.

$\therefore$ The required number = $3[(2^m - 2) + 1] = 3(2^m - 1)$

➤ **Problem 5.** Show that the number of ways in which three numbers in A.P. can be selected from 1, 2, 3,...., n is $\frac{1}{4}(n-1)^2$ or $\frac{1}{4}n(n-2)$ according as n is odd or even.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** The given numbers are 1,2, 3, ..., n
Let the three selected numbers in A.P. be a, b, c, then

$$b = \frac{a+c}{2} \text{ or } a + c = 2b \qquad ...(1)$$

From (1) it is clear that a + c should be an even integer. This is possible only when both a and c are odd or both are even.

Case I : When n is even. Let n = 2m

Then number of odd numbers = m

and number of even numbers = m

∴ The number of selections of a and c from m odd numbers = $^{m}C_2$

The number of selections of a and c from m even numbers = $^{m}C_2$

∴ The number of ways in this case = $2 . {}^{m}C_2 = m(m-1) = \frac{n}{2}\left(\frac{n}{2}-1\right) = \frac{n(n-2)}{4}$

Case II : When n is odd. Let n = 2m + 1

Then number of odd numbers = m + 1

and number of even numbers = m

∴ The required number in this case = $^{m+1}C_2 + {}^{m}C_2$

$$= \frac{(m+1)m}{2} + \frac{m(m-1)}{2} = m^2$$

$$= \left(\frac{n-1}{2}\right)^2 = \frac{1}{4}(n-1)^2$$

➤ **Problem 6.** India and South Africa play one day international series until one team wins 4 matches. No match ends in a draw. Find in how many ways the series can be won.

➤ ***Solution*** Taking I for India and S for South Africa. We can arrange I and S to show the wins for India and South Africa respectively.

For example ISSSS means first match is won by India which is followed by 4 wins by S. Africa. This is one way in which series can be won.

Suppose S. Africa wins the series, then last match is always won by S. Africa.

	Wins of I	Wins of S	No. of ways
(i)	0	4	1
(ii)	1	4	4 ! / 3 ! = 4
(iii)	2	4	$\frac{5!}{2!3!} = 10$
(iv)	3	4	$\frac{6!}{3!3!} = 20$

∴ Total no. of ways = 35

India can win the series in the same number of ways.

∴ The total no. of ways in which the series can be won = 35 × 2 = 70.

➤ **Problem 7.** How many words can be made with letters of the word INTERMEDIATE if

(i) the words neither begins with I nor ends with E
(ii) the vowels and consonants alternate in the words
(iii) the vowels are always consecutive
(iv) no vowel is between two consonants
(v) the relative order of vowels and consonants does not change
(vi) the order of vowels does not change ?

➤ ***Solution*** (i) The required number of words
= (the number of words without restriction)
– (the number of words beginning with I)
– (the number of words ending with E)
+ (the number of words beginning with I and ending with E)
(because words beginning with I as well as words ending with E contain some words beginning with I and ending with E).
The number of words without restriction

$= \frac{12!}{2!3!2!}$ (since there are 12 letters in which there are two Is, three Es and two Ts).

The number of words beginning with I = $\frac{11!}{3!2!}$

(since with I in the extreme left place we are left to arrange 11 letters NTERMEDIATE in which there are two Ts and three Es).

The number of words ending with E = $\frac{11!}{2!2!2!}$

(since with E in the extreme right place we are

www.pdfworld.net

www.pdfworld.net

left to arrange 11 letters INTERMEDIAT in which there are two Is, two Es and two Ts)
The number of words beginning with I and ending with E = $\frac{10!}{2!2!}$
(since with I in the extreme left and E in the extreme right places we are left to arrange 10 letters NTERMEDIAT in which there are two Ts and two Es).
∴ The required number of words
$= \frac{12!}{2!3!2!} - \frac{11!}{3!2!} - \frac{11!}{2!2!2!} + \frac{10!}{2!2!}$
$= \frac{10!}{2!3!2!}(12 \cdot 11 - 11 \cdot 2 - 11 \cdot 3 + 6)$
$= \frac{83 \times 10!}{24}$.

(ii) There are 6 vowels and 6 consonants. So the number of words in which vowels and consonants alternate
= (the number of words in which vowels occupy odd places and consonants occupy even places)
+ (the number of words in which consonants occupy odd places and vowels occupy even places)
$= 2.\ \frac{6!}{2!3!} \cdot \frac{6!}{2!} = 43200.$

(iii) Considering the 6 vowels IEEIAE as one thing, the number of arrangements of this with 6 consonants = $\frac{7!}{2!}$ (since there are two Ts in the consonants)
For each of these arrangement, the 6 consecutive vowels can be arranged among themselves in $\frac{6!}{3!2!}$ ways.
∴ The required number of words = $\frac{7!}{2!} \times \frac{6!}{2!3!}$
= 151200.

(iv) No vowel will be between two consonants if all the consonants become consecutive
∴ The required number of words
= the number of arrangements when all the consonants are consecutive
$= \frac{7!}{2!3!} \times \frac{6!}{2!}$ (as above) = 151200.

(v) The relative order of vowels and consonants will not change if in the arrangement of letters the vowels occupy places of vowels, i.e., 1st, 4th, 7th, 9th, 10th, 12th places and consonants occupy their places, i.e., 2nd, 3rd, 5th, 6th, 11th places.
∴ The required number of words
$= \frac{6!}{2!3!} \times \frac{6!}{2!} = 21600.$

(vi) The order of vowels will not change if no two vowels interchange places, i.e., in the arrangement all the vowels are treated as identical.
(For example LATE, ATLE, TLAE, etc., have the same order of vowels A, E. But LETA, ETLA, TLEA, etc., have changed order of vowels A, E. So, LATE is counted but LETA is not.
If A, E, are taken as identical, say V then LVTV does not give a new arrangement by interchanging V, V.)
The required number of words
= the number of arrangements of 12 letters in which 6 vowels are treated as identical
$= \frac{12!}{6!2!}$ (since there are two Ts also)

➤ **Problem 8.** How many sets of three different numbers can be chosen from numbers lying between 0 and 90 so that their average is 30.

➤ ***Solution*** Let a, b, c be three numbers such that
$\frac{a+b+c}{3} = 30$ ...(1) i.e. $a + b + c = 90$

Case I If $0 \le a \le 29$, $0 \le b \le 29$, $c > 30$, then a can be chosen in 30 ways.
b which has to be different from a can be chosen in 29 ways.
for every choice of a, b there is only one choice for c that satisfies equation (1).
Hence, number of choices = 30 × 29.

Case II If a = 30, b + c = 60, then b, c can assume different values in 30 ways.

www.pdfworld.net

www.pdfworld.net

Case III If $31 \le a \le 45, 31 \le b \le 45, c < 30$
a can be chosen in 15 ways. b which has to be different from a can be chosen in 14 ways.
For every choice of a, b there is only one choice for c.
Hence, number of choices = 15×14.
Thus, the required number of ways
$= 30 \times 29 + 30 + 15 \times 14 = 870 + 30 + 210 = 1110$.

➤ **Problem 9.** In an election, the number of contesting candidates exceeds the number of posts p for which the election is being held, by r ($r > 0$). If a voter can vote in 967 different ways by voting for atleast one of the candidates and can vote in 55 different ways to elect atleast one of $r - 1$ candidates, find the number of candidates losing the election.

➤ ***Solution*** We have
number of posts = p
and number of contesting candidates = p + r
Now, number of ways in which atleast one candidate can be selected from p + r candidates for p posts, is

$$^{p+r}C_1 + {}^{p+r}C_2 + \ldots\ldots\ldots\ldots {}^{p+r}C_p = 957 \quad \ldots(1)$$

Similarly, number of ways in which atleast one candidate can be selected from p + r candidates for $r - 1$ posts, is

$$^{p+r}C_1 + {}^{p+r}C_2 + \ldots\ldots\ldots\ldots {}^{p+r}C_{r-1} = 55$$

$$^{p+r}C_{p+r-1} + {}^{p+r}C_{p+r-2} + \ldots\ldots\ldots {}^{p+r}C_{p+1} = 55 \quad \ldots(2)$$

Adding equations (1) and (2), we have

$$^{p+r}C_1 + {}^{p+r}C_2 + \ldots\ldots\ldots\ldots {}^{p+r}C_p + {}^{p+r}C_{p+1} + \ldots\ldots\ldots + {}^{p+r}C_{p+r-2} + {}^{p+r}C_{p+r-1} = 1022$$

i.e. $2^{p+r} - ({}^{p+r}C_0 + {}^{p+r}C_{p+r}) = 1022$

i.e. $2^{p+r} = 1024 = 2^{10}, \quad p + r = 10.$

Now, putting the above value in equation (2), we have

$$^{10}C_1 + {}^{10}C_2 + \ldots\ldots\ldots\ldots + {}^{10}C_{r-1} = 55$$

Since, $^{10}C_1 + {}^{10}C_2 = 55$, therefore we have

$r - 1 = 2$ i.e. $r = 3$

which is also the number of excess candidates and hence, the number of candidates who will loose the election.

➤ **Problem 10.** Find the number of integral solutions of equation

$$x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots + x_k = n$$

if $x_1 \ge 1, x_2 \ge 2, x_3 \ge 3, \ldots\ldots\ldots, x_k \ge k$ and $n \ge \dfrac{k(k+1)}{2}$

➤ ***Solution*** Let $y_1 = x_1 - 1, y_2 = x_2 - 2, y_3 = x_3 - 3, \ldots\ldots\ldots, y_k = x_k - k$.
Thus, the given problem reduces to finding the number of integral solutions to the equation

$$(y_1 + 1) + (y_2 + 2) + (y_3 + 3) + \ldots\ldots\ldots\ldots + (y_k + k) = n$$

i.e. $y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_k = n - m$ where

$$m = \frac{k(k+1)}{2}$$

where $0 \le y_1, y_2, y_3, \ldots\ldots y_k \le n - m$
Hence, the required number of solutions

$$= \text{coeff. of } x^{n-m} \text{ in } (1 + x + x^2 + \ldots\ldots + x^{n-m})^k$$
$$= \text{coeff. of } x^{n-m} \text{ in } (1 - x^{n-m+1})^k (1 - x)^{-k}$$
$$= \text{coeff. of } x^{n-m} \text{ in } (1 - x)^{-k}$$
$$= {}^{n-m+k-1}C_{k-1} \text{ where } m = \frac{k(k+1)}{2}$$

➤ **Problem 11.** Find the number of positive unequal integral solutions of the equation $x_1 + x_2 + x_3 = 14$.

➤ ***Solution*** Let $x_2 = x_1 + y_1, \quad y_1 \ge 1$
[variables y_1, y_2 have been chosen since $x_3 = x_2 + y_2$, $y_2 \ge 1$, x_1, x_2, x_3 are given to be unequal]
Then the given equation reduces to

$$x_1 + (x_1 + y_1) + (x_1 + y_1 + y_2) = 14$$

i.e. $3x_1 + 2y_1 + y_2 = 14$, where $x_1, y_1, y_2 \ge 1$

Hence the required number of integral solutions

$$= \text{coeff. of } x^{14} \text{ in } (x^3 + x^6 + x^9 + \ldots\ldots) \times (x^2 + x^4 + x^6 + \ldots\ldots) \times (x + x^2 + x^3 + \ldots\ldots)$$
$$= \text{coeff. of } x^{14-6} \text{ in } (1 + x^3 + x^6 + \ldots\ldots) \times (1 + x^2 + x^4 + \ldots\ldots) \times (1 + x + x^2 + x^3 + \ldots\ldots)$$
$$= \text{coeff. of } x^8 \text{ in } (1 + x^2 + x^4 + x^6 + x^8 + \ldots\ldots + x^3 + x^5 + x^7 + x^9 + \ldots\ldots + x^6 + x^8 + x^{10} + \ldots\ldots) \times (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^8 + \ldots)$$
$$= \text{coeff. of } x^8 \text{ in } (1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \ldots\ldots) \times (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \ldots\ldots)$$
$$= (1 + 1 + 1 + 1 + 1 + 2 + 1 + 2) = 10$$

➤ **Problem 12.** Find the number of positive unequal integral solutions of the equation $x + y + z + t = 20$.

➤ ***Solution*** We can assume that $x < y < z < t$ without loss of generality. Now put $x_1 = x$, $x_2 = y - x$, $x_3 = z - y$ and $x_4 = t - z$.
Then $x_1, x_2, x_3, x_4 \ge 1$;
$x = x_1$, $y = x_1 + x_2$, $z = x_1 + x_2 + x_3$,
$t = x_1 + x_2 + x_3 + x_4$; and the given equation becomes $4x_1 + 3x_2 + 2x_3 + x_4 = 20$.

www.pdfworld.net

www.pdfworld.net

The number of positive integral solutions of this equation equals the coefficient of t^{20} in

$$(t^4 + t^8 + t^{12} + ...)\,(t^3 + t^6 + t^9 + ...)\,(t^2 + t^4 + t^6 + ...)\,(t + t^2 + t^3 + ...)$$
$$= t^{10}(1 + t^4 + t^8 + ...)\,(1 + t^3 + t^6 + ...)\,(1 + t^2 + t^4 + ...) \times (1 + t + t^2 + ...),$$

i.e., the coefficient of t^{10} in

$$(1 + t^4 + t^8)\,(1 + t^3 + t^6 + t^9)\,(1 + t^2 + ... + t^{10})\,(1 + t + t^2 + ... + t^{10}).$$

This is the coefficient of t^{10} in

$$(1 + t^3 + t^4 + t^6 + t^7 + t^8 + t^9 + t^{10}) \times (1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + 5t^9 + 6t^{10}),$$

which is 6 + 4 + 4 + 3 + 2 + 2 + 1 + 1 = 23. Furthermore, since x, y, z and t can be permuted in ${}^4P_4 = 4! = 24$ ways, the required number of solutions is (23) (24) = 552.

➤ **Problem 13.** Show that the number of positive integral solutions of $x + y + z + w = 26$ such $x > y$ is 1078.

➤ ***Solution*** Let n be the number of solutions of the given equation such that x > y then n should also be equal to number of solutions such that x < y (by symmetry).

Therefore if m be the number of solutions such that x = y then 2n + m = total number of positive integer solutions = ${}^{25}C_3$ (On applying the result that number of positive integer solutions of $x_1 + x_2 + ... + x_p = k$ is ${}^{k-1}C_{p-1}$)

Now if x = y the equation is 2x + y + z = 26

or z + w = 26 – 2x has ${}^{25-2x}C_1$ solutions in positive integers.

If x = 1, $y + z = {}^{24-1}C_{2-1} = 23$ has 23 solutions

If x = 2, y + z = 22 has 21 solutions

.....

The number of solutions with x = y is

$$\sum_{x=1}^{12} {}^{25-2x}C_1 = \sum_{x=1}^{12}(25 - 2x) = 144,$$

Now $2n + m = {}^{25}C_3 \Rightarrow 2n + 144 = 2300$

$\Rightarrow n = 1078$

Hence, the number of solutions is 1078.

➤ **Problem 14.** Find the number of positive integer solutions of x + 2y + 3z = 100.

➤ ***Solution*** First of all let us prove that the equation x + 2y = 2m has m – 1 solutions

while x + 2y = 2m + 1 has m solutions.

Now x + 2y = 2m has solutions namely (2, m – 1); (4, m – 2); ..... (2m – 2, 1)

and x + 2y = 2m + 1 has m solutions namely (1, m); (3, m – 1); (5, m – 2) ; ..... (2m – 1, 1). Now z is either 32 or 31 or 30 or ..... or 1.

Thus we have to find the number of positive integral solutions of

x + 2y = 4	x + 2y = 7
x + 2y = 10	x + 2y = 13
x + 2y = 16	x + 2y = 19
.....	.....
x + 2y = 94	x + 2y = 97

The total number of solutions corresponding to first column

= 1 + 4 + 7 + ... + 46 (since x + 2y = 2m has m – 1 solutions) and corresponding to second column

= 3 + 6 + 9 + ... + 48 (since x + 2y = 2m + 1 has m solutions).

Each of the above series has 16 terms. On adding term by term we get the answer as

$$4 + 10 + 16 + ... + 94 = \frac{16}{2}\,[2 \times 4 + (16 - 1)\,6] = 784.$$

➤ **Problem 15.** Prove that the number of combinations of n letters out of 3n letters of which n are a, and n are b and the rest unlike is $(n + 2)2^{n-1}$.

➤ ***Solution*** The required number

$$= \text{coefficient of } x^n \text{ in } [(1 + x + x^2 + ... + x^n)^2 (1 + x)^n]$$

$$= \text{coeff. of } x^n \text{ in } \left[\left(\frac{1 - x^{n+1}}{1 - x}\right)^2 (1 + x)^n\right]$$

$$= \text{coeff. of } x^n \text{ in } [(1 - x)^{-2} (1 + x)^n]$$

[leaving terms containing powers of x greater than n]

$$= \text{coeff. of } x^n \text{ in } [(1 - x)^{-2} \{2 - (1 - x)\}^n]$$

$$= \text{coeff. of } x^n \text{ in } [(1 - x)^{-2} \{2^n - {}^nC_1 2^{n-1}(1 - x) + {}^nC_2 2^{n-2}(1 - x)^2 - .... + (-1)^n {}^nC_n (1 - x)^n\}]$$

$$= \text{coeff. of } x^n \text{ in } [2^n (1 - x)^{-2} - {}^nC_1 2^{n-1}(1 - x)^{-1}]$$

[other terms will not contain x^n]

$$= 2^n \cdot {}^{2+n-1}C_1 - {}^nC_1 \cdot 2^{n-1} \cdot {}^{1+n-1}C_0$$

$$= 2^n \cdot (n + 1) - n \cdot 2^{n-1} = 2^{n-1}(2n + 2 - n) = 2^{n-1}(n + 2)$$

www.pdfworld.net

www.pdfworld.net

➤ **Problem 16.** A shop sells 6 different flavours of ice-cream. In how many ways can a customer choose 4 ice cream cones if

(i) they are all of different flavours :
(ii) they are not necessarily of different flavours
(iii) they contain only 3 different flavours
(iv) they contain only 2 or 3 different flavours ?

➤ ***Solution*** Let the flavours be denoted by a, b, c, d, e and f.

(i) Here we want the number of ways of choosing 4 ice cream cones of different flavours from cones of 6 different flavours.That is, we want the number of 4-combinations of 6 distinct objects without repetition. So the number is ${}^6C_4 = 15$.

(ii) The number of 4-combinations of 6 objects a, b, c, d, e, f, with repetitions allowed, is ${}^{6-1+4}C_4 = {}^9C_4 = 126$.

(iii) The number of ways of choosing 4 cones of exactly 3 different flavours, with repetitions = (the number of ways of choosing 3 flavours out of 6) × (the number of ways of choosing 3 cones of 3 chosen flavours) = ${}^6C_3 \times 3 = 60$, because there are ${}^6C_3$ ways of choosing 3 flavours out of 6 and for each choice, say a, b, c, there are 3 ways of choosing 4 cones of the 3 chosen flavours, namely a, b, c, c or a, b, b, c or a, a, b, c.

(iv) As in (iii), the number of ways of choosing 4 cones of exactly 2 different flavours, with repetitions = ${}^6C_2 \times 3 = 45$, because for each choice of 2 flavours, say a, b, there are 3 ways of choosing 4 cones of the 2 chosen flavours, namely a, a, a, b or a, a, b, b or a, b, b, b.
Hence the ways of choosing cones of 2 or 3 flavours are in all 60 + 45 = 105. Alternatively, the number of ways of choosing 4 cones of only 2 or 3 different flavours, with repetions allowed
= the number of ways of choosing 4 cones, with repetitions allowed, when 6 flavours are available
– [(the number of ways of choosing 4 cones of only 1 flavour) + (number of ways of choosing 4 cones of 4different flavours]
= 126 – (6 + 15) = 105, because there are 6 ways of choosing 4 cones of only one flavour, namely aaaa, bbbb etc.

➤ **Problem 17.** It is a rule (say) in Scotland that consonant(s) cannot be placed between a strong and a weak vowel . The strong vowels are a , o , u and the weak vowels are e & i . Show that the whole number of words that can be formed with this condition of (n + 3) letters each formed of 'n' consonants & the vowels a , e , o is $\frac{2(n+3)!}{(n+2)}$. (Assume no repetition of a letter).

➤ ***Solution*** The three vowels given are a , o (strong), e (weak) and these can occur in one of the six orders.

(i) a e o
(ii) o e a (with weak coming in the middle in (i) & (ii))
(iii) a o e (iv) e o a (v) e a o
(vi) o a e with a strong coming between a weak and a strong . (in (iii) & (vi))

In (i) & (ii) a weak one is between two strong vowels and if they get separated then there will come always one or more consonants between. Hence these three are to go together as one unit and these with the n consonants may be arranged to form ,

$(n + 1)! + (n + 1)! = 2(n + 1)!$ words .

For any one of (iii), (iv), (v) & (vi) where two strong vowels are together.

Take the case a o e
In this case o, e go together and a comes before o e, so that any arrangement can be

× × × × a × × o e × × ×

where × represents a consonant , and there are (n + 2) places for the n consonants, for a and for o e (together) .

Choose any two of these (n + 2) places ..... this can be done in ${}^{n+2}C_2$ ways ..... for n and o e and arrange the n consonants in the n places in n ! ways .

So that the corresponding number of words is ${}^{n+2}C_2 \cdot n!$ and this is repeated 4 times.

∴ Total number of words that can be formed,

$$= 2(n + 1)! + 4 \cdot \frac{(n + 2)(n + 1)}{2} n!$$

www.pdfworld.net

www.pdfworld.net

$$= 2(n+1)!\left(1+\overline{n+2}\right)$$

$$= \frac{2(n+1)!\,(n+2)\,(n+3)}{n+2} = \frac{2(n+3)!}{(n+2)}$$

➤ **Problem 18.** Find the number of all rational numbers $\frac{m}{n}$ such that

(a) $0 < \frac{m}{n} < 1$;

(b) m and n are relatively prime ;

(c) $mn = 25!$

➤ ***Solution*** We have

$$25! = 2^m . 3^n . 5^k . 7^r . 11^t . 13 . 17 . 19 . 23 \quad (1)$$

Let S denote the set $\{2^m, 3^n, ..., 23\}$ consisting of the 9 factors here. This has 2^9 subsets including the empty set. If $A \subset S$ and A is not empty let the product of the members of A be a; if A is empty choose a = 1.

The fundamental theorem of arithmetic shows that distinct subsets will give rise to distinct values of a.

thus we get 2^9 numbers a. If $b = \frac{25!}{a}$ then ab = 25!, and a and b are relatively prime. It is clear from (1) that 25! is not a perfect square. So $a \neq b$.

If $a < b$ choose m = a, n = b; if $a > b$ choose m = b, n = a. every fraction $\frac{m}{n}$ of the desired type is obtained like this.

Moreover, each $\frac{m}{n}$ is obtained exactly twice. Since we have 2^9 pairs (a, b) the number of fractions of the desired type is $\frac{1}{2}.2^9 = 256$.

➤ **Problem 19.** Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions :

(a) a, b, c are distinct ; (b) $a, b, c \in \{1, 2, 3,....,1999\}$ and (c) x + 1 divides $ax^2 + bx + c$.

➤ ***Solution*** Since x + 1 divides $ax^2 + bx + c$, we must have a + c = b. Thus we have to count the number of triples (a, b, c) with the condition that a, b, c lie in the set {1, 2, 3,...., 1999}, $a \neq c$ and a + c = b. If we take a < c, then for each a with $1 \leq a \leq 999$, c can take values from a + 1 to 1999 – a.

Thus for a = 1, c runs from 2 to 1998 giving 1997 ordered pairs (a, c) with a < c; for a = 2, c runs from 3 to 1997, giving 1995 pairs (a, c) with a < c, and so on. The number of ordered pairs (a, c) with a < c and a + c lying in the set {1, 2, 3,....,1999) is thus equal to $1997 + 1995 + 1993 + ... + 1 = 999^2$.

Similarly the number of pairs (a, c) with c < a and a + c lying in the set {1, 2, 3,....,1999} is 999^2. Hence the required number of polynomials is $2.999^2 = 1996002$.

➤ **Problem 20.** Find the number of isosceles triangles with integer sides if no side exceeds 1994.

➤ ***Solution*** Let the sides of the isosceles triangle be a, a, b. There will be such a unique triangle if and only if 2a > b.

Hence, for any a, b can vary from 1 to 2a – 1. We have additional restrictions: $a \leq 1994$ and $b \leq 1994$.

(i) If $a \leq \frac{1994}{2} = 997$, b varies from 1 to 2a – 1, i.e. b can take 2a – 1 values. Hence the number of isosceles triangles obtained is the sum of the first 997 odd natural numbers which equal $(997)^2$.

(ii) If $998 \leq a \leq 1994$, b can take any value between 1 and 1994, i.e. there are 1994 choices for b. In this case a has 997 possibilities.

So the number of isosceles triangles in this case is $997 \times 1994 = 2 \times (997)^2$. The total number of isosceles triangles is thus $3 \times (997)^2 = 3 \times 994009 = 2982027$.

➤ **Problem 21.** Show that number of integer isosceles or equilateral triangles none of whose sides exceeds 2m is $3m^2$.

➤ ***Solution*** The values of equal sides in an isosceles triangle whose sides are integers can be 1, 2, 3, 4....., 2m. A non degenerate triangle will be formed if sum of any two is greater than the third side.

Thus if the value of the equal side be k and the remaining side be x then we must have k + x > k, x + k > k, k + k > x.

The first two inequations are always true, the third inequation gives x < 2k which has 2k – 1 solutions if $2k - 1 \leq 2m$. It will have 2m solutions for other values.

Thus total numbers of solutions

$= 1 + 3 + 5 + ... + 2m - 1 + 2m + 2m + ... \text{ m times}$

$= m^2 + 2m^2 = 3m^2$.

www.pdfworld.net

www.pdfworld.net

Things to Remember

1. Mutiplication Principle
If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then total number of different ways of simultaneous occurrence of both the events in the definite order is m × n.

2. Addition Principle
If an event can occur in 'm' different ways, and another event can occur in 'n' different ways, then exactly one of the events can happen in m + n ways.

3. Complementary Principle
If a work is to be done under some restriction then the number of ways to do the work
= (the number of ways to do the work without restriction)
– (the number of ways to do the work under opposite restriction).

4. The number of permutations of n different objects taken r at a time is denoted by ${}^{n}P_{r}$.

5. The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.

6. The sum of all the numbers that can be formed using the digits $d_1, d_2, ..., d_n$ without repetition, is

$$= (n-1)!\,(d_1 + d_2 + ... + d_n)\left(\frac{10^n - 1}{9}\right).$$

7. With n distinct objects, ${}^{n}C_{r}$ counts the number of ways of choosing r of them without regard to order.

8. Standard Formulae

(i) ${}^{n}C_{r} = {}^{n}C_{n-r}$

(ii) ${}^{n+1}C_{r} = {}^{n}C_{r} + {}^{n}C_{(r-1)}$

(iii) $r.\,{}^{n}C_{r} = n.\,{}^{n-1}C_{r-1}$

(iv) The greatest value of ${}^{n}C_{r}$ is ${}^{n}C_{n/2}$ if n is even and ${}^{n}C_{\frac{n-1}{2}}$ or ${}^{n}C_{\frac{n+1}{2}}$ if n is odd.

(v) ${}^{n}C_{0}^{2} + {}^{n}C_{1}^{2} + ... + {}^{n}C_{n}^{2} = {}^{2n}C_{n}$

(vi) ${}^{n}C_{r} \cdot {}^{r}C_{s} = {}^{n}C_{s} \cdot {}^{n-s}C_{r-s}\ (n \geq r \geq s)$

(vii) $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \dfrac{n-r+1}{r}$

9. The number of permutations of n things taken all at a time when p of them are alike of one type, q of them are alike of another type, r of them are alike of a third type and the remaining n – (p + q + r) things are all different is $\dfrac{n!}{p!q!r!}$.

10. The number of permutations of n different things, taken all at a time, when m specified things always come together is m ! × (n – m + 1) !.

11. If m is the number of things on which there is no restriction and n the number of things, no two of which are to occur together, then the required number of ways = $m! \times {}^{m+1}C_{n} \times n!$

12. The total number of combinations of n different things taken one or more at a time = $2^n - 1$.

13. If out of (p + q + r + s) things, p are alike one kind, q are alike of second kind, r are alike of third kind and s are different, then the total number of combinations is (p + 1) (q + 1) (r + 1) $2^s - 1$.

14. The number of ways in which (m + n) different things can be divided into two unequal groups containing m and n things respectively is $\dfrac{(m+n)!}{m!\,n!}$.
If m = n, the groups are equal and in this case the number of division is $\dfrac{(2n)!}{n!\,n!\,2!}$

15. The number of objects having none of the properties α, β, γ .... is
$N - N(\alpha) - N(\beta) - N(\gamma) - ....$
$+ N(\alpha, \beta) + N(\alpha, \gamma) + N(\beta, \gamma) + ...$
$- N(\alpha, \beta, \gamma) - .....$ This is the general inclusion-exclusion principle.

www.pdfworld.net

www.pdfworld.net

16. The number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken as not different.
If clockwise and anticlockwise orders are not taken as different, then the number of circular permutations $= \frac{1}{2}\ {}^{n}C_{r}.\ (r-1)!$.

17. **(i)** The number of non-negative integral solutions of equation $x_1 + x_2 + ... + x_r = n$ is $= {}^{n+r-1}C_{r-1}$

(ii) The number of positive integral solutions of equation $x_1 + x_2 + ... + x_r = n$ is ${}^{n-1}C_{r-1}$.

18. The number of ways in which n different things can be arranged into r different groups is $n\ !\ .\ {}^{n+r-1}C_{r-1}$ if blank groups are allowed.

19. The number of ways in which n identical things can be distributed into r different groups is ${}^{n+r-1}C_{r-1}$ or ${}^{n-1}C_{r-1}$, according as blank groups are or are not admissible.

20. The number of ways in which n identical things can be distributed into r groups so that no group contains less than ℓ things and more than m things (ℓ < m) is coefficient of x^n in the expansion of $(x^{\ell} + x^{\ell+1} + x^{\ell+2} + ... + x^{m})^r$

21. The number of ways in which n different things can be distributed into different groups, blank groups being admissible is r^n.

22. The number of ways in which different things can be distributed into r different groups, there being no blank group, is coefficient of x^n in $n!.\ (e^x - 1)^r$
$= r^n - {}^{r}C_{1}(r-1)^n + {}^{r}C_{2}(r-2)^n - ... + (-1)^{r-1}\,{}^{r}C_{r-1}$

23. The number of permutations of r things out of n things of which p are of one kind, q are of second kind and so on = coeff of x^r in

$$r!\left[\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.....+\frac{x^p}{p!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+.....+\frac{x^q}{q!}\right).....\right]$$

24. **Factors of natural numbers**

(i) The total numbers of divisors of N including 1 and N is $= (a+1)(b+1)(c+1)...$

(ii) The sum of these divisors is
$= (p^0 + p^1 + p^2 + ... + p^a)\ (q^0 + q^1 + q^2 + ... + q^b)\ (r^0 + r^1 + r^2 + ... + r^c) ....$

(iii) Number of ways in which N can be resolved as a product of two factors is

$$= \begin{cases} \frac{1}{2}(a+1)(b+1)(c+1)\ .... & \text{if N is not a perfect square} \\ \frac{1}{2}[(a+1)(b+1)(c+1)\ ....+1] & \text{if N is a perfect square} \end{cases}$$

(iv) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

25. If out of n things, r things go to wrong places and (n – r) things go to their original places then the required number= ${}^{n}C_{n-r}\ D_r$ where D_r = No. of ways, if r things go to wrong places

$$= r!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+.....+(-1)^r\frac{1}{r!}\right).$$

26. **Selection of Squares**

Let there be n rows, I row has α_1 squares. II row has α_2 squares, III row has α_3 squares, and so on. If we placed β Xs in the squares such that each row contain atleast one X then the number of ways
= coefficient of x^{β} in

$$({}^{\alpha_1}C_1 x + {}^{\alpha_1}C_2 x^2 + ... + {}^{\alpha_1}C_{\alpha_1} x^{\alpha_1})$$
$$\times ({}^{\alpha_2}C_1 x + {}^{\alpha_2}C_2 x^2 + ... + {}^{\alpha_2}C_{\alpha_3} x^{\alpha_2})$$
$$\times ({}^{\alpha_3}C_1 x + {}^{\alpha_3}C_2 x^2 + ... + {}^{\alpha_3}C_{\alpha_3} x^{\alpha_3}) .....$$

27. n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then the number of parts into which these lines divide the plane is equal to $1 + \Sigma n$.

www.pdfworld.net

www.pdfworld.net

[Objective Exercises]

Single Correct Answer Type

1. The person is planning his journey. There are S_i (i = 1, 2, ...n) stoppages upto the destination. S_n is the final stoppage. Given that there are p_i buses and q_i trains for the S_i from the previous station. The number of ways in which the journey can be planned, is

(A) $\sum_{i=1}^{n}(p_i + q_i)$ (B) $\sum_{i=1}^{n} p_i q_i$

(C) $\prod_{i=1}^{n}(p_i + q_i)$ (D) $\prod_{i=1}^{n} p_i q_i$

2. If $\frac{{}^nP_{r-1}}{a} = \frac{{}^nP_r}{b} = \frac{{}^nP_{r+1}}{c}$, then $\frac{b^2}{a(b+c)}$ is equal to

(A) 1 (B) 2

(C) 1/2 (D) none

3. The number of ways in which a father can give 10 hotels to his 4 sons such that all the 10 hotels are not given to a particular son, is

(A) $4^{10} - 4$ (B) $10^4 - 10$

(C) ${}^{10}P_4 - 10$ (D) none

4. There are 10 lamps in a hall. Each one of them can be switched on independently. The number of ways in which the ball can be illuminated is

(A) 56 (B) 126

(C) 91 (D) none of these

5. The number of distinct rearrangements of the letters of the word "MULTIPLE" that can be made preserving the order in which the vowels occur and not counting the original arrangement is

(A) 6719 (B) 3359

(C) 6720 (D) 3214

6. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

(A) 5 (B) 7

(C) 6 (D) 4

7. The number of divisors of 6000, where 1 and 6000 are also considered as divisors of 6000, is

(A) 40 (B) 50

(C) 60 (D) 30

8. The number of four-digit numbers strictly greater than 4321 that can be formed from the digits 0, 1, 2, 3, 4, 5 allowing for repetition of digits is

(A) 310 (B) 360

(C) 288 (D) 300

9. The number of integers lying between 3000 and 8000 (including 3000 and 8000) which have atleast two digits equal is

(A) 2481 (B) 1977

(C) 4384 (D) 2755

10. A lady gives a dinner party to 5 guests to be selected from nine friends. The number of ways of forming the party of 5 given that two of the friends will not attend the party together is

(A) 56 (B) 126

(C) 91 (D) none of these

11. The number of 6 digit numbers that can be made with the digits 1, 2, 3 and 4 and having exactly two pairs of digits is

(A) 480 (B) 540

(C) 1080 (D) none

12. If m be number of different words that can be formed with the word BHARAT in which B and H are never together and n be number of different words that can be formed with the words BHARAT in which words are begin with B and end with T. Then m/n is

(A) 10 (B) 20

(C) 1 (D) 2

13. The number of six digit odd numbers, greater then 6,00,000 can be formed from the digits 5, 6, 7, 8, 9, 0 if repetition of digits is allowed, is

(A) 15120 (B) 15100

(C) 16000 (D) none of these

www.pdfworld.net

www.pdfworld.net

14. The total number of arrangements which can be made out of letters of the word 'ALGEBRA' without altering the relative position of vowels and consonants

(A) $\frac{7!}{2!}$ (B) $\frac{7!}{2!.5!}$

(C) $4! \, . \, 3!$ (D) $\frac{4!.3!}{2}$

15. The number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated is

(A) $4! \, . \, 4!$ (B) $\frac{8!}{4!}$

(C) 288 (D) none of these

16. The number of ways, in which seven persons can be arranged at a round table if two particular persons may not sit together is

(A) 480 (B) 120

(C) 80 (D) none of these

17. The number of different ways in which 8 persons can stand in a row so that between two particular persons A and B there are always two persons, is

(A) 60 (5!) (B) $15(4!) \times (51)$

(C) $4! \times 5!$ (D) none of these

18. The number of N digit quaternary sequence having exactly m 1's and m 2's and any number of 0's and 4's is

(A) ${}^{N}C_{m} \, . \, {}^{N-m}C_{m} \, . \, 2^{N-2m}$

(B) ${}^{N}P_{m} \, . \, {}^{N-m}C_{m} \, . \, 2^{N-2m}$

(C) ${}^{N}C_{m} \, . \, {}^{N-m}P_{m} \, . \, 2^{N-2m}$

(D) none of these

19. If number of divisors of a natural number 'n' is 15, then number of solutions (n, m) of $n - m^2 = 4$ $(m \in I)$ is

(A) 0 (B) 1

(C) 2 (D) ∞

20. m distinct animals of a circus have to be placed in m cages, one in each cage, if n (< m) are too small to accommodate p (n < p < m) animals, then the number of ways of putting the animals into cages is

(A) $({}^{m-n}P_{p})({}^{m-p}P_{m-p})$

(B) ${}^{m-n}C_{p}$

(C) $({}^{m-n}C_{p})({}^{m-p}C_{m-p})$

(D) None

21. The number of ordered triplets of positive integers which satisfy the inequality $20 \le x + y + z \le 50$ is

(A) ${}^{50}C_{3} - {}^{19}C_{3}$ (B) ${}^{50}C_{2} - {}^{19}C_{2}$

(C) ${}^{51}C_{3} - {}^{20}C_{3}$ (D) none of these

22. The number of different numbers which are smaller than 2×10^8 and are divisible by 3, which can be written by means of the digits 0, 1 and 2, is

(A) 4373 (B) 4374

(C) 4375 (D) none of these

23. The number of different seven digit numbers that can be written using only the three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number, is

(A) ${}^{7}P_{2} \, . \, 2^5$ (B) ${}^{7}C_{2} \, . \, 2^5$

(C) ${}^{7}C_{2} \, . \, 5^2$ (D) none of these

24. How many different nine digit numbers can be formed from the numbers 223355888 by rearranging its digits so that the odd digits occupy even positions?

(A) 16 (B) 36

(C) 60 (D) 180

25. The number of ways in which a set X, where n(X) = 12 can be partitioned in three subsets L, M, N each of 4 elements if $L \cup M \cup N = X$, $C \cap M = \phi$, $M \cap N = \phi$, is

(A) $\frac{12!}{(4!)^3}$ (B) $\frac{12!}{(3!)^4}$

(C) $\frac{12!}{(4!)^3} \cdot \frac{1}{3!}$ (D) none

26. 7 gentlemen and 4 ladies can sit at a round table so that two particular ladies may not sit together in

(A) 7 ! 3 ! 2 ! ways (B) $6 \, ! \, . \, {}^{6}P_{4}$ ways

(C) $6 \, ! \, . \, {}^{7}P_{4}$ ways (D) $6 \, ! \, . \, {}^{4}P_{2}$ ways

www.pdfworld.net

www.pdfworld.net

27. Three ladies have each brought a child for admission to a school. The head of the school wishes to interview the six people one by one, taking care that no child is interviewed before is mother. In how many different ways can the interviews be arranged?
(A) 6 (B) 36
(C) 72 (D) 90

28. There are 14 intermediate stations between Delhi and Agra on the Central Railway. A train is to be scheduled from Delhi to Agra so that it halts at exactly three intermediate stations, no two of which are consecutive. Then the number of ways of doing this is
(A) $\frac{11\times12\times13}{6}$
(B) $\frac{10\times11\times12}{6}$
(C) $\binom{14}{3}-\binom{14}{2}-\binom{14}{1}$
(D) $\binom{14}{3}+\binom{14}{2}+\binom{14}{1}$

29. Ten persons numbered 1, 2, ..., 10 play a chess tournament, each player playing against every other player exactly one game. Assume that each game results in a win for one of the players (that is, there is no draw). Let $w_1, w_2, ..., w_{10}$ be the number of games won by players 1, 2, ..., 10 respectively. Also, let $\ell_1, \ell_2, ..., \ell_{10}$ be the number of games lost by players 1, 2, ..., 10 respectively. Then
(A) $w_1^2+w_2^2+....+w_{10}^2 = 81-(\ell_1^2+\ell_2^2+.....+\ell_{10}^2)$
(B) $w_1^2+w_2^2+....+w_{10}^2 = 81+(\ell_1^2+\ell_2^2+.....+\ell_{10}^2)$
(C) $w_1^2+w_2^2+....+w_{10}^2 = \ell_1^2+\ell_2^2+.....+\ell_{10}^2$
(D) none of foregoing equalities is necessarily true.

30. The number of sequences of length five with 0 and 1 as terms which contain atleast two consecutive 0's is
(A) 4.2^3 (B) ${}^5C_2$
(C) 20 (D) 19

31. In a multiple-choice test there are 8 questions. Each question has 4 alternatives. of which only one is correct. If a candidate answers all the questions by choosing one alternative for each, the number of ways of doing it so that exactly 4 answers are correct is
(A) 70 (B) 283
(C) 5670 (D) none of these

32. The number of permutations of the letters a, b, c, d such that b does not follow a and c does not follow b and d does not follow c, is
(A) 12 (B) 11
(C) 14 (D) 13

33. The number of ways of seating three gentlemen and three ladies in a row, such that each gentleman is adjacent to atleast one lady, is
(A) 360 (B) 72
(C) 720 (D) none of these

34. The number of additional lines obtained by joining the points of intersection of n coplanar, non-parallel and non-concurrent lines, is
(A) ${}^nC_2 - {}^{n-1}C_2$
(B) ${}^{{}^nC_2}C_2 - n\left({}^{n-1}C_2\right)$
(C) ${}^{{}^nC_2}C_2$
(D) ${}^{{}^nC_2}C_2 + n\left({}^{n+1}C_2\right)$

35. The number of ways, in which one or more balls can be selected out of 10 white, 9 green and 7 blue balls, is
(A) 892 (B) 881
(C) 891 (D) 879

36. The number of ways, in which 12 identical coins can be put into 5 different purse, if none of the purse remains empty, is
(A) 660 (B) 110
(C) 165 (D) 330

www.pdfworld.net

www.pdfworld.net

37. The number of points (x, y, z) in space, whose each coordinate is a negative integer such that $x + y + z + 12 = 0$ is
(A) 385 (B) 55
(C) 110 (D) none of these

38. The number of ways of selecting n things out of 3n things, of which n are of one kind, and n are of a second kind and the rest unlike, is
(A) $n \cdot 2^{n-1}$ (B) $(n-1) \cdot 2^{n-1}$
(C) $(n+2)\,2^{n-1}$ (D) $(n+1)\,2^{n-1}$

39. The number of maps f from the set {1, 2, 3,} into the set {1, 2, 3, 4, 5} such that $f(i) \le f(j)$, whenever $i < j$, is
(A) 30 (B) 35
(C) 50 (D) 60

40. Let A_1, A_2, A_3 be three points on a straight line. Let B_1, B_2, B_3, B_4, B_5 be five points on a straight line parallel to the first one. Each of the three points on the first line is joined by a straight line to each of the five points on the second line. Further, no three or more of these joining lines meet at a point except possibly at the A's or the B's. Then the number of points of intersections of the joining line lying between the two given straight lines is
(A) 30 (B) 25
(C) 35 (D) 20

41. In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Then the number of intersection points the lines have is equal to
(A) 535 (B) 601
(C) 728 (D) none of these.

42. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$ then the number of values of the ordered quadruplet (a, b, c, d) is
(A) 165 (B) 455
(C) 310 (D) None of these

43. We are required to form different words with the help of the letters of the word INTEGER. Let m_1 be the number of words in which I and N are never together and m_2 be the number of words which start with I and end with R, then m_1/m_2 is given by
(A) 42 (B) 30
(C) 6 (D) 1/30

44. The number of ways of distributing 12 identical oranges among 4 children so that every child gets atleast one and no child more than 4 is
(A) 31 (B) 52
(C) 35 (D) 42

45. For a regular polygon with n sides ($n > 5$), the number of triangles whose vertices are joining non-adjacent vertices of the polygon is
(A) $n(n-r)(n-5)$
(B) $(n-3)(n-4)(n-5)/3$
(C) $2(n-3)(n-4)(n-5)$
(D) $n(n-r)(n-5)/6$

46. The number of ways in which a score of 11 can be made from a throw by three persons, each throwing a single dice once, is
(A) 45 (B) 18
(C) 27 (D) None of these

47. There are seven greeting cards, each of a different colour and seven envelopes of the same seven colours. The number of ways in which the cards can be put in the envelopes, so that exactly four of the cards go into the envelopes of the right colours, is
(A) $\binom{7}{3}$ (B) $2\binom{7}{3}$
(C) $(3!)\binom{4}{3}$ (D) $\binom{7}{3}\binom{4}{3}$

48. There are n white and n black balls marked 1, 2, ..., n. The number of ways in which we can arrange these balls in a row so that neighbouring balls are of different colours is
(A) n! (B) (2n)!
(C) $2(n!)^2$ (D) $(2n)!(n!)^2$

www.pdfworld.net

www.pdfworld.net

49. The number of different 5 lettered words can be formed using only the letters of the word 'ASSASSINATION' taking 2 like, 2 others alike and 1 different letter, is
(A) 720 (B) 900
(C) 1440 (D) none

50. The number of ways in which we can select four numbers from 1 to 30 as to exclude every selection of four consecutive numbers is
(A) 27378 (B) 27405
(C) 27399 (D) none of these

Multiple Correct Answer Type for JEE Advanced

51. There are counters available in 3 different colours (atleast four of each colour). Counters are all alike except for the colour. If 'm' denotes the number of arrangements of four counters if no arrangement consists of counters of same colour and 'n' denotes the corresponding figure when every arrangement consists of counters of each colour, then
(A) $m = 2n$ (B) $6m = 13n$
(C) $m = 80$ (D) $n = 36$

52. The number of ways in which 200 different things can be divided into groups of 100 pairs is
(A) $\frac{200!}{2^{100}}$
(B) $\left(\frac{101}{2}\right)\left(\frac{102}{2}\right)\left(\frac{103}{2}\right)....\left(\frac{200}{2}\right)$
(C) $\frac{200!}{2^{100}(100)!}$
(D) $(1 . 3 . 5 . ..... 199)$

53. The number of ways of arranging the letters AAAAA, BBB, CCC, D, EE & F in a row if the letter C are separated from one another is
(A) ${}^{13}C_3 . \frac{12!}{5!\,3!\,2!}$ (B) $\frac{13!}{5!\,3!\,3!\,2!}$
(C) $\frac{14!}{3!\,3!\,2!}$ (D) $11 . \frac{13!}{6!}$

54. The maximum number of permutations of 2n letters in which there are only a's & b's, taken all at a time is given by
(A) ${}^{2n}C_n$
(B) $\frac{2}{1} . \frac{6}{2} . \frac{10}{3} ..... \frac{4n-6}{n-1} . \frac{4n-2}{n}$
(C) $\frac{n+1}{1} . \frac{n+2}{2} . \frac{n+3}{3} . \frac{n+4}{4} ..... \frac{2n-1}{n-1} . \frac{2n}{n}$
(D) $\frac{2^n . [1 . 3 . 5 ..... (2n-3)(2n-1)]}{n!}$

55. You are given 8 balls of different colour (black, white, ...) . The number of ways in which these balls can be arranged in a row so that the two balls of particular colour (say red & white) may never come together is
(A) $8! - 2.7!$ (B) $6 . 7!$
(C) $2 . 6! . {}^7C_2$ (D) none

56. A women has 11 close friends. Number of ways in which she can invite 5 of them to dinner, if two particular of them are not on speaking terms & will not attend together is
(A) ${}^{11}C_5 - {}^9C_3$ (B) ${}^9C_5 + 2\,{}^9C_4$
(C) $3\,{}^9C_4$ (D) none

57. The kindergarten teacher has 25 kids in her class. She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once . Then the number of visits, the teacher makes to the garden exceeds that of a kid by
(A) ${}^{25}C_5 - {}^{24}C_4$ (B) ${}^{24}C_5$
(C) ${}^{25}C_5 - {}^{24}C_5$ (D) ${}^{24}C_4$

58. Number of permutations of the word "AUROBIND" in which vowels appear in an alphabetical order is
(A) P (8, 4) (B) C (8, 4)
(C) 4! C (8, 4) (D) C (8, 5) . 5!

59. Number of different heptagons which can be formed by joining the vertices of a polygon

www.pdfworld.net

www.pdfworld.net

having 16 sides, if none of the sides of the polygon is the side of the heptagon is

(A) $\frac{{}^{8}C_{6} \cdot 16}{7}$ (B) ${}^{10}C_{7} - {}^{8}C_{5}$

(C) 84 (D) 60

60. Two different packs each containing 52 playing cards are shuffled together. The number of ways in which a man can be dealt 26 cards so that he does not get two cards of the same suit & same denomination is

(A) $2^{26} - 1$

(B) $\frac{104 \cdot 102 \cdot 100 \cdot \ldots\ldots, 54}{26!}$

(C) ${}^{52}C_{26} \cdot 2^{26}$

(D) none

61. If N is the number of positive integral solutions of $x_1x_2x_3x_4 = 770$. Then

(A) N is divisible by a distinct primes

(B) N is a perfect square

(C) N is a perfect 4^{th} power

(D) N is a perfect 8^{th} power

62. There are 12 different balls in an urn. Number of ways in which 3 balls can be drawn from the urn, four times in succession without replacement is equal to:

(A) $\frac{12!}{(3!)^4 \cdot 4!}$ (B) $\frac{12!}{(3!)^4}$

(C) $\frac{12!}{4^2 \cdot 9^2}$ (D) $\frac{12!}{(4!)^3}$

63. There are 3 sections in a question paper each containing 5 questions. A candidate has to solve only 5 questions, choosing atleast one question from each section. The number of ways in which he can make is choices is

(A) ${}^{15}C_{5} - 3\,{}^{10}C_{5} - 3 \cdot {}^{5}C_{5}$

(B) ${}^{15}C_{5} - 3 \cdot {}^{10}C_{5} + 3 \cdot {}^{5}C_{5}$

(C) ${}^{15}C_{5} + 3 \cdot {}^{10}C_{5} - 3 \cdot {}^{5}C_{5}$

(D) $15\,({}^{5}C_{1} \cdot {}^{5}C_{3} + {}^{5}C_{2} \cdot {}^{5}C_{3})$

64. On the normal chess board, I_1 & I_2 are two insects at S.W. and N.E. corners. They start moving towards each other. Each insect moving with the same constant speed. Insect I_1 can move only to the right or upward along the lines while the insect I_2 can move only to the left or downward along the lines of the chess board. The total number of ways the two insects can meet at same point during their trip is

(A) $\left(\frac{9}{8}\right)\left(\frac{10}{7}\right)\left(\frac{11}{6}\right)\left(\frac{12}{5}\right)\left(\frac{13}{4}\right)\left(\frac{14}{3}\right)\left(\frac{15}{2}\right)\left(\frac{16}{1}\right)$

(B) $2^{8}\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)\left(\frac{7}{4}\right)\left(\frac{9}{5}\right)\left(\frac{11}{6}\right)\left(\frac{13}{7}\right)\left(\frac{15}{8}\right)$

(C) $\left(\frac{2}{1}\right)\left(\frac{6}{2}\right)\left(\frac{10}{3}\right)\left(\frac{14}{4}\right)\left(\frac{18}{5}\right)\left(\frac{22}{6}\right)\left(\frac{26}{7}\right)\left(\frac{30}{8}\right)$

(D) ${}^{16}C_{8}$

65. There are 10 seats in the first row of a theatre of which 4 are to be occupied. The number of ways of arranging 4 persons so that no two persons sit side by side is :

(A) ${}^{7}C_{4}$ (B) $4 \cdot {}^{7}P_{3}$

(C) ${}^{7}C_{3} \cdot 4!$ (D) 840

66. There are 12 points in a plane of which 5 are collinear. The maximum number of distinct quadrilaterals which can be formed with vertices at these points is :

(A) $2 \cdot {}^{7}P_{3}$ (B) ${}^{7}P_{3}$

(C) $10 \cdot {}^{7}C_{3}$ (D) 420

67. A man is at the origin on the x-axis and takes a unit step either to the left or to the right. He stops after 5 steps or if he reaches 3 or -2. Number of ways in which he

(A) reaches -2 is 3

(B) reaches 3 is 4

(C) stop exactly after walking 5 steps is 12

(D) can perform the experiment is 20

68. There are $(p + q)$ different books on different topics in Mathematics. $(p \neq q)$

If L = The number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.

M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books

www.pdfworld.net

www.pdfworld.net

and another gets q books. N = The number of ways in which these books are divided into two groups of p books and q books then,

(A) L = M (B) L = 2N
(C) 2L = M (D) M = 2N

69. A forecast is to be made of the results of five cricket matches, each of which can be a win or a draw or a loss for Indian team.
Let p = number of forecasts with exactly 1 error; q = number of forecasts with exactly 3 errors and r = number of forecasts with all five errors then the correct statements are

(A) 2q = 5r (B) 8p = q
(C) 8p = 5r (D) 2 (p + r) > q

70. Consider the expansion, $(a_1 + a_2 + a_3 + ... + a_p)^n$ where $n \in N$ and $n \leq p$. The correct statement(s) is/are

(A) number of different terms in the expansion is, ${}^{n+p-1}C_n$
(B) coefficient of any term in which none of the variables $a_1, a_2, ... , a_p$ occur more than once is 'n'
(C) coefficient of any term in which none of the variables $a_1, a_2, ... , a_p$ occur more than once is n!
(D) Number of terms in which none of the variables $a_1, a_2, ... , a_p$ occur more than once is ${}^pC_n$.

Comprehension - 1

A palindrome is a word which reeds the same when read backwards as when read forwards e.g. MADAM. Let us find the number of such words of length n which can be formed form an alphabet of k letters. If n = 2m, then one can fill the first m place in k^m ways, allowing repetition of letters. The other m places can be filled up in one way as it is required to copy the previous filling in opposite order. Thus the number of ways to form the palindromes is k^m when n = 2m. In light of this discussion, answer the following questions.

71. The number of palindromes of length 9 which can be formed using all the letters from A, A, A, B, B, C, C, C,C is

(A) 12 (B) 24
(C) 48 (D) 72

72. A family of 10 is standing for a family photograph. The grandfather and the grandmother have equal heights. Their two sons and two daughter-in-laws have equal heights but there heights are smaller than the heights of grand parents. There are four grandchildren of equal heights, they are shorter than their parents. The number of ways in which they can stand in a row so that the grandparents are in the middle followed by other members of the family with decreasing order of heights on both sides

(A) 512 (B) 1152
(C) 1024 (D) None of these

73. Let m be the number of palindromes of seven digited numbers formed by using the digits 0, 1, 2, 3, 4, ,5, 6 without repetition of digits a first for places. Let n be the number of palindromes of seven digited numbers formed by using the digits 0, 1, 2, 3, 4, 5, 6 with repetition of digits. Then the value of $\frac{m}{n}$ is

(A) $\frac{20}{49}$ (B) $\frac{720}{7^4}$
(C) $\frac{120}{343}$ (D) $\frac{35}{54}$

Comprehension - 2

A father has 7 children. He wishes to carry them to a garden three at a time. If no three same children go to the garden more than once, then

74. How many times father goes to garden

(A) 20 (B) 35
(C) 25 (D) none of these

75. How many times a child goes

(A) 20 (B) 15
(C) 10 (D) none of these

76. How many times Ram and Shyam (name of two children) go to the garden together

(A) 6 (B) 7
(C) 5 (D) none of these

www.pdfworld.net

www.pdfworld.net

Comprehension - 3

A doctor hosts a dinner to 17 guests. He is having 2 round tables which can accommodate 8 and 9 persons respectively.

77. The number of ways in which he can arrange the guests is
(A) ${}^{17}C_8\,(8!)^2$ (B) ${}^{17}C_9\,(8!\,9!)^2$
(C) $\frac{17!}{72}$ (D) $\frac{17!}{56}$

78. If two guests 'x' and 'y' are not on speaking terms and wish not to sit on same table, then number of ways of arranging them is
(A) 2(15!) (B) 15!
(C) $2.{}^{15}C_7 \,.\, 6! \,.\, 7!$ (D) ${}^{15}C_7 \,.\, 6! \,.\, 7!$

79. If two particular couples want to sit on a table allowing 9 persons to sit, then number of arrangements is
(A) 84(13!) (B) 3024(13!)
(C) 42(13!) (D) 168(13!)

Comprehension - 4

Counting by critical parenthesis method this Suppose we have to arrange n-pairs of parentheses in such a way that every arrangements is matched i.e., number of left parenthesis are always greater than or equal to number of right parenthesis in any length of the chain from start.
S is the number of ways of arranging n-right and n - left parenthesis in a row $= \frac{2n!}{n!\,n!}$. Let T be the arrangement of $(n+1)$ right and $(n-1)$ left parenthesis $= \frac{(2n)!}{(n+1)!(n-1)!}$.
It can be shown that set of mismatched arrangments of parenthesis in 'S' has bijective relation with the set of arrangements of T.
Since the set of mismatched arrangements in S has bijective relation with the set of arrangements in T.
$\therefore$ Number of the mismatched arrangements in $S = \frac{(2n)!}{(n+1)!(n-1)!}$.
$\therefore$ Number of the matched arrangements in $S = \frac{2n!}{n!\,n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{2!}{n!(n+1)!}$

80. The number of ways in which '4' pairs of parenthesis be arranged so that every arrangement is matched is :
(A) 3 (B) $\frac{{}^8C_4}{5}$
(C) ${}^8C_4$ (D) ${}^8C_5$

81. If a stamp vendor sells tickets of 1 rupee each and there are 3 persons having 1 rupee coin and 3 having 2 rupee coin standing in a row. Then the probability that stamp vendor do not run out of change if he does not have any money to start with is :
(A) 1/4 (B) 1/2
(C) 3/4 (D) None of these

82. Number of ways of arranging the 5 pairs of parenthesis, if first pair is matched but the next four pairs are not matched is :
(A) ${}^{10}C_6$ (B) ${}^8C_5$
(C) ${}^4C_2 \times {}^4C_2$ (D) None of these

Comprehension - 5

A standard 52-card deck is dealt to 4 players equally. By convention, each hand is assigned a number of points based on the formula
$4\times(\#\text{A's}) + 3\times(\#\text{K's}) + 2\times(\#\text{Q's}) + 1\times(\#\text{J's})$.
Given that a particular hand has exactly 4 cards that are A, K, Q or J.

83. The number of ways to enumerate symbols like AAAA, AAAQ, etc. (withhout permutation) to get atleast 13 points is
(A) 5 (B) 6
(C) 7 (D) 8

84. The total number of ways to get atleast 13 points is
(A) $197\,.{}^{36}C_9$ (B) $181\,.\,{}^{36}C_9$
(C) $180\,.\,{}^{36}C_9$ (D) None of these

85. The total number of ways to get exactly 13 points is
(A) $32\,.\,{}^{36}C_9$ (B) $17\,.\,{}^{36}C_9$
(C) $16\,.\,{}^{36}C_9$ (D) None of these

www.pdfworld.net

www.pdfworld.net

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true but R is not the correct explanation of A.
(C) A is true, R is false.
(D) A is false, R is true.

86. **Assertion (A) :** The number of ways of distributing n identical objects in r distinct boxes is ${}^{n+r-1}C_{r-1}$.
Reasons (R) : The number of arrangement of n objects of one kind and $r-1$ objects of another kind in a line must be $\frac{(n+r-1)!}{n!(r-1)!}$

87. **Assertion (A) :** The number of selections of four letters taken from the word PARALLEL must be 15.
Reasons (R) : Coefficient of x^4 in the expansion of $(1-x)^{-3}$ is 15

88. **Assertion (A) :** Alisha has four red socks and four blue socks. If she divides these socks into four pairs, then the probability that each pair will consist either of two red socks or two blue socks is $\frac{3}{35}$.
Reason (R) : The number of ways Alisha can divide the four red socks into two pairs is 3 and the total number of way she can divide the 8 socks into 4 pair is $\frac{\lfloor 8}{(\lfloor 2)^4 \lfloor 4}$

89. **Assertion (A) :** The number of arrangements of the numbers and letters in "VKKR2009" where one can read either VKKR or 2009 from left to right to is $23 \cdot {}^8C_4$ (For example, in 9VK0K20R one can read VKKR)
Reason (R) : To read VKKR, there are ${}^8C_4$ ways to place the letters and $\frac{\lfloor 4}{2}$ ways to place the numbers. Similarly there are ${}^8C_4 \frac{\lfloor 4}{2}$ arrangements where one can read 2009.

90. **Assertion (A) :** A candy company makes 5 colours of jelly beans, which comes in equal proportions. If I grab a random sample of 5 jellybeans, I get exactly 2 distinct colours in 300 ways.
Reason (R) : There are ${}^5C_2$ pairs of colours and each pair of colours contribute 30 sequences of beans that use both colours. Thus the answer is ${}^5C_2 \times 30$.

91. **Assertion (A) :** During the World Cup, India won 10 matches and lost 6 matches. Their wins came in three streaks of consecutive wins in ${}^9C_2 \cdot {}^7C_3$ ways.
Reason (R) : The above is equivalent to solving the equation $w_1 + w_2 + w_3 = 10$ and $\ell_1 + \ell_2 + \ell_3 + \ell_4 = 6$ where the variables are non-negative integers.

92. **Assertion (A) :** The number of subsets S of $\{1, 2, 3, ..., 10\}$ with $a < b < c$ where $a \in S$, $b \notin S$, $c \in S$, $b \in I$ is 968.
Reason (R) : Any non-empty set without the specified property must be a block of consecutive integers which count to be 45.

93. **Assertion (A) :** The number of ways in which the numbers 1, 2, 3, ..., 2010 can be placed at the vertices of a regular 2010 - gon so that no two adjacent numbers differ by more than 2 (rotations & reflections considered distinct) is 4020.
Reason (R) : The two number adjacent to 1 must be 2 and 3 and adjacent to 5 must be 3 and 7.

94. **Assertion (A) :** Given a 9× 9 chess board, we consider all rectangle whose edges lie along grid lines. For each such rectangle, we put a mark in every one of the unit squares inside it. When the process is completed, there are 25 unit squares containing an odd number of marks.
Reason (R) : The number of rectangles containing the square in the i^{th} row and j^{th} column is $i(10-i)j(10-j)$ which is odd if i and j are both odd. The number of ways of choosing $i, j \in \{1, 3, 5, 7, 9\}$ is $5 \times 5 = 25$.

95. **Assertion (A) :** The smallest positive integer n for which 360n has exactly 72 positive divisors, is 80.
Reason (R) : Let $360n = 2^a 3^b 5^c 7^d$ with $a \geq 3$, $b \geq 2, c \geq 1, d \geq 0$ and $(a+1)(b+1)(c+1)(d+1) = 72$

www.pdfworld.net

www.pdfworld.net

Match the Columns for JEE Advanced

96.

Column - I	**Column - II**
(A) The number of different 7-digit numbers that can be made by rearranging the digits of 3053354	(P) 46
(B) The largest number of pieces into which a circular pizza can be cut with 9 straight cuts is	(Q) 210
(C) The number of ordered triples (x, y, z) of positive integer which satisfy xyz = 4000 is	(R) 990
(D) Let S denote the set of all positive divisors of 60^5. If the product of all the numbers in S equals 60^k then the value of k is	(S) 360

97.

Column - I	**Column - II**
(A) The number of pairs (m, n) of positive integers for which $n! + 1 = (m! - 1)^2$, is	(P) 1
(B) The number of positive integers n for which there is an acute angled triangle with sides 10, 24 and n, is	(Q) 3
(C) The number of 3-digit number abc for which the 6-digit number 579 abc is divisible by 5, 7 and 9 is	(R) 4
(D) The number of integer x in {1, 2, 3,...., 99, 100} such that $x^2 + x^3$ is a square of an integer	(S) 9

98.

Column - I	**Column - II**
(A) Let r(n) denote the number obtained by reversing the order of the digits of n. The number of 2-digits positive number n so that n + r(n) is a perfect square, is	(P) 4
(B) If integer x,y, z satisfy $x^2 + y^2 + z^2 + 3 < xy + yz + 2z$ then a possible sum of x + y + z is	(Q) 6
(C) The digit 'a' such that the 3-digit number aaa is the sum of the first n positive integers for some n, is	(R) 5
(D) The sum of all positive integers n for which there exist a positive integer m such that $6 \mid n = m^2 - 13$, is	(S) 8

99.

Column I	**Column II**
(A) Number of increasing permutations of *m* symbols are there from the *n* set numbers $\{a_1, a_2, \ldots, a_n\}$ where the order among the numbers is given by $a_1 < a_2 < a_3 < \ldots a_{n-1} < a_n$ is	(P) n^m
(B) There are *m* men and *n* monkeys. Number of ways in which every monkey has a master, if a man can have any number of monkeys	(Q) ${}^mC_n$
(C) Number of ways in which *n* red balls and (*m* – 1) green balls can be arranged in a line, so that no two red balls are together, is (balls of the same colour are alike)	(R) ${}^nC_m$

www.pdfworld.net

www.pdfworld.net

(D) Number of ways in which '*m*' different toys can be distributed in '*n*' children if every child may receive any number of toys, is — (S) m^n

100.

	Column - I	Column - II
(A)	Number of ways of distributing 7 books to 2 children is	(P) 126
(B)	The value of ${}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots\ldots\ldots\ldots + {}^{11}C_2$ is	(Q) 128
(C)	The number of five digit numbers in which every digit exceeds the immediately preceding digit is	(R) 582
(D)	Number of permutations of {1, 2, 3, 4, 5, 6} such that the patterns 13 and 246 do not appear	(S) 220

Review Exercises for JEE Advanced

1. A man is dealt a poker hand (5 cards) from an ordinary pack of 52 playing cards. Find the number of ways in which he can be dealt a hand consisting of four cards of the same denomination.

2. There are two sets of parallel lines, their equations being $x\cos\alpha + y\sin\alpha = p$ and $y\cos\alpha - x\cos\alpha = p;\ p = 1, 2, 3, \ldots\ldots, n$ and α is a constant angle. If the number of rectangles formed by these two sets of lines is 225 then find the value of n.

3. In a certain algebraic exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation − combination and 6 examples on binomial theorem . Find the number of ways a teacher can select for his pupils atleast one but not more than 2 examples from each of these sets.

4. Find the number of permutations that can be formed from the letters of the word "THATCH" taken 3 at a time.

5. If there are three different varieties of mangoes available in the market, find the number of ways in which a man can purchase 20 mangoes, assuming that the mangoes of the same variety are identical and available in abundance.

6. Find the number of ways in which the letters of the word "ORION" can be arranged if the two consonants are not adjacent in any arrangement.

7. How many negative terms are there in the sequence (x_n), where $x_n = {}^{n+5}C_4 - \dfrac{143\times(n+5)!}{96\times(n+3)!}, n \in N$

8. Find the smallest positive integer n with 24 divisors (including 1 and n).

9. Find the number of combination of 16 things, 8 of which are alike and the rest different, taken 8 at a time.

10. We are given n A's and r B's. How many words, consisting of n A's but not necessarily all B's can be formed.

11. Show that the number of n letter words which can be formed by letters A, B, C, D, E such that each number has an even number of A's is $\dfrac{5^n + (-3)^n}{2}$.

12. If among 12 things 4 be alike and 5 be alike of another kind and the rest are different, then find the total number of combinations.

13. In maths paper there is a question on "Match the column" in which column A contains 6 entries & each entry of column A corresponds to exactly one of the 6 entries given in column B written randomly. 2 marks are awarded for each correct matching & 1 mark is deducted from each incorrect matching. A student having no subjective knowledge decides to match all the 6 entries randomly.

www.pdfworld.net

www.pdfworld.net

Find the number of ways in which he can answer, to get atleast 25% marks in this question.

14. A set 'A' consists of 6 & the set B consists of 3 distinct elements. Find the total number of functions from $A \rightarrow B$ which are onto.

15. If $N = 2^{p-1} . (2^p - 1)$, where $2^p - 1$ is a prime, then find the the sum of the divisors of N expressed in terms of N.

16. Find the number of different selections of six letters from the letters of the word "KARNATAKA".

17. Team A & B play in a tournament. The first team that wins two games in a row or wins a total of four games is considered to win the tournament. Find the number of ways in which tournament can occur.

18. How many four digit number with distinct digits can be made with the digits 0, 1, 2, 3, 4, 5 which are divisible by 3? How many of these will be divisible by 6?

19. Sixteen men compete with one another in running, swimming and riding. How many prize lists could be made if there were altogether 6 prizes of different values, one for running, 2 for swimming and 3 for riding?

20. Find the number of ways of selecting three squares on a chess board so that the three lie on a diagonal line of the board or parallel to it.

21. Three men have 6 different trousers, 5 different shirts and 4 different caps . Find the number of different ways in which they can wear them .

22. In a certain college at the B.Sc. examination, 3 candidates obtained first class honours in each of the following subjects: Physics, Chemistry and Maths, no candidates obtaining honours in more than one subject. Find the number of ways in which 9 scholarships of different value be awarded to the 9 candidates if due regard is to be paid only to the places obtained by candidates in any one subject.

23. If 'D' denotes the number of ways in which 8 different things be distributed between Ram and Shyam and 'L' denotes the corresponding figure when the things are all alike, each person receiving atleast one thing in both the cases then find the value of $D - L$.

24. How many necklaces can be made using 7 beads of which 5 are identical red beads and 2 are identical blue beads?

25. There are 8 persons.
(i) In how many ways can they be seated at a round table?
(ii) With the further condition that 2 of them, a and b, must not sit in adjacent seats?
(iii) If 4 of the persons are men and 4 ladies and if no two men are to be in adjacent seats ?
(iv) If the 8 persons are 4 married couples and if no husband and wife, as well as no two men, are to be in adjacent seats?

26. How many arrangements each consisting of 2 vowels & 2 consonants can be made out of the letters of the word 'DEVASTATION'?

27. (a) Ten boys are to be grouped into 5 tennis pairs. In how many ways can this be done?
(b) The same 10 boys are going to 5 seaside places A, B, C, D, E two to each. In how many ways can this be done?
Explain the difference between your answers.

28. The sides AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points respectively on them. Find the number of triangles that can be constructed using these interior points as vertices.

29. In how many ways can we put 6 identical white balls and 6 identical black balls in 10 different boxes if each box must contain atleast one ball.

30. In how many ways can a set with 4 elements be partitioned into
(i) Three ordered cells
(ii) Three unordered cells.

31. There are 4 pairs of hand gloves of 4 different colours. In how many ways can they be paired off so that a left-handed glove and a right-handed glove are not of the same colour ?

www.pdfworld.net

www.pdfworld.net

32. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 = 14$ in integers from 1 to 6 inclusive.

33. Find the number of solutions of $x_1 + x_2 + x_3 + x_4 = 1$ in integers between – 3and 3 inclusive.

34. There are 6 red balls and 6 green balls in a bag. Five balls are drawn out and placed in a red box. The remaining seven balls are put in a green box. Find the number of ways in which the balls can be drawn so that the number of red balls in the green box plus the number of green balls in the red box is not a prime number.

35. Find the number of 7 lettered words each consisting of 3 vowels and 4 consonants which can be formed using the letters of the word "DIFFERENTIATION".

Target Exercises for JEE Advanced

1. Find the number of numbers greater than 1000 which can be formed using only the digits 1, 1, 2, 3, 4, 0 taken four at a time.

2. There are 24 balls of 6 different sizes in a bag, there being 4 balls of each size in four different colours. In how many ways can r balls be selected so that they are of (i) different colours (ii) different sizes?

3. Three integers are selected from the integers 1, 2, ... , 1000. In how many ways can these integers be selected such that their sum is divisible by 4.

4. In how many shortest ways can we reach from the point (0, 0, 0) to point (3, 7, 11) in space where the movement is possible only along the x-axis and z-axis or parallel to them and change of axes is permitted only at integral points. (An integral point is one which has its co-ordinate as integers).

5. Consider the set of numbers {1, 2, 3, ... 5n}. Two numbers N_1 and N_2 are selected from this set with replacement. In how many ways numbers can be selected so that $N_1^4 - N_2^4$ is divisible by 5.

6. Show that there are $(n + 1)^2$ different ways in which we can get a sum of Rs. n by some rupee coins, some 50 paisa coins and some 25 paisa coins.

7. Consider n × n graph paper where n is a natural number. Consider the right angled isosceles triangles whose vertices are integer points of this graph paper and whose sides forming right angle are parallel to x and y axes. Show that the number of such triangles is $(2/3)n (n + 1) (2n + 1)$.

8. Show that the number of ways in which a hand of 13 cards can be dealt to a player in bridge game such that the hand contains all the four suits is $^{52}C_{13} - 4\ ^{39}C_{13} + 6 \times\ ^{26}C_{13} - 4$.

9. In how many ways can 3 girls and 9 boys be seated in two vans, each having numbered seats, 3 in the front and 4 at the back ? How many sitting arrangements are possible if 3 girls should sit together in the back row on adjacent seats ?

10. How many numbers of five digits can be made having exactly two identical digits ? How many of these will have the repeated digits in consecutive places ?

11. How many numbers of five digits can be made with the digits 1, 2, 3 each of which can be used atmost thrice in a number ?

12. In how many ways can 10 persons takes seats in a row of 24 fixed seats so that no two persons take consecutive seats ?

13. Prove that the total number of arrangements of things by taking any number of things from n different things is [e × n ! – 1] where [x] denotes the integer nearest to x.

14. In a certain test there are n questions. In the test, 3^{n-k} students gave wrong answers to atleast k questions, where k = 1, 2,..... n. If the total number of wrong answers is 3280, then find the value of n.

15. The number of arrangements of the letters abcd in which neither a, b nor c, d come together is $4! - {}^2C_1 \cdot 2.3! + {}^2C_2 \cdot 2^2 \cdot 2! = 8$. Explain this.

www.pdfworld.net

www.pdfworld.net

16. A party of 10 consists of 2 Englishmen, 2 Indians, 2 Americans and 4 men of other nationalities (all different). In how many ways can they stand in a row so that no two men of the same nationality are next one another ? In how many ways can they sit at a round table ?

17. Prove the number of combinations of n letters out of 3n letters of which n are a and n are b and the rest unlike is $(n + 2)2^{n-1}$.

18. In a row of 20 seats, in how many ways can three blocks of consecutive seats with five seats in each block be selected.

19. Show that if $a_1, a_2, \ldots a_m$ are distinct prime numbers other than unity, the number of solutions in integers (excluding unity) of the equation $x_1x_2x_3 \ldots x_n = a_1a_2a_3 \ldots a_n$ is n^m.
Also show that the number of solutions in which atleast one x is unity is
$${}^nC_1(n-1)^m - {}^nC_2(n-2)^m + \ldots + (-1)^{n-1} \cdot {}^nC_{n-1} \cdot 1^m$$

20. Find the number of seven digit positive integers such that the sum of the digits is 19.

21. In how many ways is it possible to separate the nine letters a, b, c, d, e, f, g, h, i into three non-empty batches ?

22. In how many ways is it possible to factor the number 30030 into three positive integer factors (a) if 1 is allowed as a factor, (b) if each factor must be greater than 1 ? (order does not count : 30.77.13 is the same factoring as 13.30.77)

23. Find the number of quintuples (x, y, z, u, v) of positive integers satisfying both equations $x + y + z + u = 30$ and $x + y + z + v = 27$.

24. Mr. A has x children by his first wife and Ms. B has x + 1 children by her first husband. They marry and have children of their own. The whole family has 10 children. Assuming that two children of the same parents do not fight, find the maximum number of fights that can take place among children.

25. Let $n_1 = x_1x_2x_3$ and $n_2 = y_1y_2y_3$ be two 3 digit numbers. How many pairs of n_1 and n_2 can be formed so that n_1 can be subtracted from n_2 without borrowing.

26. Suppose $A_1, A_2, \ldots, A_6$ are six sets each with four elements and $B_1, B_2, \ldots B_n$ are n sets each with two elements. Let $S = A_1 \cup A_2 \cup \ldots\ldots \cup A_6 = B_1 \cup B_2 \cup \ldots\ldots \cup B_n$. Given that each element of S belongs to exactly four of the A's and to exactly three of the B's. Find n.

27. The number of ways in which 2n things of one kind, 2n of another kind and 2n of a third kind can be divided between two persons so that each may have 3n things is $3n^2 + 3n + 1$.

28. Given n pairs of gloves, in how many ways can each of persons take a right handed and a left handed glove without taking a pair ?

29. Delegates from 9 countries including countries A, B, C, D are to be seated in a row. How many different seating arrangements are possible if the delegates of countries A and B are to be seated next to each other and the delegates of countries C and D are not to be seated next to each other ? How will the answer change if the seating is down at a round table ?

30. A batsman scores exactly a century by hitting fours and sixes in a twenty consecutive balls. In how many different ways can he do it if some balls may not yield runs and the order of boundaries and over-boundaries are taken into account?

31. In an exciting finish to an one-day cricket match between India and Pakistan, the Indians require 10 runs in the last 3 balls to win. If any one of the scores 0, 1, 2, 3, 4, 6 can be made from a ball and no wides or no-balls are bowled then in how many different sequences can the batsmen make exactly 10 runs?

32. How many integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 30$ with $0 \le x_i < 10$?

33. Seven people enter a lift . The lift stops at three (unspecified) floors. At each of the three floors, no one enters the lift, but atleast one person leaves the lift. After the three floor stops, the lift is empty, In how many ways can this happen?

34. Find the number of permutations of the letters of the word CONTRADICTORY such that neither the pattern "CON" nor "RAD" nor "ORY" appears.

www.pdfworld.net

www.pdfworld.net

35. Suppose a lunch buffet has 4 sections W, X, Y, Z having 10, 9, 8, 7 different items of cuisine, respectively. In how many ways, can an invitee to the lunch pick the food items if he is obliged to pick at least 4, 2, 3, 1 items from each of the sections w, x, y, z respectively.

Previous Years' Questions (JEE Advanced)

A. Fill in the blanks

1. In a certain test, a_i students gave wrong answers to atleast i question, where i= 1, 2,.....,k. No students gave more than k wrong answers. The total number of wrong answers given is..... **[IIT - 1982]**

2. The side AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points respectively on them. The number of triangles that can be constructed using these interior points as vertices is ..... **[IIT - 1984]**

3. Total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is ..... **[IIT - 1988]**

4. There are four balls of different colours and four boxes of colours, same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour is..... **[IIT - 1992]**

B. True/False

5. The product of any r consecutive natural numbers is always divisible by r!. **[IIT - 1985]**

C. Multiple Choice Questions with ONE correct answer

6. ${}^nC_{r-1} = 36$, ${}^nC_r = 84$, and ${}^nC_{r+1} = 126$, then r is **[IIT - 1979]**

(A) 1 (B) 2
(C) 3 (D) None of these

7. Ten different letters of an alphabet are given words with five letters are formed from the given letters. Then the number of words which have at least one letter repeated are **[IIT - 1980]**

(A) 69760 (B) 30240
(C) 99748 (D) None of these

8. The value of expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to **[IIT - 1989]**

(A) ${}^{47}C_5$ (B) ${}^{52}C_5$
(C) ${}^{52}C_4$ (D) None of these

9. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First women choose the chairs from amongst the chairs marked 1 to 4; and then the men select the chairs from amongst the remaining. The number of possible arrangements is **[IIT - 1982]**

(A) ${}^6C_3 \times {}^4C_2$ (B) ${}^4P_2 \times {}^4P_3$
(C) ${}^4C_2 + {}^4P_3$ (D) None of these

10. A five digit numbers divisible by 3 is to be formed using the numerals, 0, 1, 2, 3, 4 and 5, without repetition. The total number of ways this can be done is **[IIT - 1989]**

(A) 216 (B) 240
(C) 600 (D) 3125

11. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is **[IIT - 1998]**

(A) 6 (B) 7
(C) 8 (D) 9

12. Number of divisors of the form $4n + 2$ $(n \geq 0)$ of the integer 240 is **[IIT - 1998]**

(A) 4 (B) 8
(C) 10 (D) 3

13. In a college of 300 students, every student reads 5 newspapers and every newspaper is read by 60 students. The number of newspapers is **[IIT - 1998]**

(A) at least 30 (B) at most 20

www.pdfworld.net

www.pdfworld.net

(C) exactly 25 (D) None of these

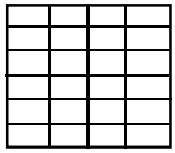
14. The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two females are not seated together is **[IIT - 1999]**
(A) 480 (B) 600
(C) 720 (D) 840

15. How many different nine digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions **[IIT - 2000]**
(A) 16 (B) 36
(C) 60 (D) 180

16. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals **[IIT - 2001]**
(A) 5 (B) 7
(C) 6 (D) 4

17. The number of arrangements of the letters of the word BANANA in which the two N's do not appear adjacently is **[IIT - 2002]**
(A) 40 (B) 60
(C) 80 (D) 100

18. A rectangle with sides 2m – 1 and 2n – 1 is divided into squares of units length by drawing parallel lines as shown in the diagram, then the number of rectangles possible with odd side lengths is



[IIT - 2005]
(A) $(m + n - 1)^2$ (B) 4^{m+n-1}
(C) m^2n^2 (D) $m(m + 1)n(n + 1)$

19. If LCM of p, q is $r^2t^4 s^2$, where r,s, t are prime numbers and p,q are the positive integers then the number of ordered pair (p, q) is **[IIT - 2006]**
(A) 252 (B) 254
(C) 225 (D) 224

20. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word COCHIN is **[IIT - 2007]**
(A) 360 (B) 192
(C) 96 (D) 48

21. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1,2 and 3 only is **[IIT - 2009]**
(A) 55 (B) 66
(C) 77 (D) 88

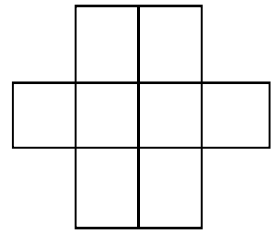
22. Let S = {1, 2, 3, 4}. The total number of unordered pairs of disjoint subsets of S is equal to **[IIT - 2010]**
(A) 25 (B) 34
(C) 42 (D) 41

23. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that eacs person gets at least one ball is **[IIT - 2012]**
(A) 75 (B) 150
(C) 210 (D) 41

D. Subjective Problems

24. **(i)** In how many ways can pack of 52 cards be divided equally amongst four players in order ?
(ii) In how many ways can you divide these cards in 4 sets, three of them having 17 cards each and the fourth one just 1 card ? **[IIT - 1979]**

25. Six X's have to be placed in the squares of figure below in such a way that each row contains at least one X. In how many different ways can this be done.



[IIT - 1978]

26. Five balls of different colours are to be placed in there boxes of different sizes. Each box can hold all five. In how many different ways can we place the balls so that no box remains empty? **[IIT - 1981]**

27. *m* men and *n* women are to be seated in a row so that no two women sit together. If m > n, then show that the number of ways in which they can be seated is $\frac{m!(m+1)!}{(m-n+1)!}$ **[IIT - 1983]**

www.pdfworld.net

www.pdfworld.net

28. 7 relatives of a man comprises 4 ladies and 3 gentlemen; his wife has also 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relative and 3 of the wife's relatives ?
[IIT - 1985]

29. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw?
[IIT - 1986]

30. A student is allowed to select atmost n books from a collection of (2n + 1) books. If the total number of ways in which he can select at least one book is 63, find the value of n.
[IIT - 1987]

31. In an examination, the maximum marks for each three paper are 50 each. Maximum marks for 4th paper are 100. Find the number of ways in which the candidate can score 60% marks in the aggregate. **[IIT - 1989]**

32. Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the sitting arrangements can be made.
[IIT - 1991]

33. A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee? In how many of these committees
(A) women are in majority
(B) men are in majority **[IIT - 1994]**

34. Let n & k be positive integers such that $n \geq \frac{k(k+1)}{2}$. The number of solutions $(x_1, x_2, ..., x_k)$, $x_1 \geq 1$, $x_2 \geq 2$, ..., $x_k \geq k$, all integers, satisfying $x_1 + x_2 + ... + x_k = n$, is .....
[IIT - 1996]

35. Find the total number of ways of selecting five letters from the letters of the word INDEPENDENT. **[IIT - 1998]**

36. Prove by permutation otherwise $\frac{(n^2)!}{(n!)^n}$ is an integer ($n \in I^+$). **[IIT - 2004]**

Paragraph for Questions 36 and 37

Let a_n denote the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

37. The value of b_6 is
(a) 7 (b) 8
(c) 9 (d) 11 **[IIT - 2012]**

38. Which of the following is correct ?
(a) $a_{17} = a_{16} + a_{15}$
(b) $c_{17} \neq c_{16} + c_{15}$
(c) $b_{17} \neq b_{16} + b_{15}$
(d) $a_{17} = c_{17} + b_{16}$ **[IIT - 2012]**

E. Match the Columns [IIT - 2008]

39. Consider all possible permutations of the letters of the word ENDEANOEL.

Column – I	Column – II
(A) The number of permutations containing the word ENDEA is	(P) 5 !
(B) The number of permutations in which the letter E occurs in the first and the last positions is	(Q) $2 \times 5!$
(C) The number of permutations in which none of the letters D, L, N occurs in the last five positions is	(R) $7 \times 5!$
(D) The number of permutations in which the letters A, E, O occur only in odd positions is	(S) $21 \times 5!$

www.pdfworld.net

www.pdfworld.net

ANSWERS

Practice Problems—A

1. Boys have 30 choices; girls have 28 choices.
2. 30 **3.** 30 **4.** 64
5. 4968 **6.** 720, 420 **7.** 243, 768.
8. (a) 99 (b) 101 **9.** 180
10. (i) 12 (ii) 4 (iii) 8

Practice Problems—B

1. (i) 132 (ii) 30 (iii) 5040
2. (i) 5 (ii) $n(n+1)$ (iii) $(n+1)(n+2)(n+3).....(2n)$
3. 720 **4.** ${}^{8}P_{3}$ **5.** ${}^{9}P_{7}$
6. 120 **7.** 24 **8.** 504
9. 5 **10.** 720 ; 6 **11.** 2730
12. 120; 30240 **13.** k_{P_n}

Practice Problems—C

1. 2^n **2.** 3374 **3.** 3^{12}
4. 6^5 **5.** 4^{10} ; 4×3^9
6. It may take more than 10 days.
7. $\frac{10(10^7-1)}{9}$ **8.** 5^3 **9.** 4096
10. (i) 4096 (ii) 1680 (iii) 4088
11. 6560 **12.** $20^3.19^2$
13. (a) 27600 (b) 35152 (c) 1104
14. (a) 17,576,000 (b) 17,576,000 (c) 456,976,000

Practice Problems—D

1. $9\,[{}^{9}P_0 + {}^{9}P_1 + {}^{9}P_2 + {}^{9}P_3 + {}^{9}P_4 + ... + {}^{9}P_9]$
2. 17. 8 ! **3.** 39996 **4.** 5199960
5. 64440 **6.** 614 **7.** 720, 120, 360
8. 375, 72 **9.** 744 **10.** 86400
11. 12 **12.** 240 **13.** 45000
14. 42 **15.** 42

Practice Problems—E

1. (i) and (iii) are combinations ; (ii), (iv) and (v) are permutations.
2. 56 **3.** 35 **4.** 70
5. 20 **6.** 2739
7. {4, 5, 6, 7, 8, 9} and {1, 15, 35, 14, 9; 1}
8. (a) {8 , 9, 10} (b) {0, 1, 2,...., 27}
9.. 5, 6, 7,..., 10 **10.** ${}^{12}C_6 \times {}^{6}C_4$
11. m = 11 ; 156 **12.** 200
13. 40. 39. ${}^{38}C_5$ **14.** ${}^{15}C_2$. ${}^{10}C_3$. ${}^{5}C_4$
15. ${}^{52}C_{10} - {}^{48}C_{10}$ **16.** 4 . ${}^{44}C_4$ **17.** 141120

Practice Problems—F

1. $\frac{12!}{(2!)^3 3!}$ **2.** (a) 120 (b) 60; (c) 10; (d) 83160
3. 2520 **4.** 22222200 **5.** $\frac{(3n)!}{(n!)^2}$
6. $\frac{12!}{2^6}$ **7.** $\frac{100!}{48!52!}$ **8.** 420; 240
9. 254 **10.** 236th

Practice Problems—G

1. (a) 240 (b)600 **2.** 24
3. (a) 2. 29 ! (b) 28. 29 ! **4.** 1225
5. (i) 1814400 (ii) 2419200 (iii) 25401600
6. 72 **7.** (i) 625 (ii) 24
8. 108 **9.** 0240
10. 200 **11.** $4!-[3!2!+3!2!-2!2!2!]=8$
12. 5760 **13.** 3! 10! 5! 5! **14.** 6! –5! 2!
15. 7560 **16.** 36 **17.** 360; 240; 12
18. (a) 9! (b) 2(5!) (c) 6! 4! (d) 5! 4!

Practice Problems—H

1. 40320
2. 6188 when all rupee coins are identical and all paise coins are identical; $16! \times {}^{17}P_{12}$, if coins of same denomination are different.
3. ${}^{m-2}C_3$ **4.** 2275 **5.** n = 8

Practice Problems—I

1. 255. **2.** 217. **3.** 624
4. 67, 70 **5.** 3720
6. (i) 2002, (ii) 364
7. 778320 **8.** 266 **9.** 2700
10. (a) 9 (b) 10 **11.** 79.
12. (a) 84 (b) 10 (c) 40 (d) 28
13. 144000 **14.** 1200, 48

www.pdfworld.net

www.pdfworld.net

Practice Problems—J

1. (a) 60 (b)10 **2.** 15 **3.** 209
4. 315 **5.** (i) $2^9 - 1$ (ii) 29.
6. 185 **7.** 113 ; 2190

Practice Problems—K

1. $\frac{12!}{3!3!3!3!} = 369600$

2. $\frac{50!}{(12!)^3(7!)^2 3!\,2!}$

3. $\frac{50!}{(12!)^3(7!)^2 3!\,2!}5!$

Practice Problems—L

1. 16000 **2.** 720 **3.** 6 ! 4!
4. 18 ! 2 ! **5.** $\frac{6!}{2} = 360$ **6.** 144
7. 12

Practice Problems—M

1. (a) ${}^{49}C_3$ (b) ${}^{53}C_3$
2. (a) ${}^{11}C_6$(b) ${}^{10}C_5 + {}^{9}C_4 + {}^{8}C_3 + {}^{7}C_2 + {}^{6}C_1 + 1$.
3. (a) ${}^{37}C_4$ (b) ${}^{30}C_4$ **4.** ${}^{16}C_3$
5. (a) ${}^{17}C_3$ (b) ${}^{11}C_3$ **8.** 66

Practice Problems—N

1. ${}^{12}C_{10}$ **2.** 3465 **3.** ${}^{7}C_2$
8. ${}^{m+a-1}C_a \cdot {}^{m+b-1}C_b \cdot {}^{m+c-1}C_c$.
9. 1024 **11.** 141 **13.** 1806

Practice Problems—O

1. 48 **2.** 23 **3.** 60
4. 22260. **5.** (i) 44 ; (ii) 20.
6. 44 **7.** 1854

Practice Problems—P

1. ${}^{m}C_2 \cdot {}^{n}C_2$ **2.** ${}^{n}C_6$ **3.** ${}^{n+k}C_k$
4. $\frac{n(n+1)(2n+1)}{6}$ **5.** 118
6. $26 \times 6!$ **7.** 18720 **8.** 36

Objective Exercises

1. C **2.** A **3.** A
4. D **5.** B **6.** B
7. A **8.** A **9.** A
10. C **11.** C **12.** B
13. D **14.** D **15.** C
16. A **17.** A **18.** A
19. A **20.** A **21.** A
22. A **23.** B **24.** C
25. C **26.** C **27.** D
28. A **29.** C **30.** D
31. C **32.** B **33.** A
34. B **35.** D **36.** D
37. B **38.** C **39.** B
40. A **41.** A **42.** A
43. B **44.** A **45.** C
46. C **47.** B **48.** C
49. A **50.** D **51.** BD
52. BCD **53.** AD **54.** ABCD
55. ABC **56.** ABC **57.** AB
58. AC **59.** AB **60.** BC
61. BCD **62.** BC **63.** BD
64. ABCD **65.** BCD **66.** AD
67. ABD **68.** CD **69.** ABD
70. ACD **71.** A **72.** B
73. C **74.** B **75.** B
76. C **77.** C **78.** A
79. C **80.** B **81.** A
82. B **83.** C **84.** A
85. D **86.** A **87.** D
88. A **89.** B **90.** A
91. C **92.** B **93.** B
94. A **95.** D
96. (A) – (S), (B) – (P), (C) – (Q), (D) – (R)
97. (A) – (P), (B) – (R), (C) – (Q), (D) – (S)
98. (A) – (S), (B) – (P), (C) – (Q), (D) – (R)
99. (A) – (R), (B) – (S), (C) – (Q), (D) – (P)
100. (A) – (Q), (B) – (S), (C) – (P), (D) – (R)

Review Exercises for JEE Advanced

1. 624 **2.** n = 6 **3.** 3150
4. 42 **5.** 231 **6.** 36

www.pdfworld.net

www.pdfworld.net

7. 3 **8.** 360 **9.** 256
10. $^{n+r+1}C_r$ **12.** 239 **13.** 56
14. 540 **15.** 2 N **16.** 19
17. 14 **18.** 96, 52
19. 12902400
20. 392 **21.** $^{6}P_3 . {}^{5}P_3 . {}^{4}P_3$ **22.** 1680
23. 247 **24.** 3
25. (i) 7! (ii) 6! (iii) 3! × 4! (iv) 12
26. 1638 **27.** 945, 113400
28. 205 **29.** 26250 **30.** 81, 14
31. 9 **32.** $^{13}C_3 - 4.\,{}^{7}C_3$
33. $^{16}C_3 - 4.\,{}^{9}C_3$
34. 96 **35.** $^{7}C_3$ .33 x 282 = 325710

Target Exercises for JEE Advanced

1. 159 **2.**(i) 1296 (ii) 3840
3. 41541750 **4.** $^{21}C_3\ {}^{18}C_7$
5. $17n^2$
9. 91× 12! ; 1320 × 9!
10. 45360; 18144
11. 210 **12.** $\frac{15!}{5!}$ **14.** 8
16. (i) 1895040; (ii) 145680
18. 56 **20.** $^{24}C_6 - {}^{15}C_6 - 6.\,{}^{14}C_6$
21. 3025 **22.** (a) 122 (b) 90
23. 2600 **24.** 33 **25.** $45.(55)^2$
26. n = 9 **28.** n! . n!
29. $6! \times {}^{7}C_2 ; 5! \times {}^{5}C_2$
30. $20!\left(\frac{1}{16!3!}+\frac{1}{14!4!2!}+\frac{1}{12!7!}+\frac{1}{10!10!}\right)$
31. 21 **32.** $^{33}C_3 - 6(^{23}C_3) + 4(^{13}C_3) - 1$
33. 1806
34. $\frac{\lfloor 3}{(\lfloor 2)^4} - \frac{5}{8}\lfloor 11 + \frac{5}{4}\lfloor 9 - \frac{\lfloor 7}{\lfloor 2}$
35. $(2^{10} - 176)(2^9 - 10)(2^9 - 10)(2^8 - 37)(2^7 - 1)$ = 11839882848

Previous Years' Questions (JEE Advanced)

1. $a_1 + a_2 + ... + a_k$ **2.**205
3. 35 **4.** 9 **5.** T
6. C **7.** A **8.** C
9. D **10.** A **11.** B
12. A **13.** C **14.** A
15. C **16.** B **17.** A
18. C **19.** C **20.** C
21. C **22.** D **23.** B
24. $\frac{52!}{(13!)4}; \frac{52!}{(17!)^3 3!}$ **25.** 26
26. 150 **28.** 485 **29.** 64
30. 3 **31.** 110551
32. $^{11}C_5(9!)^2$
33. 6062 (A) 2702, (B) 1008
34. $^{m}C_{k-1}$, $m = \frac{1}{2}(2m^2 - K^2 + K - 2)$
35. 72 **37.** B **38.** A
39. (A) – (P), (B) – (S), (C) – (Q), (D) – (Q)

pdfworld.net

www.pdfworld.net

www.pdfworld.net

5 BINOMIAL THEOREM

5.1 INTRODUCTION

In earlier classes, we have learnt how to find the squares and cubes of binomials like a + b and a – b. Using them, we could evaluate the numerical values of numbers like $(98)^2 = (100 - 2)^2$, $(999)^3 = (1000 - 1)^3$, etc. However, for higher powers like $(98)^5$, $(101)^6$, etc., the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand $(a + b)^n$, where n is an integer or a rational number. In this chapter, we will first study binomial theorem for positive integral indices.

A Binomial Expansion Pattern

In this section, we introduce a method for writing out the terms of expressions of the form $(x + a)^n$, where n is a natural number. The formula for writing out the powers of a binomial $(x + a)^n$ as a polynomial is called the binomial theorem. This theorem is important when working with probability and statistics. Some expansions of $(x + a)^n$, for various non-negative integer values of n, are given below.

$(x + a)^0 = 1$

$(x + a)^1 = x + a$

$(x + a)^2 = x^2 + 2xa + a^2$

$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$

$(x + a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$

$(x + a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$

Studying these results reveals a pattern that can be used to write a general expression for $(x + a)^n$. First , notice that after the special case $(x + a)^0 = 1$, each expression begins with x raised to the same power as the binomial itself. That is, the expansion of $(x + a)^1$ has a first term of x^1, $(x + a)^2$ has a first term of x^2, $(x + a)^3$ has a first term of x^3, and so on. also, the last term in each expansion is a to the same power as the binomial. Thus, the expansion of $(x + a)^n$ should begin with the term x^n and end with the term a^n.

Also, the exponents on x decrease by one in each term after the first, while the exponents on a, beginning with a in the second term, increase by one in each succeeding term. That is , the variables in the terms of the expansion of $(x + a)^n$ have the following pattern.

$$x^n, x^{n-1}a, x^{n-2}a^2, x^{n-3}a^3, \ldots. xa^{n-1}, a^n$$

This pattern suggests that the sum of the exponents on x and a in each term is n. For example, in the thrid term in the list above, the variable is $x^{n-2}a^2$, and the sum of the exponents is $n - 2 + 2 = n$.

www.pdfworld.net

www.pdfworld.net

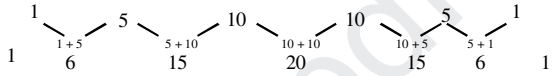
Pascal's Triangle

Now, examine the coefficients in the terms of the expansions shown above. Writing the coefficients alone gives the following pattern.

n = 0 1
n = 1 ..... 1 1
n = 2 ..... 1 2 1
n = 3 ..... 1 3 3 1
n = 4 ..... 1 4 6 4 1
n = 5 ..... 1 5 10 10 5 1
n = 6 ..... 1 6 15 20 15 6 1
n = 7 ..... 7 7 21 35 35 21 7 1
n = 8 .....1 8 28 56 70 56 28 8 1
: : : : : : : :

With the coefficients arranged in this way, we can see that each number in the triangle is the sum of the two numbers directly above it (one to the right and one to the left). For example, if we number the rows starting with row 0, in row four, 1 is the sum of 1, the only number above it, 4 is the sum of 1 and 3, 6 is the sum of 3 and 3, and so on. This triangular array of numbers is called Pascal's triangle.

If we continue with the sixth row, the first and last numbers are 1. Each of the other numbers is obtained by finding the sum of the two closest numbers above it in the fifth row.



Although it is possible to use Pascal's triangle to find the coefficients of $(x + y)^n$ for any positive integer value of n, this becomes impractical for large values of n because of the need to write out all the preceding rows. A more efficient way of finding these coefficients uses **factorial notation**. The number n! (read "n -factorial") is defined as follows :

n – Factorial

For any positive integer n

$n! = n(n-1)(n-2)....(3)(2)(1)$,

and $0! = 1$

➤ **Example 1.** If 'n' be even, show that the expression

$$\frac{n(n+2)(n+4)\ ....\ (2n-2)}{1\,.\,3\,.\,5....(n-1)}$$ simplifies to 2^{n-1}.

➤ ***Solution*** Let $n = 2m$

The given expression

$$= \frac{2m\,(2m+2)\,(2m+4)\ .....\ (4m-2)}{1\,.\,3\,.\,5\,.\,.....\,.\,(2m-1)}$$

$$= \frac{2^m\,.\,m\,(m+1)\,(m+2)\ .....(2m-1) \times (m-1)\,!\,.\,2m}{[1\,.\,3\,.\,5\,.....\,(2m-1)]\,.....\,(m-1)\,!\,\,(2m)}$$

$$= \frac{2^m\,.\,(2m)\,!}{2\,.\,m\,!\,[1\,.\,3\,.\,5\,.....\,(2m-1)]}$$

$$= \frac{2^{m-1}\,.\,2^m\ m\,!\,[1\,.\,3\,.\,5....\,(2m-1)]}{m\,!\,[1\,.\,3\,.\,5\,....\,(2m-1)]}$$

$$= 2^{2m-1} = 2^{n-1}$$

The Binomial Coefficient

Now, look at the coefficients of the expression

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

The coefficient of the second term, $5x^4y$ is 5, and the exponents on the variables are 4 and 1.

Note that $5 = \frac{5!}{4!\,1!}$.

The coefficient of the third term is 10, with exponents of 3 and 2, and $10 = \frac{5!}{3!\,2!}$

The last term (the sixth term) can be written as $y^5 = 1x^0y^5$, with the coefficient 1, and exponents of 0 and 5.

www.pdfworld.net

www.pdfworld.net

Since 0! = 1, check that $1 = \frac{5!}{0!\,5!}$

Generalizing from these examples, we find that the coefficient for the term of the expansion of $(x + a)^n$ in which the variable part is $x^r a^{n-r}$ (where $r \le n$) will be $\frac{n!}{r!\,(n-r)!}$. This number is called as binomial coefficient.

Definition of binomial coefficient

For non-negative integers n and r, with $n \ge r$, the expression ${}^nC_r$ is called a binomial coefficient and is defined by ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

$\therefore\ {}^nC_0 = \frac{n!}{0!(n-0)!} = 1$ since $0! = 1$

${}^nC_1 = \frac{n!}{1!(n-1)!} = \frac{n}{1!}$, ${}^nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2!}$

${}^nC_3 = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$, ……,

${}^nC_n = \frac{n!}{n!(n-n)!} = 1$

These binomial coefficients are just numbers from Pascal' triangle. For example , ${}^3C_0$ is the first number in the row that begins 1, 3,..... and ${}^7C_4$ is the fifth number in the row that begins 1, 7.....

The Pascal's triangle can now be rewritten as

Index	Coefficient					
0	${}^0C_0$					
	1					
1	${}^1C_0$	${}^1C_1$				
	1	1				
2	${}^2C_0$	${}^2C_1$	${}^2C_2$			
	1	2	1			
3	${}^3C_0$	${}^3C_1$	${}^3C_2$	${}^3C_3$		
	1	3	3	1		
4	${}^4C_0$	${}^4C_1$	${}^4C_2$	${}^4C_3$	${}^4C_4$	
	1	4	6	4	1	
5	${}^5C_0$	${}^5C_1$	${}^5C_2$	${}^5C_3$	${}^5C_4$	${}^5C_5$
	1	5	10	10	5	1

Pascal's triangle

Observing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For example, for the index 7 the row would be

${}^7C_0$ ${}^7C_1$ ${}^7C_2$ ${}^7C_3$ ${}^7C_4$ ${}^7C_5$ ${}^7C_6$ ${}^7C_7$

When $r \ne 0$ or $r \ne n$, there is a simple pattern for evaluating binomial coefficients.

$${}^8C_2 = \frac{\overbrace{8\ .\ 7}^{2\text{ factors}}}{\underbrace{2\ .\ 1}_{2\text{ factorial}}} \text{ and } {}^{10}C_3 = \frac{\overbrace{10\ .\ 9\ .\ 8}^{3\text{ factors}}}{\underbrace{3.\ 2\ .\ 1}_{3\text{ factorial}}}$$

In general, we have the following :

$${}^nC_r = \frac{\overbrace{n(n-1)(n-2).....}^{r\text{ factors}}}{r!}, \quad 0 < r < n$$

Note that the binomial coefficients are found in a symmetrical pattern.

For $(x + y)^5$, the pattern is

1 5 10 10 5 1

We observe that ${}^nC_r = \frac{n!}{r!(n-r)!} = {}^nC_{n-r}$

Hence, ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$,..... and thus the coefficients equidistant from the beginning and the end are equal. This is the so called **symmetry property** of the binomial coefficients.

Also note that ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$.

➤ **Example 2.** Find the binomial coefficient ${}^{12}C_{10}$

➤ ***Solution*** Rather than calculating the binomial coefficient

$${}^{12}C_{10} = \frac{12\ .\ 11\ .\ 10\ .\ 9\ .\ 8\ .\ 7\ .\ 6\ .\ 5\ .\ 4\ .\ 3}{10\ .\ 9\ .\ 8\ .\ 7\ .\ 6\ .\ 5\ .\ 4\ .\ 3\ .\ 2\ .\ 1}$$

we can use the fact that ${}^{12}C_{10} = {}^{12}C_2$ as follows :

$${}^{12}C_{10} = {}^{12}C_2 = \frac{12\ .\ 11}{2\ .\ 1} = 66$$

The property of sum of two consecutive binomial coefficients (the basis for Pascal's triangle), is

$${}^nC_k + {}^nC_{k+1} = {}^{n+1}C_{k+1}$$

www.pdfworld.net

www.pdfworld.net

It is easily derived by writing the left hand side with a common denominator

$$\frac{n(n-1).....(n-k+1)}{k!}+\frac{n(n-1).....(n-k+1)(n-k)}{(k+1)!}$$

$$=\frac{n(n-1).....(n-k+1)(k+1)+n(n-1).....(n-k+1)(n-k)}{(k+1)!}$$

$$=\frac{n(n-1).....(n-k+1)[(k+1)+(n-k)]}{(k+1)!}$$

$$=\frac{n(n-1).....(n-k+1)(n+1)}{(k+1)!}={}^{n+1}C_{k+1}$$

If we look again at the Pascal-triangle, we observe that the numbers in a given row increase for a while, and then decrease. We can prove this fact in general by considering the ratio of two successive terms,

$$\frac{{}^{n}C_{j+1}}{{}^{n}C_{j}}=\frac{n!}{(j+1)!(n-j-1)!}\cdot\frac{j!(n-j)!}{n!}=\frac{n-j}{j+1}$$

The numbers increase as long as the ratio is greater than 1, i.e,. $n-j>j+1$.

This means that $j<\frac{1}{2}(n-1)$. We must distinguish the case of an even n from an odd n.

For example, if n = 10, j must be less than $\frac{1}{2}(10-1)=4.5$. Hence for j up to 4 the terms are increasing, from j = 5 on, the terms decrease for n = 11, j must be less than $\frac{1}{2}(11-1)=5$.

For $j=5$, $(n-j)i(j+1)=1$. Hence, up to $j=5$ the terms increase, then ${}^{11}C_5={}^{11}C_6$, and then the terms decrease.

Practice Problems [A]

1. Evaluate the following: (i) $\frac{7!}{3!\,4!}$ (ii) ${}^{8}C_3$

   Solve the following equations :

2. ${}^{x}C_3={}^{x+2}C_4$
3. ${}^{x-3}C_2=21$
4. ${}^{x}C_3=\frac{5x(x-3)}{4}$
5. ${}^{x}C_3+{}^{x}C_2=15(x-1)$;
6. ${}^{2x+3}C_{2(x-1)}=16x(x+1)(2x+1)$
7. ${}^{x+8}C_{x+3}=5(x+6)(x+5)(x+4)$;

   Solve the given systems of equations :

8. $\begin{cases}{}^{x}C_{y+1}=20\\ {}^{x-1}C_{y}=10;\end{cases}$
9. $\begin{cases}{}^{x}C_{y}={}^{x}C_{y+2}\\ {}^{x}C_{2}=153;\end{cases}$
10. $\begin{cases}{}^{x}C_{y}={}^{x}C_{y+2}\\ {}^{x-1}C_{y}=10;\end{cases}$

5.2 BINOMIAL THEOREM

Binomial Theorem for Positive Integral Index

If n is a positive integer and x, $a\in C$ then

$(x+a)^n={}^{n}C_0x^{n-0}a^0+{}^{n}C_1x^{n-1}a^1+{}^{n}C_2x^{n-2}a^2+.....$
$+{}^{n}C_rx^{n-r}a^r+....+.....+{}^{n}C_{n-1}xa^{n-1}+C_nx^0a^n$

Proof: $(x+a)^n=(x+a)(x+a)(x+a)....(x+a)$, the number of factors being n.

Every term in the expansion is formed by taking one letter from each factor and multiplying them together. Choosing x from all the brackets we get the term x^n. Choosing x from (n – 1) factors and 'a' from the remaining factor we get $x^{n-1}a$. But the number of ways of doing this is equal to the number of ways of choosing one factor from n factors (i.e.,)${}^{n}C_1$. There are therefore ${}^{n}C_1$ terms, each equal to $x^{n-1}a$. Choosing x from (n – 2) factors and a from the remaining two factors, we get $x^{n-2}a^2$. But the number of ways of doing this is equal to the number of ways of choosing two factors from n factors. (i.e.,) ${}^{n}C_2$. Therefore there are ${}^{n}C_2$ terms each equal to $x^{n-2}a^2$.

www.pdfworld.net

www.pdfworld.net

Similarly there are ${}^nC_r$ terms each equal to $x^{n-r}a^r$. Finally choosing 'a' from all the factors we get the term a^n.

$\therefore (x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots\ldots\ldots$
$\ldots {}^nC_rx^{n-r}a^r + \ldots\ldots + a^n$

The theorem can also be proved by mathematical induction.

Let P(n) stand for the statement

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots\ldots\ldots {}^nC_rx^{n-r}a^r + \ldots\ldots + a^n \quad \ldots(1)$$

Note that there are (n + 1) terms in R.H.S. and all the terms are of the same degree in x and a together.

When n = 1, L.H.S. = x + a and R.H.S. = x + a (there are only 2 terms)

$\therefore$ P(1) is verified to be true.

Assume P(m) to be true

i.e. $(x+a)^m = x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a^2 + \ldots\ldots\ldots {}^mC_rx^{m-r}a^r + \ldots\ldots + a^m \quad \ldots(2)$

Multiplying equation (2) by (x + a), we have

$(x+a)^m(x+a) = (x+a)\{x^m + {}^mC_1x^{m-1}a + {}^mC_2x^{m-2}a^2 + \ldots\ldots\ldots {}^mC_rx^{m-r}a^r + \ldots\ldots + a^m\}$

$(x+a)^{m+1} = x^{m+1} + ({}^mC_1 + 1)x^ma + ({}^mC_2 + {}^mC_1)x^{m-1}a^2 + \ldots\ldots ({}^mC_r + {}^mC_{r-1})x^{m-r+1}a^r + \ldots\ldots + a^{m+1}$

$= x^{m+1} + {}^{m+1}C_1x^ma + {}^{m+2}C_2x^{m-1}a^2 + \ldots\ldots + {}^{m+1}C_rx^{m+1-r}a^r + \ldots . + a^{m+1} \quad \ldots(3)$

using the formula ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$

Equation (3) implies that P(m+1) is true and hence by induction P(n) is true.

Our observations about the expansion of $(x+a)^n$ are summarized as follows .

1. There are n + 1 terms in the expansion.
2. The first term is x^n, and the last term is a^n.
3. The exponent on x decreases by 1 and the exponent on a increases by 1 in each succeeding term.
4. The sum of the exponents on x and a in any term is n. For example, in the expansion of $(x+a)^5$, the sum of the powers of each term is 5, as follows :

$$(x+a)^5 = x^5 + \overbrace{5x^4a^1}^{4+1=5} + \overbrace{10x^3a^2}^{3+2=5} + 10x^2a^3 + 5xa^4 + a^5$$

5. The coefficient of x^ra^{n-1} is ${}^nC_r$. Here ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ..., ${}^nC_n$ are called binomial coefficients.

When it is understood that the index is n then the binomial coefficients can be written as

$$C_0, C_1, C_2, \ldots\ldots, C_n$$

The binomial theorem looks much more manageable written in summation notation. The theorem can be summarized as follows :

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}a^r \quad \ldots(1)$$

Any expansion of the difference of two terms has alternating signs.

Special Cases

(i) Replacing a by – a in (1), we get

$(x-a)^n = {}^nC_0 \cdot x^{n-0} \cdot a^0 - {}^nC_1 \cdot x^{n-1} \cdot a^1 + {}^nC_2 \cdot x^{n-2} \cdot a^2 + \ldots . + {}^nC_r(-1)^r \cdot x^{n-r} \cdot a^r + (-1)^n {}^nC_n x^0 \cdot a^n$

$$(x-a)^n = \sum_{r=0}^{n} (-1)^r {}^nC_r x^{n-r}a^r \quad \ldots(2)$$

(ii) Adding (1) and (2), we get

$(x+a)^n + (x-a)^n = 2\{x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots .\}$

$= 2$ {sum of terms at odd places}

The last term is ${}^nC_na^n$ or ${}^nC_{n-1}xa^{n-1}$ according as n is even or odd respectively.

(iii) Subtracting (2) from (1), we get

$(x+a)^n - (x-a)^n$

$= 2\{{}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + \ldots .\}$

$= 2$ {Sum of terms at even places}

The last term is ${}^nC_{n-1}xa^{n-1}$ or ${}^nC_na^n$ according as n is even or odd respectively.

(iv) Replacing x by 1 and a by x in (1), we get

$(1+x)^n = {}^nC_0x^0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots . + {}^nC_rx^r + \ldots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n$

i.e. $(1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r$

www.pdfworld.net

www.pdfworld.net

(v) Replacing x by 1 and a by –x in (1), we get

$(1-x)^n = {}^nC_0x^0 - {}^nC_1x + {}^nC_2x^2 - \ldots\ldots\ldots + {}^nC_r(-1)^rx^r + \ldots + {}^nC_{n-1}(-1)^{n-1}x^{n-1} + {}^nC_n(-1)^nx^n$

$$(1-x)^n = \sum_{r=0}^{n}(-1)^r\ {}^nC_r x^r$$

➤ **Example 1.** Expand and simplify the expansion $(2x^3 - \sqrt{5}\,x^{-1})^4$.

➤ ***Solution*** Using binomial theorem for $(x+a)^n$ with $n = 4$; x as $2x^3$ and a as $-\sqrt{5}\,x^{-1}$

$(2x^3 - \sqrt{5}\,x^{-1})^4$

$= [2x^3 + (-\sqrt{5}\,x^{-1})]^4$

$= (2x^3)^4 + 4.(2x^3)^3(-\sqrt{5}\,x^{-1}) + 6(2x^3)^2(-\sqrt{5}\,x^{-1})^2 + 4(2x^3)(-\sqrt{5}\,x^{-1})^3 + (-\sqrt{5}\,x^{-1})^4$

$= 16x^{12} - 32\sqrt{5}\,x^8 + 120x^4 - 40\sqrt{5} + 25x^{-4}.$

➤ **Example 2.** Expand the binomial $\left(\frac{2x}{3} + \frac{3y}{2}\right)^{20}$ up to four terms.

➤ ***Solution***

$$\left(\frac{2x}{3}+\frac{3y}{2}\right)^{20} = {}^{20}C_0\left(\frac{2x}{3}\right)^{20} + {}^{20}C_1\left(\frac{2x}{3}\right)^{19}\left(\frac{3y}{2}\right) + {}^{20}C_2\left(\frac{2x}{3}\right)^{18}\left(\frac{3y}{2}\right)^2 + {}^{20}C_3\left(\frac{2x}{3}\right)^{17}\left(\frac{3y}{2}\right)^3 + \ldots.$$

$$= \left(\frac{2x}{3}\right)^{20} + 20.\left(\frac{2}{3}\right)^{18}x^{19}y + 190.\left(\frac{2}{3}\right)^{16}x^{18}y^{12} + 1140\left(\frac{2}{3}\right)^{14}x^{17}y^3 + \ldots.$$

➤ **Example 3.** Using binomial theorem find the exact value of $(10.1)^5$.

➤ ***Solution*** $(10.1)^5 = (10 + 0.1)^5$

$= 10^5 + {}^5C_1\,10^4(.1) + {}^5C_2\,10^3(.1)^2 + {}^5C_3\,10^2(.1)^3 + {}^5C_4 10(.1)^4 + {}^5C_5(.1)^5$

$= 100000 + 5\times10^4(.1) + 10\times10^3(.01) + 10\times10^2(.001) + 5\times10(.0001) + 0.00001$

$= 100000 + 50000 + 100 + 1 + 0.005 + 0.00001$

$= 105101.00501$

➤ **Example 4.** Simplify : $(x + \sqrt{(x^2-1)}\,)^6 + (x - \sqrt{(x^2-1)}\,)^6$

➤ ***Solution*** $(x+y)^n + (x-y)^n = 2\{x^n + {}^nC_2x^{n-2}y^2 + {}^nC_4x^{n-4}y^4 + {}^nC_6x^{n-6}y^6 + \ldots\}$

Putting $n = 6$ and $y = \sqrt{(x^2-1)}$, then

$(x + \sqrt{(x^2-1)}\,)^6 + (x - \sqrt{(x^2-1)}\,)^6$

$= 2\{x^6 + {}^6C_2x^4(x^2-1) + {}^6C_4x^2(x^2-1)^2 + {}^6C_6(x^2-1)^3\}$

$= 2(32x^6 - 48x^4 + 18x^2 - 1)$

Symmetric Expansions

1. $(1+x)^n = C_0 + C_1x + C_2x^n + \ldots\ldots + C_nx^n$ ...(1)

$$\Rightarrow x^n\left(\frac{1}{x}+1\right)^n = x^n\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots. + \frac{C_n}{x^n}\right)$$

$= x^nC_0 + C_1 . x^{n-1} + \ldots\ldots + C_{n-1}x + C_n$ ...(2)

Since $C_0 = C_n$, $C_1 = C_{n-1}$, ... , $C_r = C_{n-r}$, the coefficients equidistant from the beginning and the end are equal. Thus, we see that the expansion (1) is symmetric.

2. Let $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_{2n}x^{2n}$ ...(3)

$$\Rightarrow x^{2n}\left(\frac{1}{x^2}+\frac{1}{x}+1\right)^n = x^{2n}\left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots. + \frac{a_{2n}}{x^{2n}}\right)^n$$

$= a_0x^{2n} + a_1x^{2n-1} + \ldots. + a_{2n}$

$\therefore\ a_0 = a_{2n}$, $a_1 = a_{2n-1}$, ... , $a_r = a_{2n-r}$ ...(4)

3. $(3+4x)^n = q_0 + q_1x + q_2x^2 + \ldots\ldots + q_nx^n$

$$\Rightarrow x^n\left(\frac{3}{x}+4\right)^n \neq x^n\left(q_0 + \frac{q_1}{x} + \frac{q_2}{x^2} + \ldots. + \frac{q_n}{x^n}\right)$$

Thus this is not a symmetric expansion.

www.pdfworld.net

www.pdfworld.net

➤ **Example 5.** Write the expansion of $(x+1)^7$.

➤ ***Solution*** $(x+1)^7 = \sum_{k=0}^{7} {}^7C_k\, x^{7-k}$

$= {}^7C_0x^7 + {}^7C_1x^6 + {}^7C_2x^5 + {}^7C_3x^4 + {}^7C_4x^3$
$+ {}^7C_5x^2 + {}^7C_6x + {}^7C_7$

$= x^7 + 7x^6 + \frac{7.6}{2}x^5 + \frac{7.6.5}{2.3}x^4 + \frac{7.6.5}{2.3}x^3 + \frac{7.6}{2}x^2$
$+ 7x + 1$

$= x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1.$

Practice Problems [B]

1. How many terms are there in the expansion of $(x+y)^8$?
2. Use the binomial theorem to expand each binomial and express the result in simplified form.
   (i) $(x^2+2y)^4$ (ii) $(a-2)^5$
   (iii) $(2a+b)^6$
3. Write the first three terms in each binomial expansion, expressing the result in simplified form.
   (i) $(a^2-b^2)^{16}$
   (ii) $(a+3b)^9$
   (iii) $\left(1+\frac{1}{y^2}\right)^7 y^7$
4. Expand the binomial $\left(\frac{2}{3}x - \frac{3}{2x}\right)^6$
5. Simplify the expressions using binomial theorem : (a) $(\sqrt{2} - \sqrt{3})^5$; (b) $(1+i\sqrt{3})^6$
6. Expand $(x^2+x+1)^4$ by writing x^2+x+1 as $x^2+(x+1)$.
7. Find the expansion of $(1+2x-x^2)^4$.
8. Solve for y : $\sum_{i=0}^{5}\binom{5}{i}(-1)^i y^{5-i}3^i = 32$

5.3 GENERAL TERM OF A BINOMIAL EXPANSION

Any single term of a binomial expansion can be determined without writing out the whole expansion. For example, the seventh term of $(x+a)^9$ has a raised to the sixth power (since a has the power 1 in the second term, the power 2 in the third term and so on). The exponents on x and a in each term must have a sum of 9, so the exponent on x in the seventh term is $9-6=3$. Thus, writing the coefficient as given in the binomial theorem, the seventh term should be

$$\frac{9!}{3!\,6!}x^3a^6$$

The general term or the $(r+1)^{th}$ term in the expansion of $(x+a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}.\, a^r$$
$$= \frac{n(n-1)....(n-r+1)}{r!}x^{n-r}a^r$$

➤ **Example 1.** Find (i) 28th term of $(5x+8y)^{30}$

(ii) 7th term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

➤ ***Solution*** (i) $T_{27+1} = {}^{30}C_{27}\,(5x)^{30-27}(8y)^{27}$

$$= \frac{30!}{3!27!}(5x)^3 .\,(8y)^{27}.$$

(ii) $T_{6+1} = {}^9C_6\left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6$

$$= \frac{9!}{3!\,6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3}.$$

➤ **Example 2.** If in the expansion of $(1+x)^{43}$, the coefficient of $(2r+1)$th term is equal to the coefficient of $(r+2)$th them find r.

➤ ***Solution*** Since in the expansion of $(1+x)^{43}$, coefficient of $(2r+1)$th term = coeff. of $(r+2)$th term

$\therefore {}^{43}C_{2r} = {}^{43}C_{r+1}$

$\therefore 2r+r+1 = 43$ or $r = 14$

www.pdfworld.net

www.pdfworld.net

STUDY TIP

p^{th} term from the end

The pth term from the end in the expansion of $(x + a)^n$ is same as the $(n + 1 - p + 1)$ i.e. $(n - p + 2)$th term from the beginning of the expansion.

➤ **Example 3.** Find the fourth term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^7$

➤ ***Solution*** The fourth term from the end
$= (7 - 4 + 2)$th term from the beginning in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^7$

$$= T_5 = {}^7C_4 \left(\frac{x^3}{2}\right)^{7-4}\left(-\frac{2}{x^2}\right)^4$$

$$= {}^7C_3 \left(\frac{x^3}{2}\right)^3\left(-\frac{2}{x^2}\right)^4$$

$$= \frac{7.6.5}{1.2.3}\cdot\frac{x^9}{8}\cdot\frac{16}{x^8}$$

➤ **Example 4.** Find the $(n + 1)$th term from the end in the expansion of $\left(2x - \frac{1}{x}\right)^{3n}$

➤ ***Solution*** The $(n + 1)$th term from the end
$= \{3n - (n + 1) + 2\}$th term from the beginning in the expansion of $\left(2x - \frac{1}{x}\right)^{3n}$

$$= T_{2n+1} = {}^{3n}C_{2n}(2x)^{3n-2n}\left(-\frac{1}{x}\right)^{2n}$$

$$= \frac{3n!}{2n!n!}\cdot 2^n \cdot x^n (-1)^{2n} \cdot \frac{1}{x^{2n}} = \frac{3n!}{2n!n!} 2^n \cdot x^{-n}$$

➤ **Example 5.** Find the seventh term of the expansion of $(y^{1/2} + x^{1/3})^n$ if the binomial coefficient of the third term from the end is equal to 45.

➤ ***Solution*** Let us first find the power of the binomial. According to the hypothesis the number n must satisfy the equation ${}^nC_{n-2} = {}^nC_2 = \frac{n(n-1)}{2} = 45$, whose roots are $n_1 = 10$, $n_2 = -9$. Since $n_2 = -9$ is not a natural number, the root is $n = 10$ and consequently, the sixth term of the expansion can be represented as

$$T_{6+1} = {}^{10}C_6 (y^{1/2})^4 (x^{1/2})^6 = {}^{10}C_4 y^2 x^3.$$
$$= 210\, y^2 x^3.$$

➤ **Example 6.** Find the coefficient of x^r in the expansion of $\left(x^2 + \frac{1}{x^3}\right)^n$.

➤ ***Solution*** Suppose that x^r occurs in the $(p + 1)^{th}$ term.

The $(p + 1)^{th}$ term $= {}^nC_P (x^2)^{n-p}\left(\frac{1}{x^3}\right)^p = {}^nC_P x^{2n-5p}$.

But this term contains x^r, and therefore $2n - 5p = r$,

or $p = \frac{2n - r}{5}$

Thus the required coefficient $= {}^nC_P = \frac{{}^nC_{2n-r}}{5}$

$$= \frac{n!}{\left(\frac{1}{5}(2n-r)\right)!\left(\frac{1}{5}(3n+r)\right)!}$$

Unless $\frac{2n - r}{5}$ is a positive integer there will be no term containing x^r in the expansion.

➤ **Example 7.** Find the term in the expansion of $\left(x + \frac{1}{x^4}\right)^{10}$ which does not contain x.

➤ ***Solution*** $T_{k+1} = {}^{10}C_k x^{10-k}\left(\frac{1}{x^4}\right)^k$

By the hypothesis, the number k must satisfy the equation $10 - k - 4k = 0$
The only root of equation is $k = 2$. Thus, the third term of the expansion

$$T_3 = {}^{10}C_2 x^8 \frac{1}{x^8} = 45$$ is the required term.

➤ **Example 8.** Find the coefficient of x^5 in the expansion of $(1 + x + x^3)^9$

➤ ***Solution*** $(1 + x + x^3)^9 = [(1 + x) + x^3]^9$
$= (1 + x)^9 + {}^9C_1(1 + x)^8 x^3 + {}^9C_2(1 + x)^7 x^6 + \ldots.$
The coefficient of x^5 in $(1 + x)^9$ is ${}^9C_5$ (i.e.,) ${}^9C_4$

www.pdfworld.net

www.pdfworld.net

The coefficient of x^5 in ${}^9C_1(1+x)^8x^3$ = coefficient of x^2 in ${}^9C_1\,(1+x)^8 = 9.{}^8C_2$

The remaining terms contain powers of x higher than the fifth.

$\therefore$ The required coefficient is ${}^9C_4 + 9 \times {}^8C_2$

$$= \frac{9.8.7.6}{1.2.3.4} + \frac{9.8.7}{1.2} = 378$$

Alternative

$[1 + x(1+x^2)]^9 = 1 + {}^9C_1x(1+x^2) + {}^9C_2x^2(1+x^2)^2 + {}^9C_3x^3(1+x^2)^3 + {}^9C_4x^4(1+x^2)^4 + ...$

x^5 occurs in 4th and 6th terms only and it is equal to $3 \times {}^9C_3 + {}^9C_5 = 252 + 126 = 378$

➤ **Example 9.** Find coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$

➤ ***Solution*** $(1 + x + x^2 + x^3)^{11} = (1+x)^{11}\,(1+x^2)^{11}$

$= (1 + {}^{11}C_1x + {}^{11}C_2x^2 + ...)$

$(1 + {}^{11}C_1x^2 + {}^{11}C_2(x^2)^2 + ...)$

So coefficient of x^4 in the above expansion

$= {}^{11}C_2 \times {}^{11}C_2 + {}^{11}C_4 + {}^{11}C_2$

$= 605 + 330 + 55 = 990$

➤ **Example 10.** For what value of x is the fourth term in the expansion of $\left\{\left(\sqrt{x}\right)^{\frac{1}{\log x+1}} + \sqrt[12]{x}\right\}^6$ equal to 200.

➤ ***Solution*** The fourth term of the expansion is

$$= {}^6C_3\sqrt{x}^{\frac{3}{\log x+1}}\left(x^{1/12}\right)^3 = 20x^{\frac{1}{4}+\frac{3}{2(1+\log x)}} = 200$$

$$\therefore\ x^{\frac{1}{4}+\frac{3}{2(1+\log x)}} = 10$$

Taking logarithm on both sides

$$\left\{\frac{1}{4} + \frac{3}{2(1+\log x)}\right\}\log x = 1$$

i.e. $\{(1 + \log x) + 6\}\log x = 4(1 + \log x)$

i.e. $(\log x)^2 + 3\log x - 4 = 0 \Rightarrow (\log x + 4)(\log x - 1) = 0$

Either $\log_{10}x = -4$ or $\log_{10}x = 1$

$\Rightarrow$ $x = 10^{-4}$ or 10

➤ **Example 11.** In the expansion of $(x+a)^n$ if the sum of odd terms be P and sum of even terms be Q. Prove that

(i) $P^2 - Q^2 = (x^2 - a^2)^n$

(ii) $4PQ = (x+a)^{2n} - (x-a)^{2n}$

➤ ***Solution*** $(x + a)^n = x^n + {}^nC_1\,x^{n-1}\,a + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + ...+{}^nC_na^n$

$= (x^n + {}^nC_2\,x^{n-2}\,a^2 + ...) + ({}^nC_1\,x^{n-1}\,a + {}^nC_3\,x^{n-3}\,a^3 + ...)$

$= P + Q$ ...(1)

and $(x - a)^n = x^n - {}^nC_1\,x^{n-1}a + {}^nC_2\,x^{n-2}\,a^2 - {}^nC_3\,x^{n-3}\,a^3 + ... + (-1)^n\,{}^nC_n\,a^n$

$= (x^n + {}^nC_2\,x^{n-2}\,a^2 + ...)$

$-\,({}^nC_1\,x^{n-1}\,a + {}^nC_3\,x^{n-3}\,a^3 + ...) = P - Q$ ...(2)

(i) $P^2 - Q^2 = (P+Q)(P-Q)$

$= (x+a)^n\,(x-a)^n$ [From (1) and (2)]

$= (x^2 - a^2)$

(ii) $4PQ = (P+Q)^2 - (P-Q)^2 = (x+a)^{2n} - (x-a)^{2n}$ [From (1) and (2)]

➤ **Example 12.** The sum of coefficients in the first three terms of the expansion of $(x^2 - 2/x)^m$ is 97. Show that the coefficient of x^4 is 1120.

➤ ***Solution*** We have $(x^2 - 2/x)^m = (x^2)^m + {}^mC_1\,(x^2)^{m-1}\,(-2/x) + {}^mC_2(x^2)^{m-2}\,(-2/x)^2 + ...$

Now, the sum of coefficients in the first three terms $= 1 - {}^mC_1\,.\,2 + {}^mC_2\,.\,4$

Therefore $1 - 2m + 2m(m-1) = 97$

$\Rightarrow m^2 - 2m - 48 = 0 \quad \Rightarrow m = 8, -6$

The value -6 is not admissible since m must be a positive integer.

Thus the given binomial is $(x^2 - 2/x)^8$ in which the general term

$$T_{r+1} = {}^8C_r(x^2)^{8-r}(-2/x)^r = {}^8C_r\,(-2)^r x^{16-3r}$$

The value of r corresponding to the term containing x^4 is given by $16 - 3r = 4$

$\Rightarrow r = 4$. Thus the coeff. of x^4 is ${}^8C_r(-2)^4 = 70 \times 16 = 1120$.

➤ **Example 13.** Find the coefficient of x^{-1} in $(1 + 3x^3 + x^4)\left(1 + \frac{1}{x}\right)^8$.

➤ ***Solution*** coefficient of x^{-1} in

$$(1 + 3x^3 + x^4)\left(1 + \frac{1}{x}\right)^8$$

$$= \text{coefficient of } x^{-1} \text{ in } \left(1 + \frac{1}{x}\right)^8 + \text{coefficient of } x^{-1}$$

$$\text{in } \ 3x^2\left(1 + \frac{1}{x}\right)^8 + \text{coefficient of } x^{-1} \text{ in } x^4\left(1 + \frac{1}{x}\right)^8$$

$$= \text{coefficient of } x^{-1} \text{ in} \left(1 + \frac{1}{x}\right)^8 + 3 \text{ coefficient of } x^{-3}$$

www.pdfworld.net

www.pdfworld.net

in $\left(1+\frac{1}{x}\right)^8$ + coefficient of x^{-5} in $\left(1+\frac{1}{x}\right)^8$ ...(1)

In the expansion of $\left(1+\frac{1}{x}\right)^8$, rth term is given by

$$t_r = {}^nC_{r-1}\left(\frac{1}{x}\right)^{r-1} = {}^nC_{r-1}x^{-(r-1)}$$

Putting $r-1=1$ i.e. $r=2$, we get coefficient of x^{-1} in $\left(1+\frac{1}{x}\right)^8 = {}^8C_1$

Putting $r-1=3$ i.e. $r=1$, we get coefficient of x^{-3} in $\left(1+\frac{1}{x}\right)^8 = {}^8C_3$

Putting $r-1=5$ i.e. $r=6$, we get coefficient of x^{-5} in $\left(1+\frac{1}{x}\right)^8 = {}^8C_5$

$\therefore$ From (1), required coefficient of x^{-1}

$${}^8C_1 + 3\,.\,{}^8C_3 + {}^8C_5 = 8 + 3\times 56 + 56 = 232.$$

➤ **Example 14.** Find the sum of the series

$$\sum_{r=0}^{n}(-1)^r\,.\,{}^nC_r\left\{\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\text{...upto m terms}\right\}$$

➤ ***Solution*** We have

$$\sum_{r=0}^{n}(-1)^r\left\{\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\text{....upto m terms}\right\}$$

$$= \sum_{r=0}^{n}(-1)^r\,.\,{}^nC_r\left\{\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\left(\frac{15}{16}\right)^r+\text{....+ to m terms}\right\}$$

$$= \sum_{r=0}^{n}(-1)^r\,.\,{}^nC_r\left(\frac{1}{2}\right)^r + \sum_{r=0}^{n}(-1)^r\,.\,{}^nC_r\left(\frac{3}{4}\right)^r + \sum_{r=0}^{n}(-1)^r\,.\,{}^nC_r\left(\frac{7}{8}\right)^r + \sum_{r=0}^{n}(-1)^r\,.\,{}^nC_r\left(\frac{15}{16}\right)^r + \text{.... to m terms}$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \text{.... to m terms}$$

$$= \frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{8^n}+\cdots\cdots \text{ to m terms}$$

$$= \frac{\frac{1}{2^n}\left(1-\left(\frac{1}{2^n}\right)^m\right)}{\left(1-\frac{1}{2^n}\right)} = \frac{(2^{mn}-1)}{2^{mn}(2^n-1)}$$

Practice Problems [C]

1. Find the term indicated in each expansion

(i) $(2a+b)^6$; third term

(ii) $\left(1-\frac{a^2}{2}\right)^{14}$; eighth term

(iii) $(4a-b)^{10}$; term containing a^2 as a factor

2. Find (i) the seventh term of the expansion $(2x-3)^{10}$

(ii) the fourth term of the expansion $\left(\sqrt[3]{a}+\frac{1}{\sqrt[3]{a}}\right)^9$

3. Find the term indicated in each expansion

(i) $\left(\frac{a}{3}+9b\right)^{10}$; fourth term

(ii) $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}$; 13th term

(iii) $\left(\sqrt{x}+\sqrt{2}\right)^{15}$; term containing x^5.

4. Find the coefficient of

(i) x^{18} in $\left(x^2+\frac{3a}{x}\right)^{15}$

(ii) x^7 in $\left(3x^2+\frac{1}{5x}\right)^{11}$

(iii) x^{-7} in $\left(2x-\frac{1}{3x^2}\right)^{11}$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

5. Find the value of n for which the coefficients of the fifth and eighth terms in the expansion of $(x+y)^n$ are the same.

6. Find the term independent of x in the expansion

of (i) $\left(x^3-\frac{3}{x^2}\right)^{15}$ (ii) $\left(\frac{1}{2}x^{\frac{1}{3}}+x^{-\frac{1}{5}}\right)^8$

7. Find the term independent of x in

(i) $\left(x-\frac{1}{x^2}\right)^{3n}$ (ii) $\left(\frac{m}{\sqrt[3]{m}}+m^{-\frac{2}{7}}\right)^{10}$

8. Find the $(p+2)^{th}$ term from the end in $\left(x-\frac{1}{x}\right)^{2n+1}$

9. Find the exponent of the binomial if in the expansion of $\left(9x-\frac{x^{-\frac{1}{2}}}{\sqrt{3}}\right)^n$ the binomial coefficient of the third term is equal to 105.

10. Find the coefficient of x^4 in the expansion of (i) $(1+x+x^2+x^3)^{11}$ (ii) $(2-x+3x^2)^6$

11. The fourth term in the expansion of $\left(2x+\frac{1}{x^2}\right)^m$ does not contain x. For which x is that term equal to the third term in the expansion of $(1+x^3)^{30}$?

5.4 MIDDLE TERM(S) AND RATIONAL TERMS

Middle term(s)

The middle term(s) is the expansion of $(x+a)^n$ is (are) :

(a) If n is even, there is only one middle term since the number of terms is odd.

It is given by $T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot a^{n/2}$

(b) If n is odd, there are two middle terms since the number of terms is even.

They are given by $T_{(n+1)/2}$ and $T_{[(n+1)/2]+1}$

For example, in the expansion of $(x+2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5th term.

In the expansion $(2x-y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$, ie. 4th and $\left(\frac{7+1}{2}\right)^{th}$, i.e. 5th term

➤ **Example 1.** Find the middle term(s) in the expansion of (i) $\left(1-\frac{x^2}{2}\right)^{14}$ (ii) $\left(3a-\frac{a^3}{6}\right)^9$

➤ ***Solution*** (i) $\left(1-\frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)$th term which means T_8 is the middle term.

$$T_8 = {}^{14}C_7\left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}x^{14}.$$

(ii) $\left(3a-\frac{a^3}{6}\right)^9$ Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+1}{2}+1\right)$th terms.

It means T_5 and T_6 are the middle terms

$$T_5 = {}^9C_4(3a)^{9-4}\left(-\frac{a^3}{6}\right) = \frac{189}{8}a^{17}$$

$$T_6 = {}^9C_5(3a)^{9-5}\left(-\frac{a^3}{6}\right) = \frac{21}{16}a^{19}$$

Rational terms

To find the rational terms i.e. terms free from radicals in the expansion of $(a^{1/p}+b^{1/p})^N$ where a, b are prime numbers first find the general term

www.pdfworld.net

www.pdfworld.net

$$T_{r+1} = {}^{N}C_r \left(a^{1/p}\right)^{N-r} \left(b^{1/q}\right)^{r}$$

$$\therefore \quad T_{r+1} = {}^{N}C_r a^{(N-r)/p} \cdot b^{r/q}$$

By inspection choose the values of r ($0 \le r \le N$) so that the indices of a and b are integers.

➤ **Example 2.** Find the number of rational terms terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

➤ ***Solution*** The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r \left(9^{\frac{1}{4}}\right)^{1000-r} \left(8^{\frac{1}{6}}\right)^{r}$$

$$= {}^{1000}C_r \, 3^{\frac{1000-r}{2}} 2^{\frac{r}{2}}$$

The above term will be rational if exponent of 3 and 2 are integers.

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4,...., 1000)
Hence, number of rational terms is 501.

➤ **Example 3.** In the binomial expansion of $\left(\sqrt[3]{3} + \sqrt{2}\right)^5$, find the terms which do not contain irrational expressions.

➤ ***Solution*** A general term in the expansion is

$$T_k = {}^{5}C_k \left(\sqrt[3]{3}\right)^{5-k} (\sqrt{2})^{k}$$

$$= {}^{5}C_k . 3^{\frac{5-k}{3}} 2^{\frac{k}{2}}$$

The resulting expression is rational if $\frac{5-k}{3}$ and $\frac{k}{2}$ are integers. The number k must evidently, be an even number smaller than 5. We make sure by direct verification that the only value it can assume is 2. Consequently, there is only one such term in the binomial expansion, that is

$$T_3 = {}^{5}C_2 . 3.2 = 5.\frac{5.4}{2} = 60$$

Practice Problems [D]

1. Find (i) the middle term in the expansion of $\left(\frac{1}{\sqrt[5]{x}} + \sqrt[3]{x}\right)^{10}$
   (ii) the middle terms in the expansion of $\left(\sqrt{a} - \sqrt[3]{b}\right)^{13}$
2. Find the middle term of $(3x^7 + 2y^3)^8$.
3. Find the two middle terms of $\left(3a - \frac{a^3}{6}\right)^9$.
4. Without expanding the binomial $\left(\sqrt[5]{x} + \sqrt[10]{y}\right)^{55}$, find the number of the terms in its expansion which are free of radicals.
5. Find the terms which do not contain irrational expressions in the expansion of $\left(\sqrt[5]{3} + \sqrt[7]{2}\right)^{24}$
6. How many rational terms are there in the expansion of $\left(\sqrt{2} + \sqrt[4]{3}\right)^{100}$?

5.5 RATIO OF TWO CONSECUTIVE TERMS

The ratio of two consecutive terms in the expansion of $(x + a)^n$ is

$$\frac{T_{r+1}}{T_r} = \frac{{}^{n}C_r x^{n-r} a^{r}}{{}^{n}C_{r-1} x^{n-r+1} a^{r-1}} = \frac{(n-r+1)}{r}\frac{a}{x}$$

Note that $\frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}$

➤ **Example 1.** If three consecutive coefficients in the expansion of $(1 + x)^n$ be 165, 330 and 462, then find the number of terms in expansion of $(1 + x)^n$.

➤ ***Solution*** Let the three consecutive coefficients in the expansion of $(1 + x)^n$ be r^{th}, $(r + 1)^{th}$ & $(r + 2)^{th}$ respectively.

$$\therefore \quad {}^{n}C_{r-1} = 165 \qquad ...(1)$$

$$ {}^{n}C_r = 330 \qquad ...(2)$$

www.pdfworld.net

www.pdfworld.net

$${}^{n}C_{r+1} = 462 \qquad ...(3)$$

Dividing (1) by (2) $\frac{r}{n-r+1} = \frac{1}{2} \Rightarrow 3r = n+1$ ...(4)

Dividing (2) by (3) $\frac{r+1}{n-r} = \frac{5}{7} \Rightarrow 12r = 5n-7$ ...(5)

From (4) & (5), we get n = 11

$\therefore$ number of terms in $(1+x)^n = 11 + 1 = 12$

➤ **Example 2.** If the coefficient of the second, third and fourth terms in the expansion of $(1+x)^n$ are in A.P. show that n = 7.

➤ ***Solution*** According to the question ${}^{n}C_1$, ${}^{n}C_2$, ${}^{n}C_3$ are in A.P.

$$\frac{2n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$$

$n^2 - 9n + 14 = 0 \Rightarrow (n-2)(n-7) = 0 \Rightarrow n = 2$ or 7.

Since the symbol ${}^{n}C_3$ demands that n should be ≥ 3 n cannot be 2, $\therefore$ n =7 only.

➤ **Example 3.** Find the ratio of the product of co-efficients in the expansion of the binomial $(1+x)^{n+1}$ to the product of the binomial co-efficients in $(1+x)^n$ in simplest form, $n \in N$.

➤ ***Solution*** ratio

$$= \frac{{}^{n+1}C_0 \cdot {}^{n+1}C_1 \cdot {}^{n+1}C_2 \ldots\ldots {}^{n+1}C_r \ldots\ldots {}^{n+1}C_n \cdot {}^{n+1}C_{n+1}}{{}^{n}C_0 \cdot {}^{n}C_1 \cdot {}^{n}C_2 \ldots\ldots {}^{n}C_r \ldots\ldots {}^{n}C_{n-1} \cdot {}^{n}C_n}$$

$$= \frac{{}^{n+1}C_1}{{}^{n}C_0} \cdot \frac{{}^{n+1}C_2}{{}^{n}C_1} \cdot \ldots\ldots \cdot \frac{{}^{n+1}C_r}{{}^{n}C_r} \cdot \ldots\ldots \cdot \frac{{}^{n+1}C_{n+1}}{{}^{n}C_n}$$

$$= \prod_{r=1}^{n+1} \frac{{}^{n+1}C_r}{{}^{n}C_r} = \prod_{r=1}^{n+1} \frac{n+1}{r} = \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)^n}{n!}$$

➤ **Example 4.** The coefficients of three consecutive terms in the expansion of $(1+a)^n$ are in the ratio 1 : 7 : 2. Find n.

➤ ***Solution*** Suppose the three consecutive terms in the expansion of $(1+a)^n$ are $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms.

The $(r-1)^{th}$ term is ${}^{n}C_{r-2}a^{r-2}$ and its coefficient is ${}^{n}C_{r-2}$. Similarly, the coefficients of r^{th} and $(r+1)^{th}$ terms are ${}^{n}C_{r-1}$ and ${}^{n}C_r$, respectively.

Since the coefficients are in the ratio 1 : 7 : 42, so we have,

$$\frac{{}^{n}C_{r-2}}{{}^{n}C_{r-1}} = \frac{1}{7}, \text{ i.e., } n - 8r + 9 = 0 \qquad ...(1)$$

and $$\frac{{}^{n}C_{r-1}}{{}^{n}C_r} = \frac{7}{42}, \text{ i.e., } n\ 7r + 1 = 0 \qquad ...(2)$$

Solving equations (1) and (2), we get n = 55.

➤ **Example 5.** If a, b, c, d be four consecutive coefficients in the binomial expansion of $(1+x)^n$ then prove that

$$\left(\frac{b}{b+c}\right)^2 > \frac{ac}{(a+b)(c+d)} \text{ if } x > 0.$$

➤ ***Solution*** Let the four coefficients be (r + 1)th, (r+2)th, (r+3)th and (r+4)th coefficients respectively.

$\therefore$ $a = {}^{n}C_r$, $b = {}^{n}C_{r+1}$, $c = {}^{n}C_{r+2}$, $d = {}^{n}C_{r+3}$,

Now, $$\frac{a}{a+b} = \frac{{}^{n}C_r}{{}^{n}C_r + {}^{n}C_{r+1}} = \frac{{}^{n}C_r}{{}^{n+1}C_{r+1}}$$

$$= \frac{n!}{r!(n-r)!} \cdot \frac{(r+1)!(n-r)!}{(n+1)!} = \frac{r+1}{n+1}$$

$$\frac{b}{b+c} = \frac{{}^{n}C_{r+1}}{{}^{n}C_{r+1} + {}^{n}C_{r+2}} = \frac{(r+1)+1}{n+1}$$

(replacing r by r + 1 in the above)

$$\frac{c}{c+d} = \frac{{}^{n}C_{r+2}}{{}^{n}C_{r+2} + {}^{n}C_{r+3}} = \frac{(r+2)+1}{n+1}$$

$\therefore$ $$\frac{a}{a+b} + \frac{c}{c+d} = \frac{r+1}{n+1} + \frac{r+3}{n+1}$$

$$= \frac{2(r+2)}{n+1} = 2.\frac{b}{b+c}$$

$\therefore$ $\frac{b}{b+c}$ is the AM of $\frac{a}{a+b}$, $\frac{c}{c+d}$

As A.M. > G.M., $\frac{b}{b+c} > \sqrt{\frac{a}{a+b} \cdot \frac{c}{c+d}}$

$\therefore$ $$\left(\frac{b}{b+c}\right)^2 > \frac{ac}{(a+b)(c+d)}$$

www.pdfworld.net

www.pdfworld.net

➤ **Example 6.** Sum the series $\left(\frac{{}^nC_1}{{}^nC_0}\right)^2 + \left(\frac{2\,{}^nC_2}{{}^nC_1}\right)^2 + \left(\frac{3\,{}^nC_3}{{}^nC_2}\right)^2 + \ldots\ldots\ldots$ upto n terms.

➤ ***Solution*** We have $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \Rightarrow \frac{r.{}^nC_r}{{}^nC_{r-1}} = (n-r+1)$.

$$\text{Let } S = \sum_{r=1}^{n}(n-r+1)^2 = \sum_{r=1}^{n}[n^2 + (r-1)^2 - 2n(r-1)]$$

$$= n^2\sum_{r=1}^{n}1 + \sum_{r=1}^{n}(r-1)^2 - 2n\sum_{r=1}^{n}(r-1)$$

$$= n^2 n + \frac{(n-1)n(2n-1)}{6} - n^2(n-1)$$

$$= \frac{n}{6}(14n^2 - 9n + 1).$$

➤ **Example 7.** Let C_r stands for ${}^nC_r$, show that $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)\ldots\ldots\ldots (C_{n-1} + C_n) = \frac{(n+1)^n}{n!}(C_0C_1C_2\ldots C_{n-1})$

➤ ***Solution*** We have

$$\left(\frac{C_0+C_1}{C_0}\right)\left(\frac{C_1+C_2}{C_1}\right)\left(\frac{C_2+C_3}{C_2}\right)\ldots\left(\frac{C_{n-1}+C_n}{C_{n-1}}\right)$$

$$= \left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\left(1+\frac{C_3}{C_2}\right)\ldots\left(1+\frac{C_n}{C_{n-1}}\right)$$

$$= \left(1+\frac{n}{1}\right)\left(1+\frac{n-1}{2}\right)\left(1+\frac{n-2}{3}\right)\ldots\left(1+\frac{1}{n}\right)$$

$$\left(\text{since } \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right)$$

$$= \left(\frac{n+1}{1}\right)\left(\frac{n+1}{2}\right)\left(\frac{n+1}{3}\right)\ldots\ldots\left(\frac{n+1}{n}\right) = \frac{(n+1)^n}{n!}$$

$\therefore$ $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)\ldots\ldots\ldots (C_{n-1} + C_n) = \frac{(n+1)^n}{n!}(C_0C_1C_2\ldots C_{n-1})$

Alternative

$$\text{L.H.S.} = (C_0 + C_1)(C_1 + C_2)(C_2 + C_3)\ldots\ldots\ldots (C_{n-1} + C_n)$$
$$= ({}^{n+1}C_1)({}^{n+1}C_2)({}^{n+1}C_3)\ldots\ldots ({}^{n+1}C_n)$$

$$= \left(\frac{n+1}{1}.{}^nC_0\right)\left(\frac{n+1}{2}.{}^nC_1\right)\left(\frac{n+1}{3}.{}^nC_2\right)\ldots\ldots$$

$$\left(\frac{n+1}{n}.{}^nC_{n-1}\right)$$

$$= \frac{(n+1)^n}{n!}(C_0C_1C_2\ldots C_{n-1}) = \text{R.H.S.}$$

➤ **Example 8.** If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ $(n \in N)$ then show that

$$\sum_{k=1}^{n} k^3\left(\frac{C_k}{C_{k-1}}\right)^2 = \frac{1}{12}n(n+1)^2(n+2)$$

➤ ***Solution*** For $k \geq 1$, we have $\frac{C_k}{C_{k-1}} = \frac{n-k+1}{k}$ so that $k^3\left(\frac{C_k}{C_{k-1}}\right)^2 = k^3\left(\frac{n-k+1}{k}\right)^2 = k(n-k+1)^2$

$$= k[(n+1)^2 - 2k(n+1) + k^2] \Rightarrow \sum_{k=1}^{n} k^3\left(\frac{C_k}{C_{k-1}}\right)^2$$

$$= (n+1)^2\sum_{k=1}^{n}k - 2(n+1)\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k^3$$

$$= (n+1)^2\left[\frac{n(n+1)}{2}\right] - 2(n+1)\left[\frac{n(n+1)(2n+1)}{6}\right] + \left[\frac{n^2(n+1)^2}{4}\right] = \frac{n(n+1)^2}{12}[6n+6-8n-4+3n]$$

$$= \frac{1}{12}n(n+1)^2(n+2).$$

➤ **Example 9.** Show that no three consecutive binomial coefficients can be in (i) G.P. and (ii) H.P.

➤ ***Solution*** (i) Suppose that the rth, (r + 1)th and (r + 2)th coefficients of $(1 + x)^n$ are in G.P. i.e., C_{r-1}, C_r, C_{r+1} are in G.P.

$$\Rightarrow \quad \frac{C_r}{C_{r-1}} = \frac{C_{r+1}}{C_r} \quad \Rightarrow \quad \frac{n-r+1}{r} = \frac{n-r}{r+1}$$

www.pdfworld.net

www.pdfworld.net

$$\left(\text{since } \frac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}\right)$$

$\Rightarrow$ $(n-r+1)(r+1) = r(n-r)$

$\Rightarrow$ $nr + n - r^2 - r + r + 1 = nr - r^2$

$n + 1 = 0$

$n = -1$ which is not possible since n is a positive integer.

(ii) Suppose that rth, (r + 1)th and (r + 2)th coefficients of $(1+x)^n$ are in H.P., i.e., C_{r-1}, C_r, C_{r+1} are in H.P. then

$$\frac{2}{C_r} = \frac{1}{C_{r-1}} + \frac{1}{C_{r+1}} \Rightarrow 2 = \frac{C_r}{C_{r-1}} + \frac{C_r}{C_{r+1}}$$

$$\Rightarrow \quad 2 = \frac{n-r+1}{r} + \frac{r+1}{n-r}$$

$\Rightarrow$ $2r(n-r) = (n-r+1)(n-r) + r(r+1)$

$\Rightarrow$ $2nr - 2r^2 = n^2 - nr - nr + r^2 + n - r + r^2 + r$

$\Rightarrow$ $n^2 - 4nr + 4r^2 + n = 0 \quad (n-2r)^2 + n = 0$

This is not possible, as $(n-2r)^2 \geq 0$ and n is a positive integer.

5.6 NUMERICALLY GREATEST TERM

To find the numerically greatest term in the expansion of $(x+a)^n$

Let T_r and T_{r+1} be the rth and (r + 1)th terms respectively

$T_r = {}^{n}C_{r-1}\, x^{n-(r-1)}\, a^{r-1}$ $T_{r+1} = {}^{n}C_r\, x^{n-r}\, a^r$

Now,

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^{n}C_r}{{}^{n}C_{r-1}} \frac{x^{n-r}a^r}{x^{n-r+1}a^{r-1}}\right| = \frac{n-r+1}{r} \cdot \left|\frac{a}{x}\right|$$

Consider $\left|\frac{T_{r+1}}{T_r}\right| \geq 1 \Rightarrow \left(\frac{n-r+1}{r}\right) \cdot \left|\frac{a}{x}\right| \geq 1$

$$\Rightarrow \frac{n+1}{r} - 1 \geq \left|\frac{x}{a}\right| \Rightarrow \quad r \leq \frac{n+1}{1+\left|\frac{x}{a}\right|}$$

Case-I. : When $\frac{n+1}{1+\left|\frac{x}{a}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3, \ldots, m-1)$
i.e. $T_2 > T_1, T_3 > T_2, \ldots\ldots, T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$ i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ $(r = m+1, m+2, \ldots.n)$
i.e. $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, \ldots\ldots T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1+\left|\frac{x}{a}\right|}$ is an integer, equal to m, then T_m and T_{m+1} will be the numerically greatest terms (both terms are equal in magnitude).

Case-II : When $\frac{n+1}{1+\left|\frac{x}{a}\right|}$ is not an integer

(Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r < \frac{n+1}{1+\left|\frac{x}{y}\right|}$ $(r = 1, 2, 3, \ldots\ldots, m-1,$ m) i.e. $T_2 > T_1, T_3 > T_2, \ldots\ldots, T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r > \frac{n+1}{1+\left|\frac{x}{a}\right|}$ $(r = m+1, m+2, \ldots.n)$

i.e. $T_{m+2} < T_{m+1}, T_{m+2} < T_{m+2}, \ldots\ldots, T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1+\left|\frac{x}{a}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

In short, to find the numerically greatest term in the expansion of $(x+a)^n$

(a) Calculate $\frac{n+1}{1+\left|\frac{x}{a}\right|} = m$ (say)

(b) If m is an integer, then T_m and T_{m+1} are numerically equal and greatest terms.

(c) If m is not integer, then $T_{[m]+1}$ is the numerically greatest term, where [.] denotes the greatest integer function.

www.pdfworld.net

www.pdfworld.net

CAUTION It should be clearly borne in that the answer makes sense in the original expansion only. It may become different if the expansion is reversed. For example

$$\left(1+\frac{1}{2}\right)^5 = 1 + {}^5C_1 \cdot \frac{1}{2} + {}^5C_2 \cdot \frac{1}{2^2} + {}^5C_3 \cdot \frac{1}{2^3}$$

$$+ {}^5C_4 \cdot \frac{1}{2^4} + \frac{1}{2^5} = 1+\frac{5}{2}+\frac{10}{4}+\frac{10}{8}+\frac{5}{16}+\frac{1}{32}$$

The greatest term $= \frac{5}{2} = 2^{nd}$ term

If we reverse the order of the same expansion $\left(\frac{1}{2}+1\right)^5$

$$= {}^5C_0\left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^4 + {}^5C_2\left(\frac{1}{2}\right)^3 + {}^5C_3\left(\frac{1}{2}\right)^2$$

$$+ {}^5C_4\left(\frac{1}{2}\right)^1 + {}^5C_5$$

Here 5^{th} term is greatest.

STUDY TIP

To find the greatest coefficient in the expansion of $(1 + x)^n$.

The coefficient of the general term of $(1+x)^n$ is ${}^nC_r$; and we have only to find for what value of r this is greatest.

(a) If n is even, then the greatest coefficient = ${}^nC_{n/2}$

(b) If n is odd, then the greatest coefficients are ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

➤ **Example 1.** Find the numerically greatest term in the expansion of $(4 - 3x)^7$ when $x = \frac{2}{3}$

➤ ***Solution*** $\left|\frac{T_{r+1}}{T_r}\right| = \frac{{}^7C_r\, 4^{7-r}(3x)^r}{{}^7C_{r-1}\, 4^{8-r}(3x)^{r-1}} = \frac{8-r}{r}\frac{3x}{4}$

$= \frac{8-r}{r}\cdot\left(\frac{2}{4}\right)$ since $x = 2/3$ $= \frac{8-r}{2r}$

Now $|T_{r+1}| \geq |T_r|$ if $8 - r \geq 2r$ or $\frac{8}{3} \geq r$

This inequality is valid only for r = 1 or 2

Thus for r = 1, 2 ; $|T_{r+1}| > |T_r|$ and

for r = 3, 4; $|T_{r+1}| < |T_r|$

$\therefore$ $|T_1| < |T_2| < |T_3| > |T_4| > |T_5| > \ldots..$

The greatest term $= |T_3| = {}^7C_2.4^5.(3x)^2$ where $x = \frac{2}{3}$

$= 21 \times 4^5 \times 2^2 = 86016$

➤ **Example 2.** Find the numerically greatest term(s) in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$.

➤ ***Solution*** Let r^{th} and $(r+1)^{th}$ be the two consecutive terms in the expansion of $(3-5x)^{15}$ $|T_{r+1}| \geq |T_r|$

${}^{15}C_r\, 3^{15-r}\,(|-5x|)^r \geq {}^{15}C_{r-1}\, 3^{15-(r-1)}\,(|-5x|)^{r-1}$

$$\frac{(15)}{(15-r)!r!}|-5x| \geq \frac{3.(15)!}{(16-r)!(r-1)!};$$

$5 \cdot \frac{1}{5}(16-r) \geq 3r$; $16 - r \geq 3r$;

$4r \leq 16$; $r \leq 4$

Explanation : For $r \leq 4$, $T_{r+1} \geq T_r$

$\Rightarrow$ $|T_2| > |T_1|$ $|T_3| > |T_2|$ $|T_4| > |T_3|$ $|T_5| = |T_4|$

For $r > 5$, $T_{r+1} < T_r \Rightarrow$ $|T_6| < |T_5|$ $|T_7| < |T_6|$ and so on

Hence, T_4 and T_5 are the numerically greatest terms and both are equal.

➤ **Example 3.** Find the numerically greatest term in the expansion of $(2 + 3x)^9$, when x = 3/2

➤ ***Solution*** Now, in the expansion of $(2 + 3x)^9$, we have

$$\left|\frac{T_{r+1}}{T_r}\right| = \frac{(9-r+1)}{r}\left|\frac{3}{2}x\right| = \frac{10-r}{r}\left|\frac{3}{2}\times\frac{3}{2}\right|$$

$(\because x = 3/2)$ $= \frac{90-9r}{4r}$

$$\left|\frac{T_{r+1}}{T_r}\right| \geq 1 \Rightarrow \frac{90-9r}{4r} \geq 1 \Rightarrow r \leq \frac{90}{13} = 6\frac{12}{13}$$

$\therefore$ The maximum value of r satisfying $\left|\frac{T_{r+1}}{T_r}\right| \geq 1$ is 6

So the numerically greatest term $= T_{6+1}$

$$= {}^9C_6\, 2^3\left(\frac{3.3}{2}\right)^6 = \frac{7\times 3^{13}}{2}$$

www.pdfworld.net

www.pdfworld.net

Alternative

On calculating $\frac{n+1}{1+\left|\frac{x}{a}\right|}$ we get $\frac{90}{13} = 6\frac{12}{13} = m \neq$ Integer. Hence the greatest term in the expansion is $T_{[m]+1} = T_{6+1} = T_7$

Thus the greatest term $= \frac{7 \times 3^{13}}{2}$

➤ **Example 4.** Find the algebraically greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \frac{1}{5}$.

➤ ***Solution*** First we find the numerically greatest term. Now in the expansion of $(3 - 5x)^{11}$, we have

$$\left|\frac{T_{r+1}}{T_r}\right| = \frac{(11-r+1)}{r}\left|-\frac{5x}{3}\right| = \left(\frac{12-r}{r}\right)\left|-\frac{5}{3}\times\frac{1}{5}\right|$$

$$\left(\because x = \frac{1}{5}\right) = \left(\frac{12-r}{3r}\right)\left|\frac{T_{r+1}}{T_r}\right| \geq 1 \Rightarrow \frac{12-r}{3r} \geq 1$$

$$\Rightarrow 4r \leq 12 \Rightarrow r \leq 3 \quad \therefore r = 2, 3$$

So, the numerically greatest terms are T_3 and T_4. Since the terms of the expansion occur with alternate signs, T_3 is positive and the algebraically greatest term.

$\therefore$ The algebraically greatest term T_3

$$= 3^{11}\,{}^{11}C_2\left(-\frac{5}{3}x\right)^2 = 3^{11}.{}^{11}C_2\left(-\frac{5}{3}\times\frac{1}{5}\right)^2 = 55 \times 3^9$$

➤ **Example 5.** Find the algebraically greatest term in the expansion of $(a - 2x)^9$ when $a = 1$ and $x = \frac{1}{4}$.

➤ ***Solution*** First we find the numerically greatest term.

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^9C_r a^{9-r}.(2x)^r}{{}^9C_{r-1}a^{9-r+1}(2x)^{r-1}}\right| = \frac{9-r+1}{r}\left|\frac{2x}{a}\right|$$

$$= \frac{9-r+1}{r}\,\frac{2}{4} = \frac{10-r}{r}\,\frac{1}{2}$$

$$\therefore \left|\frac{T_{r+1}}{T_r}\right| > 1 \text{ if } \frac{10-r}{r}\,\frac{1}{2} > 1$$

$$\Rightarrow 10 > 3r \Rightarrow r < \frac{10}{3}$$

Hence for values of 'r' upto 3, $\left|\frac{T_{r+1}}{T_r}\right| > 1$

$\therefore$ The numerically greatest term in the expansion is $T_{3+1} = T_4$.

T_4 is a negative term. The two neighbouring terms T_3 and T_5 are positive and numerically close to T_4. Hence we must compare them.

$$\frac{T_5}{T_3} = \frac{T_5}{T_4}\frac{T_4}{T_3} = \frac{10-4}{4}.\frac{1}{2}.\frac{10-3}{3}.\frac{1}{2} = \frac{7}{8} < 1$$

Hence T_3 is the algebraically greatest term.

Practice Problems [E]

1. Find the exponent of the binomial if in the expansion of $\left(ax^{-\frac{1}{2}} + xa^{-\frac{1}{2}}\right)^n$ the ratio of the coefficient of the third term to that of the second is equal to $\frac{11}{2}$.
2. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^{2n}$ are in A.P., show that $2n^2 - 9n + 7 = 0$
3. Find out which term in the expansion of the binomial $\left(\sqrt{x} + \sqrt[4]{x^{-3}}\right)^n$ contains $x^{6.5}$ if the tenth term has the largest coefficient.
4. Which is the greatest term in the expansion of $\left(\frac{9}{10} + \frac{1}{10}\right)^{100}$?
5. Find the greatest term(s) in the expansion of $(a + x)^{13}$,when $a = 5$ & $x = 2$.
6. Find the greatest coefficient of the given polynomial :

www.pdfworld.net

www.pdfworld.net

(i) $\left(\frac{1}{2}+\frac{1}{2}x\right)^{100}$ (ii) $\left(\frac{1}{10}+\frac{9}{10}x\right)^{12}$

(iii) $\left(\frac{1}{4}+\frac{2}{3}x\right)^{105}$ (iv) $(\sqrt{5}+\sqrt{2}x)^{20}$

7. For which positive values of x is the fifth term in the expansion of $(5 + 3x)^{10}$ the greatest?

8. Find x for which the fifty-first term in the expansion of $(x + y)^{100}$ is the greatest if it is known that $x + y = 1, x > 0, y > 0$.

9. Find x for which the (k+1)th term of the expansion of $(x + y)^n$ is the greatest if $x + y = 1$ and $x > 0, y > 0$.

5.7 DIVISIBILITY USING BINOMIAL THEOREM

From the expansion,

$(1 + \alpha)^n = 1 + {}^nC_1\alpha + {}^nC_2\alpha^2 + ... + {}^nC_n\alpha^n$

we can say that $(1 + \alpha)^n - 1$ is divisible by α.

Similarly, $(1 + \alpha)^n - 1 - n\alpha$

$= {}^nC_2\alpha^2 + {}^nC_3\alpha^3 + ... + {}^nC_n\alpha^n$ is divisible by α^2.

$$(1 + \alpha)^n - 1 - n\alpha - \frac{n(n-1)}{2}\alpha^2$$

$= {}^nC_3\alpha^3 + {}^nC_4\alpha^4 + ... + {}^nC_n\alpha^n$ is divisible by α^3.

For example $(1 + 5)^{42} - 1 = 6^{42} - 1$ is divisible by 6.

In other words, if we divide 6^{42} by 6 we get remainder

$= 1 (1 + 7)^{35} - 1 - 35 \times 7 = 8^{35} - 245$ is divisible by 7^2 i.e. 49

In other words if we divide 8^{51} by 49, we get remainder = 15

$$(1 + 7)^{51} - 1 - 51 \times 7 - \frac{51\times 50}{2} \times 7^2$$

$= 8^{51} - 358 - 62475 = 8^{51} - 62833$ is divisible by 7^3 i.e. 343

In other words if we divide 8^{51} by 343, we get remainder = 64

STUDY TIP In such divisibility problems, we will have to identify α and n. The number by which division is to be made can be α or α^2 or α^3 but the number in the base is mostly expressed in the form $1 \pm \alpha$.

➤ **Example 1.** If n is any positive integer, show that $2^{3n+3} - 7n - 8$ is divisible by 49.

➤ ***Solution*** The given expression.

$= 2^{3n+3} - 7n - 8 = 2^{3n} .2^3 - 7n - 8$

$= 8^n . 8 - 7n - 8 = 8(1 + 7)^n - 7n - 8$

$= 8\{1 + {}^nC_1 7 + {}^nC_2 7^2 + ... + {}^nC_n 7^n\} - 7n - 8$

$= 8 + 56n + 8({}^nC_2 . 7^2 + ... + {}^nC_n 7^n) - 7n - 8$

$= 49n + 8({}^nC_2 . 7^2 + ... + {}^nC_n 7^n)$

$= 49\{n + 8({}^nC_2 + ... + {}^nC_n 7^{n-2})\}$

Hence $2^{3n+3} - 7n - 8$ is divisible by 49.

➤ **Example 2.** Using binomial theorem, show that $3^{4n+1} + 16n - 3$ is divisible by 256 if n is a positive integer.

➤ ***Solution*** Given expression $= 3^{4n+1} + 16n - 3$

$= 3^{4n} . 3 + 16n - 3 = 3(3^{4n} - 1) + 16n$

$= 3[(3^4)^n - 1] + 16n = 3[(81)^n - 1] + 16n$

$= 3[(1 + 80)^n - 1] + 16n$

[since 80^2 contains 256 as a factor]

$= 3[\{1 + {}^nC_1(80) + {}^nC_2(80)^2 + {}^nC_3(80)^3 + ...\} - 1] + 16n$

$= 3\{80n + {}^nC_2(80)^2 + {}^nC_3(80)^3 + ...\} + 16n$

$= 240n + 3[{}^nC_2(80)^2 + {}^nC_3(80)^3 + ...] + 16n$

$= (240n + 16n) + 3.80^2[{}^nC_2 . 80 + ... + 80^{n-2}]$

$= 256[n + 75\{{}^nC_2 + {}^nC_2 . 80 + ... + 80^{n-2}\}]$

$\therefore$ If $n \geq 2$, $3^{4n+1} + 16n - 3$ is divisible by 256 and for $n = 1$, $3^{4n+1} + 16n - 3 = 256$ which is divisible by 256.

Hence $3^{4n+1} + 16n - 3$ is divisible by 256 for every positive integer n,

➤ **Example 3.** Using binomial theorem, show that $3^{2n+2} - 8n - 9$ is divisible by 64 for all $n \in N$.

➤ ***Solution*** $3^{2n+2} - 8n - 9 = (3^2)^{n+1} - 8n - 9$;

Now express 3^2 or 9 as (1 + 8) since 64 is power of 8.

$= (1 + 8)^{n+1} - 8n - 9$

$= \{1 + {}^{(n+1)}C_1 8 + {}^{(n+1)}C_2 8^2 + {}^{(n+1)}C_3 8^3 + ...$

$+ {}^{(n+1)}C_{n+1} 8^{n+1}\} - 8n - 9$

$= 8^2\{{}^{(n+1)}C_2 + {}^{(n+1)}C_3 8 + ... + {}^{(n+1)}C_{n+1} 8^{n-1}\}$

Here every term is divisible by 64

➤ **Example 4.** Show that $11^{n+2} + 12^{2n+1}$ is divisible by 133

➤ ***Solution*** $11^{n+2} + 12^{2n+1} = 11^2 . 11^n + 12(144)^n$

Now 144 and 121 should be expressed in terms of 133; 144 as (133 + 11) and 121 as (133 – 12)

www.pdfworld.net

www.pdfworld.net

$= 121.11^n + 12(11 + 133)^n$

$= 121.11^n + 12\{11^n + {}^nC_1 11^{n-1} . 133 + {}^nC_2 . 11^{n-2} (133)^2 + ...\}$

$= 11^n (121 + 12)$ + terms containing 133 as a factor

$= 11^n . 133$ + terms containing 133 as a factor

Hence the expression is divisible by 133.

➤ **Example 5.** Prove that $13^{99} - 19^{93}$ is divisible by 162.

➤ ***Solution*** $162 = 2 \times 81 = 2 \times 3^4$

Hence 13 should be written as $(1 + 4 \times 3)$

and 19 as $(1 + 2 \times 3^2)$

In the two expansions $(1 + 4 \times 3)^{99}$ and $(1 + 2 \times 3^2)^{93}$ terms containing powers of 3 less than 4 will be equal or their difference will contain $3^4 \times 2$ $13^{99} - 19^{93}$

$= (1+4\times3)^{99} - (1+2\times3^2)^{93}$

$= \{1 + {}^{99}C_1 \times 4 \times 3 + {}^{99}C_2 \times (4 \times 3)^2 + ...\}$

$\quad - \{1 + {}^{93}C_1 \times 2 \times 3^2 + {}^{93}C_2 \times (2 \times 3^2)^2 + ...\}$

$= (99 \times 4 \times 3 - 93 \times 2 \times 3^2) +$

$$\left(\frac{99.98}{2} \times 4^2 \times 3^2 - \frac{93 \times 92}{2} \times 2^2 \times 3^4\right)$$

$\quad$ + terms containing 2×3^4 as a factor

$= 2 \times 3^3 (22 - 31) + 2.3^4 (11 \times 98 \times 8 - 93 \times 92) + ...$

$= -2 \times 3^5 + 2 \times 3^4 \times$ integer + ...

$= 2 \times 3^4 \times$ an integer and hence divisible by 162.

➤ **Example 6.** Show that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for all n.

➤ ***Solution*** Let us try to create powers of 43 in the expression $6^{n+2} + 7^{2n+1}$

Indeed $6^{n+2} + 7^{2n+1} = 6^{n+2} + 7.49^n = 6^{n+2} + 7.(6 + 43)^n$

$= 6^{n+2} + 7.[6^n + {}^nC_1 6^{n-1}(43)^1 + {}^nC_2 6^{n-2}(43)^2 + ... + {}^nC_n (43)^n]$

Now ${}^nC_1 6^{n-1}(43)^1 + {}^nC_2 6^{n-2}(43)^2 + ... + {}^nC_n (43)^n = 43\lambda$ where λ is an integer

$\Rightarrow 6^{n+2} + 7^{2n+1} = 6^{n+2} + 7.[6^n + 43\lambda] = 6^{n+2} + 7.6^n + 7 \times 43\lambda$

$= 6^n(36 + 7) + 7 \times 43\lambda = 43.6^n + 43.7\lambda = 43(6^n + 7\lambda)$

$\Rightarrow 6^{n+2} + 7^{2n+1}$ is divisible by 43 for all natural numbers n.

➤ **Example 7.** Show that $2^{n+1} - 9n^2 + 3n - 2$ is divisible by 54 for all n.

➤ ***Solution*** $2^{2n+1} - 9n^2 + 3n - 2 = 2.4^n - 9n^2 + 3n - 2$

$= 2.(1+3)^n - 9n^2 + 3n - 2$

$= 2\left[1 + {}^nC_1 3 + {}^nC_2 .3^2 + 27\lambda\right] - 9n^2 + 3n - 2$

(Note that $27\lambda = {}^nC_3 .3^3 + {}^nC_4 . 3^4 + ... + {}^nC_n . 3^n$)

$$= 2\left[1 + 3n + \frac{n(n-1)}{2} 9 + 27\lambda\right] - 9n^2 + 3n - 2 = 54\lambda$$

which proves our assertion.

Finding the last few digits of a number

➤ **Example 8.** Find the last two digits of the number $(17)^{10}$.

➤ ***Solution*** $(17)^{10} = (289)^5$

$= (290 - 1)^5$

$= {}^5C_0 (290)^5 - {}^5C_1 (290)^4 + ... + {}^5C_4 (290)^1 - {}^5C_5 (290)^0$

$= {}^5C_0 (290)^5 - {}^5C_1 . (290)^4 + ... + {}^5C_3 (290)^2 - 5 \times 290 - 1$

= a multiple of 1000 + 1449

Hence, last two digits are 49.

We can also conclude that last three digits are 449.

Finding the unit digit of a number

Let us find the unit digit of a natural number N^m, where m is a positive integer.

Let the unit digit of N be k. The unit digit of N^m is the unit digit of k^m. For instance the unit digit of $(37)^{21}$ is same as the unit digit of 7^{21}.

If k = 0, 1, 5, 6 then the unit digit of k^m is 0, 1, 5, 6 respectively and the same happens with N^m.

Consider k = 2

2^1 has unit digit 2; 2^2 has unit digit 4 ;

2^3 has unit digit 8; 2^4 has unit digit 6 ;

2^5 has unit digit 2; 2^6 has unit digit 4 ;

2^7 has unit digit 8; 2^8 has unit digit 6 ; and so on.

Thus, 2^{4n+1} has unit digit 2; 2^{4n+2} has unit digit 4; 2^{4n+3} has unit digit 8; 2^{4n} has unit digit 6

For example the unit digit of $2^{37} = 2^{4.9+1}$ is 2.

Consider k = 3

3^1 has unit digit 3; 3^2 has unit digit 9;

3^3 has unit digit 7; 3^4 has unit digit 1 and so on.

Thus, 3^{4n+1} has unit digit 3; 3^{4n+2} has unit digit 9; 3^{4n+3} has unit digit 7; 3^{4n} has unit digit 1.

For example the unit digit of $(143)^{43}$ is same as the unit digit of $3^{43} = 3^{4.10+3}$ which is 7.

Similarly,

7^{4n+1} has unit digit 7; 7^{4n+2} has unit digit 9; 7^{4n+3} has unit digit 3; 7^{4n} has unit digit 1.

www.pdfworld.net

www.pdfworld.net

Similarly,
8^{4n+1} has unit digit 8 ; 8^{4n+2} has unit digit 4 ; 8^{4n+3} has unit digit 2 ; 8^{4n} has unit digit 6.
Consider k = 4
4^1 has unit digit 4 ; 4^2 has unit digit 6 ;
4^3 has unit digit 4 ; 4^4 has unit digit 6 and so on.
Thus, 4^{2n+1} has unit digit 4 ; 4^{2n} has unit digit 6.
For example the unit digit of $(34)^{27}$ is same as the unit digit of $4^{2.13+1}$ which is 4.
Similarly,
9^{2n+1} has unit digit 9 ; 9^{2n} has unit digit 1.

We can use binomial theorem for finding the unit digit of N^m, when the unit digit of N is 3,7 or 9.

➤ **Example 9.** Find the unit digit of the number $(27)^{50} + (18)^{50}$.
➤ ***Solution*** The unit digit of $(27)^{50}$ is same as the unit digit of $7^{4.12+2}$ which is 9.
The unit digit of $(18)^{50}$ is same as the unit digit of $8^{4.12+2}$ which is 4.
Hence, the unit digit of the number $(27)^{50} + (18)^{50}$ is unit digit of (9 + 4) which is 3.

➤ **Example 10.** Prove that if n > 4, then the number 1! + 2! + 3! + ...n! is never a square.
➤ ***Solution*** The units digit of the square of an integer must be one of 1, 4, 5, 6, 9.
But when n > 4., the number a = 5! + ...n! is divisible by 10 and so the units digit of
1! + 2! + 3! + ...n! = 1! + 2! + 3! + 4! + a = 33 is 3.
Hence the given number is not a square.

Divisibility of $x^n \pm y^n$ by $x \pm y$, $x, y, n \in N$

(i) If n is even, then $x^n - y^n$ is divisible by $x \pm y$.
(ii) If n is odd, then $x^n - y^n$ is divisible by $x - y$.
(iii) If $n \in N$, then $x^n - y^n$ is divisible by $x - y$.
(iv) If n is odd, then $x^n + y^n$ is divisible by $x + y$.
These statements are easily verified using factor theorem.
For example, $15^6 - 7^6$ is divisible by 15 ± 7 i.e. it is divisible by 22 as well as 8.
We also conclude that it is divisible by L.C.M (22, 8) i.e. 88.

➤ **Example 11.** Show that $1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$ is divisible by 1998.
➤ ***Solution*** Using the remainder theorem we know that $x^n - a^n$ is divisible by (x – a) for any positive integer n
$1992^n - 1938^n$ is divisible by (1992 – 1938) i.e. 54
$1955^n - 1901^n$ is divisible by (1955 – 1901) i.e. 54
∴ The given number is divisible by 54.
and $1992^n - 1955^n$ is divisible by (1992 – 1955) i.e. 37 $1938^n - 1901^n$ is divisible by (1938 – 1901) i.e. 37
∴ The given number is divisible by 37.
Hence the given number is divisible by $54 \times 37 = 1998$.

➤ **Example 12.** Show that $1^{1997} + 2^{1997} + ... + 1996^{1997}$ is divisible by 1997.
➤ ***Solution*** We shall make groups of the terms of the expression as follows :
$(1^{1997} + 1996^{1997}) + (2^{1997} + 1995^{1997}) + ...$
$+ (998^{1997} + 999^{1997})$

Here each bracket is of the form $(a_i^{2n+1} + b_i^{2n+1})$ is divisible by $(a_i + b_i)$
But $(a_i + b_i) = 1997$ for all i.
∴ each bracket and hence, their sum is divisible by 1997.

The product of r consecutive integers is divisible by r!.
Let p be the product of r consecutive integers beginning with n.
Then p = n(n + 1)(n + 2).... (n + r – 1)

$$\text{Now } \frac{p}{r!} = \frac{n(n+1)(n+2).....(n+r-1)}{r!}$$

$$= \frac{[1.2.3.....(n-1)][n(n+1)(n+2).....(n+r-1)]}{[1.2.3.....(n-1)]r!}$$

$$= \frac{(n+r-1)!}{(n-1)!r!} = {}^{n+r-1}C_r$$

www.pdfworld.net

www.pdfworld.net

$\therefore \dfrac{p}{r!}$ = an integer (${}^nC_r$ is a natural number)

$\therefore$ p is divisible by r!

➤ **Example 13.** Suppose m, n are integers and $m = n^2 - n$. Then show that $m^2 - 2m$ is divisible by 24.

➤ ***Solution*** Given m, n are integers and $m = n^2 - n$. Then $m^2 - 2m = (n-2)(n-1)n(n+1)$. It is a product of 4 consecutive integers.

Hence $m^2 - 2m$ is divisible by 4! i.e. 24.

Practice Problems [F]

1. Find the remainder when 7^{98} is divided by 5.
2. Using binomial theorem prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.
3. Find the last digit, last two digits and last three digits of the number $(27)^{27}$.
4. Show that $53^{53} - 33^3$ is divisible by 10.
5. Show that $2^{4n} - 2^n(7n+1)$ is divisible by square of 14, where n is a positive integer.
6. Prove that $11^{10} - 1$ is divisible by 100.
7. Find the last digit of the number $9^{(9^9)}$ and of the number $2^{(3^4)}$
8. Find the last two digits of 3^{999}.

5.8 ANALYSING INTEGRAL AND FRACTIONAL PART OF A NUMBER

1. If $(\sqrt{A} + B)^n = I + f$ where I and n are positive integers, n being odd and $0 < f < 1$, then show that $(I + f)f = k^n$ where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$.

Proof : Given $\sqrt{A} - B < 1$

$\therefore 0 < (\sqrt{A} - B)^n < 1$

Now let $(\sqrt{A} - B)^n = g$ where $0 < g < 1$

$\therefore I + f - g = (\sqrt{A} + B)^n - (\sqrt{A} - B)^n$

Since R.H.S. contains even powers of $\sqrt{A}$, hence R.H.S. is integer.

$\therefore$ L.H.S. is also integer ($\because$ n is odd). Since I is an integer, f – g must be an integer. Now $-1 < f - g < 1$

Hence $f - g = 0 \quad \Rightarrow \quad f = g$

$\therefore (I+f)f = (I+f)g = (\sqrt{A} + B)^n (\sqrt{A} - B)^n$

$= (A - B^2)^n = k^n$.

2. If $(\sqrt{A} + B)^n = I + f$ where I and n are positive integers, n being even and $0 < f < 1$, then show that $(I + f)(1 - f) = k^n$ where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$.

Proof : Given $\sqrt{A} - B < 1$

$\therefore 0 < (\sqrt{A} - B)^n < 1$

Now let $(\sqrt{A} - B)^n = g$ where $0 < g < 1$

$\therefore I + f + g = (\sqrt{A} + B)^n + (\sqrt{A} - B)^n$

Since R.H.S. contains even powers of $\sqrt{A}$, hence R.H.S. is integer.

$\therefore$ L.H.S. is also integer ($\because$ n is odd).

Since I is an integer, f – g must be an integer.

Now $0 < f + g < 2$

Hence $f + g = 1$

$\Rightarrow \quad g = 1 - f$

Hence $(I + f)(1 - f) = (I + f)g$

$= (\sqrt{A} + B)^n (\sqrt{A} - B)^n = (A - B^2)^n = k^n$.

➤ **Example 1.** Show that the integral part of $(5 + 2\sqrt{6})^n$ is odd, where $n \in N$.

➤ ***Solution*** Let $(5 + 2\sqrt{6})^n = I + f$, where I is an integer and $0 \le f < 1$

Now, let $g = (5 - 2\sqrt{6})^n$, $0 < g < 1$

$I + f + g = (5 + 2\sqrt{6})^n + (5 - 2\sqrt{6})^n$

$= 2\{5^n + {}^nC_2 5^{n-2}(2\sqrt{6})^2 + {}^nC_4 5^{n-4}(2\sqrt{6})^4 + ...\}$

= Even Integer

www.pdfworld.net

www.pdfworld.net

$= 2k \qquad (k \in N)$

Hence $f + g = 2k - I$ is an integer, but $0 < f + g < 2$ therefore $f + g = 1$

$1 = 2k - I$

$I = 2k - 1$ = odd integer

Hence the integral part of $(5 + 2\sqrt{6})^n$ is an odd integer.

➤ **Example 2.** Show that the integral part of $(5\sqrt{5} + 11)^{2n+1}$ is even where $n \in N$.

➤ ***Solution*** Let $(5\sqrt{5} + 11)^{2n+1} = I + f$, where I is an integer and $0 \le f < 1$

Now, let $g = (5\sqrt{5} - 11)^{2n+1}$, $0 < g < 1$

Also, $I + f - g = (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$

$= 2\{{}^{2n+1}C_1(5\sqrt{5})^{2n}(11) + {}^{2n+1}C_3(5\sqrt{5})^{2n-2}(11)^3 + ...\}$

= Even integer

$= 2k \ (k \in N)$

Hence $f - g = 2k - I$ is an integer but $-1 < f - g < 1$ therefore $0 = 2k - I$

$\therefore I = 2k$ = even integer.

Hence the integral part of $(5\sqrt{5} + 11)^{2n+1}$ is an even integer.

➤ **Example 3.** If $x = (8 + 3\sqrt{7})^n$, $n \in N$, prove that the integral part of x is an odd integer and also show that $x - x^2 + x[x] = 1$, where [.] denotes the greatest integer function.

➤ ***Solution*** Since $x = (8 + 3\sqrt{7})^n = [x] + f$

where $[x] \le x$ and $0 \le f < 1$

Now, let $g = (8 - 3\sqrt{7})^n$ where $0 < g < 1$

Also, $[x] + f + g = (8 + 3\sqrt{7})^n + (8 - 3\sqrt{7})^n$

$= 2\{8^n + {}^nC_2(8)^{n-2}(3\sqrt{7})^2 + {}^nC_4(8)^{n-4}(3\sqrt{7})^4 + ...\}$

= 2 (Integer)

$\therefore$ $[x] + f + g$ = Even integer = 2k (say)

$\therefore$ L.H.S. is also an integer.

$\therefore$ $f + g$ is an integer therefore $f + g = 1$ ($\because 0 < f + g < 2$)

Hence $[x] + 1 = 2k$

$[x] = 2k - 1$ = odd integer.

Now $x - x^2 + x[x] = x - x(x - [x])$

$= x - xf = x(1 - f)$

$= xg \qquad (\because f + g = 1)$

$= (8 + 3\sqrt{7})^n (8 - 3\sqrt{7})^n = (64 - 63)^n = 1$

STUDY TIP Some times students find it difficult to decide whether in the problem addition is required or subtraction. Now if $x = [x] + f$, and $0 < g < 1$ and if $[x] + f + g$ = Integer, then addition is required and if $[x] + f - g$ = Integer, then subtraction is required and the values of $f + g$ and $f - g$ are 1 and 0 respectively.

➤ **Example 4.** If $(7 + 4\sqrt{3})^n = x + t$, where n and x are positive integers and t is a proper fraction, show that $(1 - t)(x + t) = 1$.

➤ ***Solution*** Since t is a proper fraction $0 < t < 1$

given $(7 + 4\sqrt{3})^n = s + t$

Now, let $t' = (7 + 4\sqrt{3})^n$, $0 < t' < 1$

Also, $x + t + t' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$

$= 2(7^n + {}^nC_2(7)^{n-2}(4\sqrt{3})^2 + {}^nC_4(7)^{n-4}(4\sqrt{3})^4 + ...]$

$= 2$ (integer) $= 2k \qquad (k \in N)$ = Even integer

Hence $t + t'$ = Even integer $- s$, but $0 < t + t' < 2$

$\because$ R.H.S. is integer, hence L.H.S. is also integer.

Therefore $t + t' = 1$

$\therefore t' = (1 - t)$

Hence $(1 - t)(x + t) = t'(x + t)$

$= (7 - 4\sqrt{3})^n (7 + 4\sqrt{3})^n = (49 - 48)^n = 1$.

➤ **Example 5.** Find the integer just greater than the number $\left(3+\sqrt{5}\right)^5$

➤ ***Solution*** Let $I + f = \left(3+\sqrt{5}\right)^5$ and $f' = \left(3-\sqrt{5}\right)^5$

As before, we can prove that $f + f' = 1$.

Hence the integer just greater than the number $\left(3+\sqrt{5}\right)^5$ is $I + f + f'$

$= \left(3+\sqrt{5}\right)^5 + \left(3-\sqrt{5}\right)^5$

$= 2\left[{}^5C_0\,3^5 + {}^5C_2\,3^3\left(\sqrt{5}\right)^2 + {}^5C_4\,3\left(\sqrt{5}\right)^4\right]$

$= 2(243 + 1350 + 375) = 2(1968) = 3936$

www.pdfworld.net

www.pdfworld.net

➤ **Example 6.** Show that the integer just above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1} for all $n \in N$.

➤ ***Solution*** Let $(\sqrt{3}+1)^{2n} = (4+2\sqrt{3})^n$

$$= 2^n (2+\sqrt{3})^n = I + f \qquad ...(1)$$

where I and f are its integral & fractional parts respectively

$0 < f < 1$. Now $0 < \sqrt{3} - 1 < 1$

$$0 < (\sqrt{3}-1)^{2n} = (4-2\sqrt{3})^n = 2^n (2-\sqrt{3})^n = f' \ldots(2)$$

$0 < f' < 1$ adding (1) and (2)

$$I + f + f' = (\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n}$$

$$= 2^n [(2+\sqrt{3})^n + (2-\sqrt{3})^n]$$

$$= 2.2^n [{}^nC_0\, 2^n + {}^nC_2\, 2^{n-2} (\sqrt{3})^2 + \ldots.]$$

$I + f + f' = 2^{n+1} k$ (where k is a positive integer)

$0 < f < f' < 2 \quad \Rightarrow \quad f + f' = 1 \quad I + 1 = 2^{n+1} k.$

$I + 1$ is the integer just above $(\sqrt{3}+1)^{2n}$ and which is divisible by 2^{n+1}.

Practice Problems [G]

1. Show that the integral part of $\left(6+\sqrt{35}\right)^n$ is odd, $n \in N$.
2. Show that the integral part of $\left(3\sqrt{3}+5\right)^{2n+1}$ is even, $n \in N$.
3. If $\left(3\sqrt{3}+5\right)^{2n+1} = p+f$ where p is an integer and f is a proper fraction then find the value of $f(p+f)$. $n \in N$
4. Find the greatest integer less than or equal to $\left(\sqrt{3}+1\right)^6$.
5. If $\left(6\sqrt{6}+14\right)^{2n+1} = N$ and F be the fractional part of N, prove that $NF = 20^{2n+1}$ $(n \in N)$
6. Let $R = \left(5\sqrt{3}+8\right)^{2n+1}$ and $f = R - [R]$, where [ ] denotes greatest integer function, prove that $Rf = 11^{2n+1}$ $(n \in N)$.

5.9 COMPARISON BETWEEN TWO NUMBERS

Binomial theorem can be used to compare two numbers. This is illustrated with the examples that follow.

➤ **Example 1.** Show that $(101)^{50} > (100)^{50} + (99)^{50}$

➤ ***Solution*** Since

$$(101)^{50} = (100+1)^{50} = 100^{50} + {}^{50}C_1 100^{49} + {}^{50}C_2 100^{48} + \ldots. + 1 \qquad ...(1)$$

and $$(99)^{50} = (100-1)^{50} = 100^{50} - {}^{50}C_1 100^{49} + {}^{50}C_2 100^{48} - \ldots. + 1 \qquad ...(2)$$

Subtracting (2) from (1), we get

$$(101)^{50} - (99)^{50} = 2\{{}^{50}C_1 100^{49} + {}^{50}C_3 100^{47} + \ldots.\}$$
$$= 2 \times {}^{50}C_1 100^{49} + {}^{50}C_3 100^{47} + \ldots.$$
$$= 100 \times 100^{49} + (\text{a positive number})$$
$$= (100^{50} + (\text{a positive number})$$
$$> 100^{50}$$

Hence $(101)^{50} - (99)^{50} > (100)^{50}$.

$(101)^{50} > (100)^{50} + (99)^{50}$.

➤ **Example 2.** Which is larger $(1.01)^{1000000}$ or 10,000?

➤ ***Solution*** Splitting 1.01 and using binomial theorem to write the first few terms we have

$(1.01)^{1000000} = (1+0.01)^{1000000}$

$= {}^{1000000}C_0 + {}^{1000000}C_1 (0.01)$ + other positive terms

$= 1 + 1000000 \times 0.01$ + other positive terms

$= 1 + 10000$ + other positive terms > 10000

Hence $(1.01)^{1000000} > 10000$

➤ **Example 3.** (i) If $a_n = \left(1+\frac{1}{n}\right)^n$ then prove that $2 < a_n < 3 \ \forall \ n \in N$.

(ii) Using binomial theorem, prove the inequality $n^{n+1} > (n+1)^n$, $n \geq 2$, $n \in N$

(iii) Using binomial theorem, prove the inequality $n^n > (n+1)^{n-1}$, $n \geq 3$, $n \in N$.

➤ ***Solution*** (i) We have $a_n = \left(1+\frac{1}{n}\right)^n$

$$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{1.2} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3} \cdot \frac{1}{n^3} + \ldots.$$

$$= 1 + 1 + \frac{\left(1+\frac{1}{n}\right)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \ldots.$$

$= 2 +$ positive quantity $(\because n \in N) \ \therefore \ a_n > 2 \qquad ...(1)$

www.pdfworld.net

www.pdfworld.net

Also $a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}$

$< 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + ... \infty$

$< 1 + \frac{1}{1-\frac{1}{2}} < 3$ ...(2)

from (1) and (2), we get $2 < a_n < 3$

(ii) From part (i) $\left(1+\frac{1}{n}\right)^n < 3$

$\Rightarrow \left(1+\frac{1}{n}\right)^n < n \Rightarrow \frac{(n+1)^n}{n^n} < n$

$\Rightarrow (n+1)^n < n^{n+1}$

(iii) From part (ii) $\frac{(n+1)^n}{n^n} < n$

$\Rightarrow n^n > \frac{(n+1)^n}{n^n} > \frac{(n+1)^n}{n+1}$

$\left(\text{since } \frac{1}{n} > \frac{1}{n+1}\right)$

$\Rightarrow n^2 > \frac{(n+1)^n}{(n+1)} \quad \Rightarrow n^n > (n+1)^{n-1}.$

➤ **Example 4.** Which of the two numbers $(1.000\,001)^{1\,000\,000}$ and 2 is greater ?

➤ ***Solution*** We have $(1.000001)^{1\,000\,000}$

$= \left(1+\frac{1}{1000\,000}\right)^{1000000} > 2$

which follows from the inequality $2 \le (1 + 1/n)^n < 3$.

➤ **Example 5.** Which of the two numbers 1000^{1000} and 10001^{999} is greater?

➤ ***Solution*** We obviously have

$\frac{(1001)^{999}}{(1000)^{1000}} = \left(\frac{1001}{1000}\right)^{1000} \cdot \frac{1}{1001}$

$= \left(1+\frac{1}{1000}\right)^{1000} \cdot \frac{1}{1001}$

$< 3 . \frac{1}{1001} < 1$ and consequently

$1000^{1000} > 1001^{999}$

➤ **Example 6.** Prove that for any integer $n > 6$ we have $\left(\frac{n}{2}\right)^n > n ! > \left(\frac{n}{3}\right)^n$

➤ ***Solution*** Let us suppose that the inequalities indicated in the condition of the problem hold for some n. To prove that they hold for n + 1 as well, it is sufficient to verify the validity of the following inequalities:

$\left(\frac{n+1}{2}\right)^{n+1} : \left(\frac{n}{2}\right)^n \ge n+1 \ge \left(\frac{n+1}{3}\right)^{n+1} : \left(\frac{n}{3}\right)^n$

On cancelling these inequalities by n + 1 we obtain the equivalent inequalities

$(1 + 1/n)^n/2 \ge 1 \ge (1 + 1/n)^n/3$ which follow from the inequality $2 \le (1 + 1/n)^n < 3$.

Now it only remains to note that the assertion of the problem holds for n = 6 because

$\left(\frac{6}{2}\right)^6 = 3^6 = 729, \quad 6 ! = 720$ and $\left(\frac{6}{3}\right)^6 = 2^6 = 64$

➤ **Example 7.** Which is larger 300! or $\sqrt{300^{300}}$?

➤ ***Solution*** From the previous example,

$300! > (100)^{300}$ ...(1)

Clearly $(100)^{150} > 3^{150}$

$\Rightarrow (100)^{150}(100)^{150} > 3^{150} .(100)^{150}$

$\Rightarrow (100)^{300} > (300)^{150}$

$\Rightarrow (100)^{300} > \sqrt{300^{300}}$ ...(2)

From (1) and (2), we get $300! > (100)^{300} > \sqrt{300^{300}}$

Hence $300! > \sqrt{300^{300}}$

Hence the larger number is 300!.

5.10 SUMMATION OF SERIES INVOLVING BINOMIAL COEFFICIENTS

Consider the following expansions :

$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$ ...(1)

$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$ ...(2)

Now, we shall discuss some techniques to obtain series involving the coefficients of these expansions.

www.pdfworld.net

www.pdfworld.net

Method of Particular Values

Here we put particular values of x on both sides of the expansion to obtain different series involving binomial coefficients. Further we add or substract two or more series to obtain new series.

(i) Putting x = 0, we get the constant term.

$C_0 = 1 \quad a_0 = 1$

(ii) Putting x = 1, we get the sum of all coefficients

$C_0 + C_1 + C_2 + ... + C_n = 2^n$ ...(1)

$a_0 + a_1 + a_2 + ... + a_{2n} = 3^n$ ...(2)

(iii) Putting x = –1, we get the sum of all coefficients with alternate signs

$C_0 - C_1 + C_2 - .... + (-1)^n C_n = 0$ ...(3)

$a_0 - a_1 + a_2 - ... + a_{2n} = 1$ ...(4)

(iv) Putting x = 1 and –1 and adding

$2(C_0 + C_2 + ...) = 2^n + 0 \Rightarrow C_0 + C_2 + ... = 2^{n-1}$ ...(5)

$2(a_0 + a_2 + ... + a_{2n}) = 3^n + 1$

$\Rightarrow \quad a_0 + a_2 + ... + a_{2n} = \dfrac{3^n - 1}{2}$ ...(6)

(v) Putting x = 1 and – 1 and substracting

$2(C_1 + C_3 + ...) = 2^n - 0 \Rightarrow C_1 + C_3 + ... = 2^{n-1}$ ...(7)

$2(a_1 + a_3 + ... + a_{2n-1}) = 3^n - 1$

$\Rightarrow \quad a_1 + a_3 + ... + a_{2n-1} = \dfrac{3^n - 1}{2}$ ...(8)

(vi) Putting x = i

$C_0 + C_1 i - C_2 - C_3 i + C_4 .... = (1 + i)^n$

$= 2^{n/2} \left(\cos \dfrac{n\pi}{4} + i \sin \dfrac{n\pi}{4}\right)$ ...(9)

$a_0 + a_1 i - a_2 - a_3 i + a_4 + ... = i^n$ ...(10)

(vii) Putting x = i and equating real parts on both sides

$C_0 - C_2 + C_4 - .... = \text{Re}(1 + i)^n = 2^{n/2} \cos \dfrac{n\pi}{4}$ ...(11)

$a_0 - a_2 + a_4 - .... = \text{Re}\, i^n = \text{Re}(e^{i\pi/2})^n = \cos \dfrac{n\pi}{2}$ ...(12)

(viii) Putting x = i and equating imaginary parts

$C_1 - C_3 + C_5 - .... = \text{Im}(1 + i)^n = 2^{n/2} \sin \dfrac{n\pi}{4}$ ...(13)

$a_1 - a_3 + a_5 - ..... = \text{Im}\, i^n = \sin \dfrac{n\pi}{2}$ ...(14)

(ix) Adding (5) and (9) and dividing by 2

$C_0 + C_4 + C_8 + ... = \dfrac{1}{2}\left(2^{n-1} + 2^{n/2} \cos \dfrac{n\pi}{4}\right)$

Substracting (5) and (9) and dividing by 2

$C_2 + C_6 + C_{10} + ... = \dfrac{1}{2}\left(2^{n-1} - 2^{n/2} \cos \dfrac{n\pi}{4}\right)$

(x) Adding (7) and (11) and dividing by 2

$C_1 + C_5 + C_9 + ... = \dfrac{1}{2}\left(2^{n-1} + 2^{n/2} \sin \dfrac{n\pi}{4}\right)$

(xi) Evaluating square of modulus on both sides of (9) and (10)

$(C_0 - C_2 + C_4 - ....)^2 + (C_1 + C_5 - .....)^2$

$= |(1 + i)^n|^2 = 2^n$

$(a_0 - a_2 + a_4 - ....)^2 + (a_1 - a_3 + a_5 - ....)^2$

$= |i^n|^2 = (|i|^2)^n = 1$

STUDY TIP

$(1 + i)^n = \left[\sqrt{2}\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)\right]^n = 2^{n/2}\left\{\cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4}\right\}^n$

$= 2^{n/2}\left\{\cos \dfrac{n\pi}{4} + i \sin \dfrac{n\pi}{4}\right\}$ using DeMoivre's Theorem.

➤ **Example 1.** Find the sum :

$${}^nC_0 + {}^nC_1 \frac{1}{3} + {}^nC_2 \frac{1}{3^2} + ...+ {}^nC_{n-1} \frac{1}{3^{n-1}} + {}^nC_n \frac{1}{3^n}$$

➤ ***Solution*** Consider the expansion $(1 + x)^n$

$= {}^nC_0 x^0 + {}^nC_1 x + {}^nC_2 x^2 + ...+ {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$

Put $x = \dfrac{1}{3}$ on both sides $(1 + \dfrac{1}{3})^n$

$= {}^nC_0 + {}^nC_1 \dfrac{1}{3} + {}^nC_2 \dfrac{1}{3^2} + ... + {}^nC_{n-1} \dfrac{1}{3^{n-1}} + {}^nC_n \dfrac{1}{3^n}$

Hence the sum is $\left(\dfrac{4}{3}\right)^n$.

➤ **Example 2.** If the sum of the coefficient in the expansion of $(\alpha x^2 + 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, then find the value of α.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$
= Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$
Putting x = y = 1
$\therefore \quad (\alpha - 1)^{35} = (1 - \alpha)^{35}$
$\Rightarrow \quad (\alpha - 1)^{35} = -(\alpha - 1)^{35}$
$\Rightarrow \quad 2(\alpha - 1)^{35} = 0$
$\therefore \quad \alpha - 1 = 0 \quad \therefore \alpha = 1$

➤ **Example 3.** If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + ... + a_{40}x^{40}$ then find the value of $a_0 + a_2 + a_4 + ... a_{38}$.
➤ ***Solution*** Since $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + ... + a_{38}x^{38} + a_{39}x^{39} + a_{40}x^{40}$
Putting x = 1 and x = –1, we get
$4^{20} = a_0 + a_1 + a_2 + ... + a_{38} + a_{39} + a_{40}$ ....(1)
$2^{20} = a_0 - a_1 + a_2 - ..... + a_{38} - a_{39} + a_{40}$ ....(2)
Adding (1) and (2), we get
$2^{20}(2^{20} + 1) = 2(a_0 + a_2 + ... + a_{38} + a_{40})$
$\therefore \quad 2^{19}(2^{20} + 1) = a_0 + a_2 + ... + a_{38} + a_{40}$
$\Rightarrow \quad a_0 + a_2 + ... + a_{38} = 2^{19}(2^{20} + 1) - a_{40}$ ...(3)
but a_{40} = coefficient of x^{40} in $(1 + x + 2x^2)^{20} = 2^{20}$
from (3), $a_0 + a_2 + ... + a_{38} = 2^{19}(2^{20} + 1) - 2^{20}$
$= 2^{19}(2^{20} - 1)$.

➤ **Example 4.** If $t_0, t_1, t_2, t_3, ...$ be the terms in the expansion of $(a + x)^n$, prove that
$(t_0 - t_2 + t_4 -)^2 + (t_1 - t_3 + t_5 - ...)^2 = (a^2 + x^2)^n$
➤ ***Solution*** $(a + x)^n$
$= a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + ... + {}^nC_nx^n$ ...(1)
$= t_0 + t_1 + t_2 + t_3 + ... + t_n$
$\therefore t_0 = a^n, \; t_1 = {}^nC_1a^{n-1}x, \; t_2 = {}^nC_2a^{n-2}x^2$ etc.
Now putting ix in place of x in (1), we get
$(a + ix)^n = a^n + {}^nC_1a^{n-1}(ix) + {}^nC_2a^{n-2}(ix)^2 + ... + {}^nC_n(ix)^n$
$= (a^n - {}^nC_2a^{n-2}x^2 + ...) + i\,({}^nC_1a^{n-1}x - {}^nC_3a^{n-2}x^3 + ...)$
$= (t_0 - t_2 + ...) + i(t_1 - t_3 + ...)$ ...(2)
Taking modulus and then squaring, we get
$(a^2 + x^2)^n = (t_0 - t_2 + ...)^2 + (t_1 - t_3 + ...)^2$

➤ **Example 5.** Prove that

$$ {}^nC_3 + {}^nC_7 + {}^nC_{11} + ...= \frac{1}{2}\left\{2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right\}. $$

➤ ***Solution*** In the given series difference of suffices is 4.
i.e. $7 - 1 = 11 - 7 = ... = 4$
Consider the expansion $(1 + x)^n$
$= {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$
Putting x = 1 and – 1 and substracting
$2({}^nC_1 + {}^nC_3 + ...) = 2^n - 0$
$\Rightarrow \quad {}^nC_1 + {}^nC_3 + ... = 2^{n-1}$ ...(1)
Putting x = i and equating imaginary parts

$$ {}^nC_1 - {}^nC_3 + {}^nC_5 - .... = \text{Im}(1 + i)^n = 2^{n/2}\sin\frac{n\pi}{4} \quad ...(2) $$

Substracting (1) and (2), $2({}^nC_3 + {}^nC_7 + {}^nC_{11} + ...)$

$$ = \left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right) $$

$$ \Rightarrow \quad {}^nC_3 + {}^nC_7 + {}^nC_{11} + ...= \frac{1}{2}\left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right) $$

➤ **Example 6.** Prove the inequality,

$$ \prod_{k=0}^{n} {}^nC_k < \left(\frac{2^n - 2}{n - 1}\right)^{n-1}, \; n \geq 4 . $$

➤ ***Solution*** To prove that

$$ C_0 \cdot C_1 \cdot C_2 \cdot ..... C_n < \left(\frac{2^n - 2}{n - 1}\right)^{n-1} $$

consider $C_1, C_2, ..., C_{n-1}$ and apply A.M. > G.M.

$$ \frac{C_1 + C_2 + ... + C_{n-1}}{n-1} > \left(C_1.C_2.....C_{n-1}\right)^{\frac{1}{n-1}} > \left(C_0.C_1.C_2.....C_n\right)^{\frac{1}{n-1}} $$

$$ \Rightarrow \quad \frac{2^n - 2}{n - 1} > \left(C_0.C_1.....C_n\right)^{\frac{1}{n-1}} $$

➤ **Example 7.** Prove that $\sum_{r=1}^{k}(-3)^{r-1}.{}^{3n}C_{2r-1} = 0$
where $k = \frac{3n}{2}$ and n is an even positive integer.
➤ ***Solution*** Given n is an even postive integer.
Let n = 2m $\quad \therefore k = 3m, m \in N$

$$ \text{L.H.S.} = \sum_{r=1}^{k}(-3)^{r-1}.{}^{3n}C_{2r-1} = \sum_{r=1}^{3m}(-3)^{r-1}.{}^{6m}C_{2r-1} $$

www.pdfworld.net

www.pdfworld.net

$= {}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2.{}^{6m}C_5 - .... (-3)^{3m-1}\,{}^{6m}C_{6m-1}$

Consider $(1+i\sqrt{3})^{6m}$

$= {}^{6m}C_0 + {}^{6m}C_1 (i\sqrt{3}) + {}^{6m}C_2 (i\sqrt{3})^2 + {}^{6m}C_3 (i\sqrt{3})^3$
$+ {}^{6m}C_4 (i\sqrt{3})^4 + {}^{6m}C_5 (i\sqrt{3})^5 .....$
$+ {}^{6m}C_{6m-1}(i\sqrt{3})^{6m-1} + {}^{6m}C_{6m}(i\sqrt{3})^{6m}$

$\Rightarrow 2^{6m}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{6m}$

$= {}^{6m}C_0 + {}^{6m}C_1 (i\sqrt{3}) + {}^{6m}C_2 (i\sqrt{3})^2 + {}^{6m}C_3 (i\sqrt{3})^3$
$+ {}^{6m}C_4 (i\sqrt{3})^4 + {}^{6m}C_5 (i\sqrt{3})^5 .....$
$+ {}^{6m}C_{6m-1}(i\sqrt{3})^{6m-1} + {}^{6m}C_{6m}(i\sqrt{3})^{6m}$

$\Rightarrow 2^{6m}[\cos 2\pi m + i \sin 2\pi m]$

$= ({}^{6m}C_0 - 3.{}^{6m}C_2 + 3^2.{}^{6m}C_4 - .... + (-3)^{3m}\,{}^{6m}C_{6m})$
$+ i\sqrt{3}\,({}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - .....$
$+ (-3)^{3m-1}\,{}^{6m}C_{6m-1})$

Comparing the imaginary part on both sides we get

$\sqrt{3}\,({}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - .... + (-3)^{3m-1}\,{}^{6m}C_{6m-1}) = 0$

$\Rightarrow {}^{6m}C_1 - 3.{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - .... + (-3)^{3m-1}\,{}^{6m}C_{6m-1} = 0$

$\Rightarrow \sum_{r=1}^{3m}(-3)^{r-1}.{}^{6m}C_{2r-1} = 0$

$\Rightarrow \sum_{r=1}^{k}(-3)^{r-1}.{}^{3n}C_{2r-1} = 0$

(where $n = 2m$).

Summation upto middle term

➤ **Example 8.** Sum ${}^{15}C_0 + {}^{15}C_1 + {}^{15}C_2 + ...{}^{15}C_7$

➤ ***Solution*** We have ${}^{15}C_0 + {}^{15}C_1 + ...{}^{15}C_7 + {}^{15}C_8 + ... + {}^{15}C_{14} + {}^{15}C_{15} = 2^{15}$

where ${}^{15}C_0 = {}^{15}C_{15}$; ${}^{15}C_1 = {}^{15}C_{14}$; ..... ; ${}^{15}C_7 = {}^{15}C_8$

using the formula ${}^nC_r = {}^nC_{n-r}$

Hence $2({}^{15}C_0 + {}^{15}C_1 + {}^{15}C_2 + ...{}^{15}C_7) = 2^{15}$

${}^5C_0 + {}^{15}C_1 + {}^{15}C_2 + ...{}^{15}C_7 = 2^{14}$

➤ **Example 9.** Evaluate

${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n$

➤ ***Solution*** We have

${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n$
$+ {}^{2n+1}C_{n+1} + ... + {}^{2n+1}C_{2n+1} = 2^{2n+1}$

Hence

$2({}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n) = 2^{2n+1}$

${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n = 2^{2n}$

➤ **Example 10.** Prove that ${}^{52}C_0 + {}^{52}C_1 + {}^{52}C_2 + ...{}^{52}C_{25} = \frac{1}{2}(2^{56} - {}^{52}C_{26})$

➤ ***Solution*** We have ${}^{52}C_0 + {}^{52}C_1 + ... {}^{52}C_{25} + {}^{15}C_{26} + {}^{15}C_{27} + ... + {}^{15}C_{51} + {}^{15}C_{52} = 2^{52}$

where ${}^{52}C_0 = {}^{52}C_{52}$; ${}^{52}C_1 = {}^{52}C_{51}$; ..... ; ${}^{15}C_{25} = {}^{15}C_{27}$

using the formula ${}^nC_r = {}^nC_{n-r}$

Hence $2({}^{52}C_0 + {}^{52}C_1 + {}^{52}C_2 + ...{}^{52}C_{25}) + {}^{15}C_{26} = 2^{15}$

${}^{52}C_0 + {}^{52}C_1 + {}^{52}C_2 + ...{}^{52}C_{25} = \frac{1}{2}(2^{56} - {}^{52}C_{26})$

Summation using the formula ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

➤ **Example 11.** Find the summation of the following series

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

➤ ***Solution*** Using formula, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + ... + {}^nC_m$

$(\because {}^mC_m = {}^{m+1}C_{m+1})$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + ... + {}^nC_m$

$= {}^{m+3}C_{m+1} + ... + {}^nC_m$

$= {}^nC_{m+1} + {}^nC_m = {}^{n+1}C_{m+1}$

Alternative

${}^mC_m + {}^{m+1}C_m + {}^{n+2}C_m + ... + {}^nC_m$

The above series can be obtained by writing the coefficient of x^m in

$(1+x)^m + (1+x)^{m+1} + ... + (1+x)^n$

Let $S = (1+x)^m + (1+x)^{m+1} + ... + (1+x)^n$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

$$= \frac{(1+x)^m[(1+x)^{n-m+1}-1]}{x} = \frac{(1+x)^{n+1}-(1+x)^m}{x}$$

The series sum is equal to the coefficient of x^m in

$$\frac{(1+x)^{n+1}-(1+x)^m}{x}$$

i.e. the coefficient of x^{m+1} in $(1+x)^{n+1} - (1+x)^m$
i.e. the coefficient of x^{m+1} in $(1+x)^{n+1} = {}^{n+1}C_{m+1}$
Hence, the required sum is ${}^{n+1}C_{m+1}$.

➤ **Example 12.** Find the sum to 'n' terms of the series $(m \in N)$,

$$m! + \frac{(m+1)!}{1!} + \frac{(m+2)!}{2!} + \frac{(m+3)!}{3!} + \ldots.$$

➤ ***Solution*** $S = m! + \frac{(m+1)!}{1!} + \frac{(m+2)!}{2!} + \ldots.. + \frac{(m+n-1)!}{(n-1)!}$

$$= m!\left[{}^{m+1}C_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + \ldots. + {}^{m+n-1}C_{n-1}\right]$$

$$= m!\left[{}^{m+1}C_1 + {}^{m+2}C_2 + \ldots..\right]$$

$$= m!\left[{}^{m+3}C_2 + \ldots. \text{ and so on}\right] = m!\left[{}^{m+n}C_{n-1}\right]$$

$$= m!\ \frac{(n+m)!}{(n-1)!\ (m+1)!} = \frac{(m+n)!}{(m+1)\ .\ (n-1)!}$$

Summation using Series Sum

In some series, each of the binomial coefficients are picked as coefficient of certain power of x in the some binomial expansions. We first select such binomial expansions which can be summed easily by the methods of summing a G.P, A.G.P or binomial theorem itself. Then we find the coefficient of the same power of x from the sum of various terms.

➤ **Example 13.** Find the coefficient of x^{50} in the polynomials

(i) $(1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \ldots. + x^{1000}$

(ii) $(1+x) + 2(1+x)^2 + 3(1+x)^3 + \ldots. + 1000(1+x)^{1000}$

➤ ***Solution*** (i) Using the formula for the sum of a geometric progression we find

$$(1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \ldots. + x^{1000}$$

$$= \frac{(1+x)^{1000}\left[1-\left(\frac{x}{1+x}\right)^{1001}\right]}{\left[1-\left(\frac{x}{1+x}\right)\right]} = \frac{(1+x)^{1000} - \frac{x^{1001}}{(1+x)}}{\frac{x+1-x}{(1+x)}}$$

$$= (1+x)^{1001} - x^{1001}$$

Hence the coefficient of $x^{50} = {}^{1001}C_{50} = \frac{1001!}{50!\,951!}$

(ii) Let $P(x) = (1+x) + 2(1+x)^2 + 3(1+x)^3 + \ldots. + 999(1+x)^{999} + 1000(1+x)^{1000}$

Now $(1+x)P(x) - P(x)$

$$= [(1+x)^2 + 2(1+x)^3 + 3(1+x)^4 + \ldots.. + 999(1+x)^{1000} + 1000(1+x)^{1001}]$$
$$-[(1+x) + 2(1+x)^2 + 3(1+x)^3 + \ldots. + 999(1+x)^{999} + 1000(1+x)^{1000}]$$

$$= 1000(1+x)^{1001} - [(1+x) + (1+x)^2 + (1+x)^3 + \ldots. + (1+x)^{1000}]$$

$$= 1000(1+x)^{1001} - \left[\frac{(1+x)^{1001}-(1+x)}{(1+x)-1}\right]$$

$$= 1000(1+x)^{1001} - \left[\frac{(1+x)^{1001}-(1+x)}{x}\right]$$

$$\therefore P(x) = \frac{1000(1+x)^{1001}}{x} - \frac{(1+x)^{1001}}{x^2} + \frac{(1+x)}{x^2}$$

The coefficient of x^{50} in P(x)

$$= (1000)^{1001}C_{51} - {}^{1001}C_{52}$$

$$= \frac{(1000)1001!}{51!\,950!}[52000-950] = \frac{(51050)1001!}{52!\,950!}$$

➤ **Example 14.** Find the coefficient in x^m in the expansion of the expression
$(1+x)^k + (1+x)^{k+1} + \ldots.. + (1+x)^n$ in powers of x.
Consider the cases $m < k$ and $m \geq k$.

➤ ***Solution*** The summands of the given sum form a progression with common ratio 1 + x. Therefore,

$$(1+x)^k + (1+x)^{k+1} + \ldots. + (1+x)^n = \frac{(1+x)^{n+1}-(1+x)^k}{x} \ldots(1)$$

Writing the sum in the form of a polynomial
$a_0 + a_1x + \ldots. + a_mx^m + \ldots.. + a_nm^n$,

www.pdfworld.net

www.pdfworld.net

and removing brackets in the right hand member of equality (1) we see that if $m < k$, then

$a_m = {}^{n+1}C_{m+1} - {}^{k}C_{m+1}$,

and if $m \geq k$, then $a_m = {}^{n+1}C_{m+1}$

➤ **Example 15.** Find the sum of the series

${}^{2n}C_m + {}^{2n-1}C_m + {}^{2n-2}C_m + \ldots\ldots + {}^{m}C_m$, $n < m < 2n$.

➤ ***Solution*** We observe that ${}^{2n}C_m$ is the coefficient of x^m in the expansion of $(1 + x)^{2n}$.

Similarly, ${}^{2n-1}C_m$ is the coefficient of x^m in the expansion of $(1 + x)^{2n-1}$

.....

${}^{n}C_m$ is the coefficient of x^m in the expansion of $(1 + x)^n$

Hence ${}^{2n}C_m + {}^{2n-1}C_m + {}^{2n-2}C_m + \ldots\ldots + {}^{n}C_m$ is equal to the coefficient of x^m in $(1 + x)^{2n} + (1 + x)^{2n-1}$

$+ \ldots\ldots\ldots + (1 + x)^n$

It is also equal to the coefficient of x^m in

$(1 + x)^{2n} + (1 + x)^{2n-1} + \ldots\ldots\ldots + (1 + x)^m$

$+ (1 + x)^{m-1} + \ldots\ldots\ldots + (1 + x) + 1.$

(This is done to obtain the sum easily, without any effect on sum)

Hence, it is equal to the coefficient of x^m in

$\dfrac{(1+x)^{2n+1} - 1}{1 + x - 1}$ (G.P. summation)

which is same as the coefficient of x^{m+1} in $(1 + x)^{2n+1}$ which is ${}^{2n+1}C_{m+1}$.

Thus, ${}^{2n}C_m + {}^{2n-1}C_m + {}^{2n-2}C_m + \ldots\ldots + {}^{m}C_m = {}^{2n+1}C_{m+1}$.

➤ **Example 16.** Find the summation of the following series

${}^{n}C_3 + 2 \,.\, {}^{n+1}C_3 + 3 \,.\, {}^{n+2}C_3 + \ldots\ldots\ldots + n \,.\, {}^{2n-1}C_3$

➤ ***Solution*** The series

${}^{n}C_3 + 2 \,.\, {}^{n+1}C_3 + 3 \,.\, {}^{n+2}C_3 + \ldots\ldots\ldots + n \,.\, {}^{2n-1}C_3$

can be obtained by finding the coefficient of x^3 in

$(1+x)^n + 2.(1+x)^{n+1} + 3.(1+x)^{n+2} + \ldots\ldots + n.(1+x)^{2n-1}$

Let $S = (1 + x)^n + 2 \,.\, (1 + x)^{n+1} + 3 \,.\, (1 + x)^{n+2}$

$+ \ldots\ldots\ldots + n\,(1 + x)^{2n-1}$ ...(1)

$(1 + x)S = (1 + x)^{n+1} + 2 \,.\, (1 + x)^{n+2} + \ldots\ldots\ldots$

$+ (n - 1)\,(1 + x)^{2n-1} + n\,(1 + x)^{2n}$ ...(2)

Subtracting (2) from (1)

$-xS = (1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + \ldots\ldots\ldots$

$+ (1 + x)^{2n-1} - n\,(1 + x)^{2n}$

$= \dfrac{(1 + x)^n \left[(1 + x)^n - 1\right]}{x} - n\,(1 + x)^{2n}$

$S = \dfrac{-(1 + x)^{2n} + (1 + x)^n}{x^2} + \dfrac{n(1 + x)^{2n}}{x}$

Hence the coefficient of x^3 in S

$= \dfrac{-(1 + x)^{2n} + (1 + x)^n}{x^2} + \dfrac{n(1 + x)^{2n}}{x}$ is

$-{}^{2n}C_5 + {}^{n}C_5 + n \,.\, {}^{2n}C_4$.

➤ **Example 17.** Sum the series

${}^{1000}C_{50} + 2 \,.\, {}^{999}C_{49} + 3 \,.\, {}^{998}C_{48} + \ldots\ldots + 51 \,.\, {}^{950}C_0$.

➤ ***Solution*** Sum $= {}^{1000}C_{950} + 2 \,.\, {}^{998}C_{950} + \ldots\ldots$

$+ 51 \,.\, {}^{950}C_{950}$ $\quad (\because {}^{n}C_r = {}^{n}C_{n-r})$

= coefficient of x^{950} in $\{(1 + x)^{1000} + 2(1 + x)^{999}$

$+ 3(1 + x)^{998} + \ldots\ldots + 51.(1 + x)^{950}\}$.

Let $S = (1 + x)^{1000} + 2(1 + x)^{999} + 3(1 + x)^{998} + \ldots\ldots$

$+ 51(1 + x)^{950}$

$\dfrac{S}{1+x} = (1 + x)^{999} + 2(1 + x)^{998} + 3(1 + x)^{997} + \ldots\ldots$

$+ 50(1 + x)^{950} + 51(1 + x)^{949}$

Subtracting,

$S\left(1 - \dfrac{1}{1+x}\right) = (1 + x)^{1000} + (1 + x)^{999} + (1 + x)^{998} +$

$\ldots\ldots + (1 + x)^{950}\} - 51(1 + x)^{949}$

$= (1 + x)^{950} \,.\, \dfrac{\{1 - (1 + x)^{51}\}}{1 - (1 + x)} - 51(1 + x)^{949}$

or $\dfrac{x}{1+x}.S = (1 + x)^{950} \,.\, \dfrac{(1 + x)^{51} - 1}{x} - 51(1 + x)^{949}$

$\therefore\ S = \dfrac{(1 + x)^{951}}{x^2}\{(1 + x)^{51} - 1\} - 51.\dfrac{(1 + x)^{950}}{x}$

$\therefore$ The required sum = coeffcient of x^{950} in

$\left[\dfrac{(1 + x)^{951}}{x^2}.\{(1 + x)^{51} - 1\} - 51.\dfrac{(1 + x)^{950}}{x}\right]$

= coefficient of x^{952} in $(1 + x)^{951} \,.\, \{(1 + x)^{51} - 1\}$

$- 51 \times$ coefficient of x^{951} in $(1 + x)^{950}$

= coefficient of x^{952} in $\{(1 + x)^{1002} - (1 + x)^{951}\} - 51 \times 0$

$= {}^{1002}C_{952} = {}^{1002}C_{50}$.

➤ **Example 18.** If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ and $a_k = 1$ for all $k \geq n$, then show that $b_n = {}^{2n+1}C_{n+1}$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** $\because \sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$

$y = x - 3 \Rightarrow y + 1 = x - 2$

So the given expression reduces to :

$$\sum_{r=0}^{2n} a_r (1+y)^r = \sum_{r=0}^{2n} b_r y^r$$

$\Rightarrow \quad a_0 + a_1 (1+y) + a_2 (1+y)^2 + ... + a_{2n}(1+y)^{2n}$
$= b_0 + b_1 y + ... + b_{2n} y^{2n}$

using $a_k = 1$ for all $k \geq n$, then we get

$\Rightarrow a_0 + a_1 (1+y) + a_2 (1+y)^2 + ... + a_{n-1}(1+y)^{n-1}$
$+ (1+y)^n + (1+y)^{n+1} + ... + (1+y)^{2n}$
$= b_0 + b_1 y + ... + b_n y^n + ... + b_{2n} y^{2n}$

Compare the coefficients of y^n on both sides, we get

${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + ... + {}^{2n}C_n = b_n$

$\Rightarrow \quad {}^{n+1}C_{n+1} + {}^{n+1}C_n + {}^{n+2}C_n + ... + {}^{2n}C_n = b_n$

(adding the first two terms using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)

$\Rightarrow \quad {}^{n+2}C_{n+1} + {}^{n+2}C_n + ... + {}^{2n}C_n = b_n$

If we continue combining the terms on L.H.S. we finally get ${}^{2n+1}C_{n+1} = b_n$

Alternative

b_n = co-efficient of y^n in $(1+y)^n + (1+y)^{n+1} + ... + (1+y)^{2n}$

$$= \text{co-efficient of } y^n \text{ in } \sum_{r=n}^{2n} (y+1)^r$$

$$= \text{co-efficient of } y^n \text{ in } \frac{[(y+1)^{n+1} - 1](y+1)^n}{(y+1) - 1}$$

$$= \text{co-efficient of } y^n \text{ in } \frac{[(y+1)^{2n+1} - (y+1)^n]}{y},$$

co-efficient of y^{n+1} in $(y+1)^{2n+1} - (y+1)^n = {}^{2n+1}C_{2n+1}$

Reversal Method

➤ **Example 19.** Prove that

(i) $C_1 + 2C_2 + 3C_3 + ... + C_n = n.2^{n-1}$

(ii) $C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n = (n+2)\, 2^{n-1}$

➤ ***Solution*** (i) Denoting the given sum by S, and writing the series on the left in the given order as well as in the reverse order, we have

$S = C_1 + 2C_2 + ...nC_n,$

$S = nC_n + (n-1)C_{n-1} + ...+C_n,$

By adding we get

$2S = nC_0 + nC_1 + ... + nC_n,$ since $C_r = C_{n-r}$,

$= n(C_0 + C_1 + C_2 + ... + C_n),$

$= n.\, 2^n,$ so that $S = n.2^{n-1}$.

(ii) Proceeding as above, and denoting the given sum by S', we have

$2S' = (n+2)\,[C_0 + C_1 + ... + C_n],$

$= (n+2)\,.\,2^n,$

$S' = (n+2).\, 2^{n-1}.$

➤ **Example 20.** Evaluate the sum

$S = 3\,.\,{}^nC_1 + 7\,.\,{}^nC_2 + 11\,.\,{}^nC_3 + ... + (4n-1)\,{}^nC_n$

➤ ***Solution*** Let $y = (-1)\,.\,{}^nC_0 + 3\,.\,{}^nC_1 + 7\,.\,{}^nC_2 + ... + (4n-1)\,.\,{}^nC_n$ where $y = S - 1$

Also $y = (4n-1)\,{}^nC_0 + (4n-5)\,{}^nC_1 + ... + (-1)\,{}^nC_n$

$2y = (4n-2)\,[\,C_0 + C_1 + C_2 + ... + C_n\,]$

$\Rightarrow \quad y = (2n-1)\,2^n \quad \Rightarrow \quad S = 1 + (2n-1)\,2^n$

STUDY TIP

Let $S = \sum_{r=0}^{n} t_r = t_0 + t_1 + t_2 + ... + t_{n-1} + t_n$

This is also equal to

$t_n + t_{n-1} + ... + t_2 + t_1 + t_0 = \sum_{r=0}^{n} t_{n-r}$

Hence $\sum_{r=0}^{n} t_r = \sum_{r=0}^{n} t_{n-r}$

Since ${}^nC_{n-r} = {}^nC_r$, the above manipulation helps in summing several series.

➤ **Example 21.** Evaluate the sum

$$\sum_{k=0}^{n} {}^nC_k \sin kx \cos(n-k)x.$$

➤ ***Solution*** Let $S = \sum_{k=0}^{n} {}^nC_k \sin kx \cos(n-k)x$

$$S = \sum_{k=0}^{n} {}^nC_{n-k} \sin(n-k)x \cos kx$$

$$\Rightarrow \quad 2S = \sum_{k=0}^{n} {}^nC_k (\sin kx \cos(n-k)x + \sin(n-k)x \cos kx)$$

www.pdfworld.net

www.pdfworld.net

$$= \sum_{k=0}^{n} {}^nC_k \sin(kx + (n-k)x) = \sum_{k=0}^{n} {}^nC_k \sin nx$$

$$= \sin nx \cdot \sum_{k=0}^{n} {}^nC_k = 2^n \sin nx$$

Hence $S = 2^{n-1} \sin nx$.

➤ **Example 22.** If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then prove that $\sum_{r=0}^{n} \frac{r}{{}^nC_r} = \frac{1}{2} n.a_n$

➤ ***Solution*** We have $S = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_{n-r}}$

$$= \sum_{r=0}^{n} \frac{n}{{}^nC_r} - \sum_{r=0}^{n} \frac{r}{{}^nC_r} = n.a_n - S$$

Hence $2S = n.\, a_n \Rightarrow S = \frac{1}{2} n.a_n$

Practice Problems [H]

1. Which is larger, $(101\,!)^{100}$ or $(100!)^{101}$?
2. Prove that the sum of all the coefficients of the expansion of $(2y - x)^k$ is equal to 1 for any natural k.
3. Find the sum of the coefficients of the given polynomial;
   (i) $(2x-1)^{100}$
   (ii) $(x^3 - x - 1)^{99}$
4. Find the sum of the coefficients of all the terms containing odd powers of 'x' in the expansion of $(2x + 3y - z)^{20}$.
5. The sum of the binomial coefficients in an expansion is equal to 1024. Find the term of the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ which contains x raised to the eleventh power.
6. Compute the indicated sums :

   (i) $\sum_{k=2}^{n-2} {}^nC_k$ (ii) $\sum_{k=0}^{n} (-1)^k \, {}^nC_k$

   (iii) $\sum_{k=0}^{n} {}^{2n}C_{2k}$ (iv) $\sum_{k=1}^{n} {}^{2n}C_{2k-1}$

7. Prove that the equality

   $$1 - 10\binom{2n}{1} + 10^2\binom{2n}{2} - 10^3\binom{2n}{3} + ... - 10^{2n-1}\binom{2n}{1} + 10^{2n} = (81)^n \text{ holds true.}$$

8. Prove that

   $$\binom{n}{m} + \binom{n-1}{m} + ... + \binom{n-10}{m} = \binom{n+1}{m+1} - \binom{n-10}{m+1}$$

9. If $\sum_{r=0}^{n} \{a_r . (x - \alpha + 2)^r - b_r (\alpha - x - 1)^r\} = 0$, then prove that $b_n - (-1)^n a_n = 0$.

5.11 ALGEBRAIC METHOD

Let us consider the summation of the series $\sum_{r=0}^{n} f(r) . {}^nC_r$, where f(r) is a polynomial. We need to eliminate r in the multiplier of binomial coefficient f(r) using $r \cdot {}^nC_r = n \, {}^{n-1}C_{r-1}$

➤ **Example 1.** Sum $C_1 + 2C_2 + 3C_3 + ... + ... + n.\, C_n$

➤ ***Solution*** The rth term is given by $t_r = r. \, {}^nC_r = n \cdot {}^{n-1}C_{r-1}$

Now $C_1 + 2C_2 + 3C_3 + ... + ... + n.\, C_n = \sum_{r=1}^{n} r. {}^nC_r$

$$= \sum_{r=1}^{n} n. {}^{n-1}C_{r-1} = n \sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$$= n[{}^{n-1}C_0 + {}^{n-1}C_1 + ... + {}^{n-1}C_{n-1}]$$

www.pdfworld.net

www.pdfworld.net

$= n(1+1)^{n-1} = n \cdot 2^{n-1}$

Hence, $\sum_{r=1}^{n} r.{}^{n}C_r = n \cdot 2^{n-1}$

➤ **Example 2.** Sum $C_0 + 2.C_1 + 3.C_2 + ... + (n+1).C_n$

➤ ***Solution*** $C_0 + 2.C_1 + 3.C_2 + ... + (n+1).C_n$

$= \sum_{r=0}^{n}(r+1).C_r = \sum_{r=0}^{n} r.C_r + \sum_{r=0}^{n} C_r$

$= n \cdot 2^{n-1} + 2^n = (n+2)2^{n-1}$

➤ **Example 3.** Sum

$C_1 - 2.C_2 + 3.C_2 - 4.C_4 + ... + (-1)^{n-1} n.C_n$

➤ ***Solution*** The rth term is given by

$t_r = (-1)^{r-1} . r . {}^{n}C_r = (-1)^{r-1}. n. {}^{n-1}C_n$

Now $C_1 - 2.C_2 + 3.C_3 - 4.C_4 + ... + (-1)^{n-1}. n.{}^{n}C_n$

$\sum_{r=1}^{n} t_r = \sum_{r=1}^{n}(-1)^{r-1}.n.{}^{n-1}C_{r-1} = n\sum_{r=1}^{n}(-1)^{r-1}.{}^{n-1}C_{r-1}$

$= n({}^{n-1}C_0 - {}^{n-1}C_1 + {}^{n-1}C_2 - {}^{n-1}C_3 + ... + (-1)^{n-1}.{}^{n-1}C_{n-1}$

$= n(1-1)^{n-1} = 0.$

➤ **Example 4.** Sum

${}^{2n}C_0 + 3.{}^{2n}C_1 + 5.{}^{2n}C_2 + ... + (4n+1).{}^{2n}C_{2n}$

➤ ***Solution*** ${}^{2n}C_0 + 3.{}^{2n}C_1 + 5.{}^{2n}C_2 + ... + (4n+1).{}^{2n}C_{2n}$

$= \sum_{r=0}^{2n}(2r+1).{}^{2n}C_r$

$= 2\sum_{r=0}^{2n} r.{}^{2n}C_r + \sum_{r=0}^{2n} {}^{2n}C_r = 2\sum_{r=1}^{2n} 2n.{}^{2n-1}C_{r-1} + 2^{2n}$

$= 4n \cdot 2^{2n-1} + 2^{2n} = (2n+1)2^{2n}.$

➤ **Example 5.** If $a_1, a_2, a_3, ..., a_{n+1}$ are in A.P. show that $\sum_{k=0}^{n} {}^{n}C_k a_{k+1} = 2^{n-1}(a_1 + a_{n+1})$.

➤ ***Solution*** Let $a_1 = a, a_2 = a + d, ..., a_{n+1} = a + nd.$

$\sum_{k=0}^{n} {}^{n}C_k a_{k+1} = \sum_{k=0}^{n} {}^{n}C_k (a + kd) = a\sum_{k=0}^{n} {}^{n}C_k$

$+ d\sum_{k=0}^{n} k.{}^{n}C_k = (2a + nd)2^{n-1},$

since $\sum_{k=0}^{n} {}^{n}C_k = 2^n$, $\sum_{k=0}^{n} k.{}^{n}C_k = n.2^{n-1}.$

Alternative : Write the series in the reverse order and add to the original series.

➤ **Example 6.** Sum $C_1 + 2^2 C_2 + 3^2 C_3 + ... + n^2 C_n$

➤ ***Solution*** The rth term is given by $t_r = r^2.{}^{n}C_r$

$= [r(r-1) + r]{}^{n}C_r$

$= n(r-1).{}^{n-1}C_{r-1} + r.{}^{n}C_r$

$= n(n-1).{}^{n-2}C_{r-2} + n.{}^{n-1}C_{r-1}$

Now $C_1 + 2^2 C_2 + 3^2 C_3 + ... + n^2 C_n$

$= \sum_{r=1}^{n} r^2.{}^{n}C_r = n(n-1)\sum_{r=2}^{n} {}^{n-2}C_{r-2} + \sum_{r=1}^{n} n.{}^{n-1}C_{r-1}$

$= n(n-1)2^{n-2} + n \cdot 2^{n-1} = n(n+1)2^{n-2}$

Hence, $\sum_{r=1}^{n} r^2.{}^{n}C_r = n \cdot n(n+1)2^{n-2}$

➤ **Example 7.** Show that, $C_0(0-n)^2 + C_1(2-n)^2 + C_2(4-n)^2 + ... + C_n(2n-n)^2 = n \cdot 2^n$.

➤ ***Solution*** $\sum_{r=0}^{n} C_r (2r-n)^2$

$= 4\sum_{r=0}^{n} r^2 . C_r - 4n\sum_{r=0}^{n} r . C_r + n^2 \sum_{r=0}^{n} C_r$

$= 4n(n+1) \cdot 2^{n-2} - 4n \cdot n\, 2^{n-1} + n^2 \cdot 2^n = n \cdot 2^n$

➤ **Example 8.** Sum the series

$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1}$

➤ ***Solution*** The (r + 1)th term is given by

$t_{r+1} = \frac{{}^{n}C_r}{r+1}$

Now, $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1}$

$= \sum_{r=0}^{n} \frac{{}^{n}C_r}{r+1} = \sum_{r=0}^{n} \frac{{}^{n+1}C_{r+1}}{n+1}$ since $\frac{{}^{n}C_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$

$= \frac{1}{n+1}({}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1})$

$= \frac{1}{n+1}({}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 ..... + {}^{n+1}C_{n+1} - {}^{n+1}C_0)$

$= \frac{1}{n+1}(2^{n+1} - 1) = \frac{2^{n+1} - 1}{n+1}$

Hence $\sum_{r=0}^{n} \frac{{}^{n}C_r}{r+1} = \frac{2^{n+1} - 1}{n+1}$

www.pdfworld.net

www.pdfworld.net

➤ **Example 9.** Sum the series $\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+.....$

➤ ***Solution*** The rth term is given by $t_r = \frac{{}^nC_{2r-1}}{2r}$

$$= \frac{{}^nC_{2r-1}}{(2r-1)+1} = \frac{{}^{n+1}C_{2r}}{n+1}$$

Now $\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+.....=\sum_{r=1} t_r = \frac{1}{n+1}\sum_{r=1} {}^{n+1}C_{2r}$

$$= \frac{1}{n+1}\,[{}^{n+1}C_2 + {}^{n+1}C_4 + {}^{n+1}C_6 + ...]$$

$$= \frac{1}{n+1}\,[{}^{n+1}C_0 + {}^{n+1}C_2 + {}^{n+1}C_4 + ...) - {}^{n+1}C_0]$$

$$= \frac{1}{n+1}\,(2^n - 1)$$

$[\because\ {}^nC_0 + {}^nC_2 + {}^nC_4 + ... = 2^{n-1}]$

➤ **Example 10.** Evaluate $\sum_{r=1}^{n} \frac{{}^{2n}C_{2r-1}}{r}$.

➤ ***Solution*** $\sum_{r=1}^{n} \frac{{}^{2n}C_{2r-1}}{r} = \sum_{r=1}^{n} \frac{1}{r}\frac{2r}{2n+1}\,{}^{2n+1}C_{2r}$

$$\left(\text{As}\quad {}^{2n}C_{2r-1} = \frac{2r}{2n+1}\,{}^{2n+1}C_{2r}\right)$$

$$= \frac{2}{2n+1}\,({}^{2n+1}C_2 + {}^{2n+1}C_4 + ... {}^{2n+1}C_{2n})$$

$$= \frac{2}{2n+1}\,({}^{2n+1}C_0 + {}^{2n+1}C_2 + ... {}^{2n+1}C_{2n} - 1)$$

$$= \frac{2}{2n+1}\left(\frac{2^{2n+1}}{2} - 1\right) = \frac{2}{2n+1}\,(2^{2n} - 1).$$

➤ **Example 11.** Evaluate

$$\frac{C_0}{1.2}+\frac{C_1}{2.3}+\frac{C_2}{3.4}+.....+\frac{C_n}{(n+1)(n+2)}$$

➤ ***Solution*** The (r + 1)th term is given by t_{r+1}

$$= \frac{{}^nC_r}{(r+1)(r+2)}$$

$$= \frac{{}^{n+1}C_{r+1}}{(n+1)(r+2)} = \frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)}$$

Now $\frac{C_0}{1.2}+\frac{C_1}{2.3}+\frac{C_2}{3.4}+.....+\frac{C_n}{(n+1)(n+2)}$

$$= \sum_{r=0}^{n} \frac{{}^nC_r}{(r+1)(r+2)} = \sum_{r=0}^{n} \frac{{}^nC_r}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}\,({}^{n+2}C_2 + {}^{n+2}C_3 + {}^{n+2}C_4 + ... + {}^{n+2}C_{n+2})$$

$$= \frac{1}{(n+1)(n+2)}\,(2^{n+2} - {}^{n+2}C_0 - {}^{n+2}C_1)$$

$$= \frac{2^{n+2} - n - 3}{(n+1)(n+2)}$$

5.12 DIFFERENTIATION METHOD

We use some of the following tips along with differentiation to obtain the sum $\sum_{r=0}^{n} f(r).{}^nC_r$

(i) If the last term of the series leaving the plus or minus sign is m, multipled by some binomial coefficient, then express m as (an + b) where a be the quotient and b the remainder when m is divided by n. then replace x by x^a in the expansion $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ and multiply both sides of the relation by x^b.

(ii) Now differentiate both sides w.r.t. x and put x = 1, –1 or i according to the given series.

(iii) If product of two numerals (or square of numerals) or three numerals (or cube of numerals) also appear then differentiate twice or thrice.

➤ **Example 1.** Prove that

$C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n = (n+2)2^{n-1}$

➤ ***Solution*** Here last term of $C_0 + 2C_1 + 3C_2 + ...$ $+ (n+1)\,C_n$ is $(n+1)C_n$ then n + 1 = n.1 + 1

Thus a = 1 and b = 1

Consider $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

Now replacing x by x^1 and multiplying both sides by x^1, then

$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_nx^{n+1}$

Now differentiating both sides w.r.t. x, we get

www.pdfworld.net

www.pdfworld.net

$x \cdot n(1+x)^{n-1} + (1+x)^n \cdot 1$
$= C_0 + 2C_1x + 3C_2x^2 + ... + (n+1)C_nx^n$
Putting x = 1, we get
$n(2)^{n-1} + 2^n = C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n$
or $C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n = (n+2)2^{n-1}$

➤ **Example 2.** Prove that
$C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n = (n+1)2^n$
➤ ***Solution*** Here last term of $C_0 + 3C_1 + 5C_2 + ...$ $+ (2n+1)C_n$ is $(2n+1)C_n$ i.e. $(2n+1)$ and last term with positive sign, then $2n+1 = n.2+1$
Thus a = 2 and b = 1
Consider $(1+x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$,
Now replacing x by x^2
$(1+x^2)^n = C_0x + C_1x^3 + C_2x^5 + ...+ C_nx^{2n+1}$
Then differentiating both sides w.r. to x, we get
$x \cdot n(1+x^2)^{n-1} \cdot 2x + (1+x^2)^n \cdot 1$
$= C_0 + 3C_1x^2 + 5C_2x^4 + ...+ (2n+1)C_nx^{2n}$
Putting x = 1, then we get
$n \cdot 2^{n-1} \cdot 2 + 2^n = C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n$
or $C_0 + 3C_1 + 5C_2 + ... + (2n+1)C_n = (n+1)2^n$.

➤ **Example 3.** If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3$ $+...+ C_nx^n$, then prove that
$C_0 - 2C_1 + 3C_2 - 4C_3 + ... + (-1)^n(n+1)C_n = 0$
➤ ***Solution*** The last term is $(-1)^n(n+1)C_n$ and $n = n \cdot 1 + 1$ Here a = 1 and b = 1
Consider $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 +...+$ C_nx^n,
Multiplying both sides by x, then,
$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + C_3x^4 +...+ C_nx^{n+1}$,
Differentiating both sides w.r.t x then we get
$x.n(1+x)^{n-1} + (1+x)^n \cdot 1$
$= C_0 + 2C_1x + 3C_2x^2 + 4C_3x^3 +...+ (n+1)C_nx^n$,
Putting x = –1, then we get
$0 = C_0 - 2C_1 + 3C_2 - 4C_3 +...+ (-1)^n(n+1)C_n$
$C_0 - 2C_1 + 3C_2 - 4C_3 +...+ (-1)^n(n+1)C_n = 0$

➤ **Example 4.** If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3$ $x^3 + ... + C_nx^n$, then prove that

$C_0 - 3C_1 + 5C_2 - ..... + (-1)^n(2n+1)C_n = 0$

➤ ***Solution*** The numerical value of last term is $(2n+1)$ and $2n+1 = 2.n+1$
Here a = 2 and b = 1
Consider $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ... +$ C_nx^n
Replacing x by x^2 we get
$(1+x^2) = C_0 + C_1x^2 + C_2x^4 + ... + C_nx^{2n}$
Multiplying both sides by x, then
$x(1+x^2)^n = C_0x + C_1x^3 + C_2x^5 + ... + C_nx^{2n+1}$
Differentiating both sides w.r. to x , we get
$x.n(1+x^2)^{n-1}\,2x + (1+x^2)^n.1$
$= C_0 + 3C_1x^2 + 5C_2x^4 + ... + (2n+1)C_nx^{2n}$
Putting x = i on both sides, we get
$0 + 0 = C_0 - 3C_1 + 5C_2 - ..... + (2n+1)(-1)^nC_n$
or $C_0 - 3C_1 + 5C_2 - .... + (-1)^n(2n+1)C_n = 0$

➤ **Example 5.** If $(1+x)^n = C_0 + C_1x + C_2x^2 + ...$ $+ C_nx^n$ then evaluate $C_1 - 3C_3 + 5C_5 - .....$

➤ ***Solution*** $(1+x)^n$
$= C_0 + C_1x + C_2x^2 + C_3x^3 + ...+ C_nx^n$ ... (1)
Differentiating (1) w.r.t. x
$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + 5C_5x^4 +....(2)$
Put x = i in equation (2)
$n(1+i)^{n-1}$
$= C_1 + 2C_2i - 3C_3 - 4C_4i + 5C_5 + 6C_6i....$ ...(3)
Equating real part of (3)

$C_1 - 3C_3 + 5C_5$ ..... real part of $n(\sqrt{2}e^{(in\pi/4)})^{n-1}$

$= n \cdot 2^{\frac{n-2}{2}} \cdot \cos\dfrac{(n-1)\pi}{4}$

➤ **Example 6.** Prove that $C_1 + 2^2C_2 + 3^2C_3 + ...$ $+ n^2C_n = n(n+1)2^{n-2}$
➤ ***Solution*** Consider $(1+x)^n$
$= C_0 + C_1x + C_2x^2 + ... + C_nx^n$ ...(1)
Differentiating (1) w.r.t. x $n(1+x)^{n-1}$
$= C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + 5C_5x^4 + ...$ ...(2)
Multiplying both sides of (2) by x,
$nx(1+x)^{n-1} = C_1x + 2C_2x^2 + 3C_3x^3 + 4C_4x^4$
$+ 5C_5x^5 + ...$ ...(3)
Now differentiating both sides of (3) w.r.t. x again, we get
$n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2}$
$= C_1 + 2^2C_2x + 3^2C_3x^2 + 4^2C_4x^3 +.....$
Putting x = 1, we get
$n(n+1)2^{n-2} = C_1 + 2^2C_2 + 3^2C_3 + ... + n^2C_n$

➤ **Example 7.** If k be any real number, then prove that $k^2C_0 - (k-1)^2C_1 + (k-2)^2C_2 + ... (-1)^n(k-n)^2C_n = 0.$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** $S = k^2 [C_0 - C_1 + C_2 - ...]$
$+ 2k [C_1 - 2C_2 + 3C_3 - ....] - [1^2C_1 - 2^2C_2 + 3^2C_3 - ...],$
$= k^2 . 0 + 2k . 0 - S' = - S'$
where $S' = 1^2C_1 - 2^2C_2 + 3^2C_3 - .... (-1)^{n+1} n^2C_n$.
Differentiating both sides of the relation
$(1 - x)^n = C_0 - C_1x + C_2x^2 - ... + (-1)^n C_nx^n,$
we have
$-n(1-x)^{n-1} = -C_1 + 2C_2x - 3C_3x^2 + ... + (-1)^n nC_nx^{n-1}.$
Multiplying throughout by x and differentiating w.r.t. x, we have
$-n(1-x)^{n-1} + n(n-1)x(1-x)^{n-2}$
$= -C_1 + 2^2C_2x - 3^2C_3x^2 + ...$
Putting x = 1 throughout, we get S' = 0. Hence S = 0.

➤ **Example 8.** If $(1 + px + x^2)^n = 1 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$, prove that
$1 + 3a_1 + 5a_2 + 7a_3 + ... + (4n+1) a_{2n} = (2n+1)(2+p)^n.$
➤ ***Solution*** Consider $(1 + px + x^2)^n = 1 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$ ...(1)
Put x = 1
$(2 + p)^n = 1 + a_1 + a_2 + a_3 + ... + a_{2n}$ ...(2)
Differentiate (1) w.r.t. x
$n(1 + px + x^2)^{n-1} (p + 2x)$
$= a_1 + 2a_2 x + 3a_3 x^2 + ... + 2n a_{2n} x^{2n-1}.$
Put x = 1
$n . (2 + p)^n = a_1 + 2a_2 + 3a_3 + ... + 2na_{2n}$ ...(3)
2 × (3) + (1) gives the result.
Alternative
Replace both sides of (1) by x^2, then multiply both sides by x. Now differentiate both sides and put x = 1.

➤ **Example 9.** If n is positive integer and if
$(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$ then prove that
(i) $a_r = a_{2n-r}$ for $0 \le r \le 2n$
(ii) $a_0 + a_1 + a_2 + ... + a_{n-1} = \frac{1}{2}(3^n - a_n)$
(iii) $(r+1) a_{r+1} = (n-r)a_r + (2n-r+1) a_{r-1} (0 < r < 2n)$

➤ ***Solution*** (i) We have $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$ ...(1)

$\therefore \quad \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} a_r \left(\frac{1}{x}\right)^r$

$\Rightarrow \quad (x^2 + x + 1)^n = \sum_{r=0}^{2n} a_r x^{2n-r}$

$\Rightarrow \quad \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r}$ {using (1)}

Equating the coefficient of x^{2n-r} on both sides, we get
$a_{2n-r} = a_r$ for $0 \le r \le 2n$
Hence $a_r = a_{2n-r}$
(ii) Putting x = 1 in given series, we have
$a_0 + a_1 + a_2 + a_3 + ... + a_{2n} = (1 + 1 + 1)^n$
$a_0 + a_1 + a_2 + a_3 + ... + a_{2n} = 3^n$
But $a_r = a_{2n-r}$ for $0 \le r \le 2n$
$a_0 = a_{2n}$
$a_1 = a_{2n-1}$
$a_2 = a_{2n-2}$
.....
$a_n = a_n$
.....
$a_{2n} = a_0$
then series (1) reduces to
$2(a_0 + a_1 + a_2 + ... + a_{n-1}) + a_n = 3^n$
$\therefore \quad a_0 + a_1 + a_2 + ... + a_{n-1} = \frac{1}{2}(3^n - a_n)$
(iii) Differentiate the given series w.r.t. x, we get
$n(1 + 2x)(1 + x + x^2)^{n-1} = \sum_{r=0}^{2n} r a_r x^{r-1}$
$\Rightarrow n(1 + 2x)(1 + x + x^2)^n = (1 + x + x^2) \sum_{r=0}^{2n} r a_r x^{r-1}$
$\Rightarrow n(1+2x) \sum_{r=0}^{2n} a_r x^r = (1+x+x^2) \sum_{r=0}^{2n} r a_r x^{r-1}$ (using (1))
$\Rightarrow$ Equating the coefficient of x^r, $(0 < r < 2n)$, we get
$n a_r + 2n a_{r-1} = (r+1)a_{r+1} + r a_r + (r-1)a_{r-1}$
$(r+1)a_{r+1} = (n-r)a_r + (2n-r+1)a_{r-1}$

Practice Problems [I]

1. If $(1+x)^{15} = C_0 + C_1 . x + C_2 . x^2 + ... + C_{15} . x^{15}$ then find the value of $C_2 + 2C_3 + 3C_4 + ... + 14C_{15}$
2. If $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + a_2x^2 + ... + a_{np} . x^{np}$, then find the value of $a_1 + 2a_2 + 3a_3 + ... + np . a_{np}$

www.pdfworld.net

www.pdfworld.net

3. If $(1+x)^n + C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then show that

(i) $C_0 + 3C_1 + 3^2C_2 + ... + 3^nC_n = 4^n$.

(ii) $C_0 + 2C_1 + 3 . C_2 + ... + (n+1)C_n = 2^{n-1}(n+2)$.

4. Prove that : $(n-1)^2 . C_1 + (n-3)^2 . C_3 + (n-5)^2 . C_5 + ... = n(n+1)2^{n-3}$

5. Prove that : $1^2 . C_0 + 2^2 . C_1 + 3^2 . C_2 + 4^2 . C_3 + ... + (n+1)^2 C_n = 2^{n-2}(n+1)(n+4)$.

6. Show that $\sum_{r=0}^{n} C_r (2r-n)^2 = n.2^n$

5.13 INTEGRATION METHOD

We use some of the following tips along with integration to obtain the sum $\sum_{r=0}^{n} \frac{{}^nC_r}{f(r)}$

(i) If the last term of the series leaving the plus or minus sign is m, multiplied by some binomial coefficient, then express m as (an + b) where a be the quotient and b the remainder when m is divided by n. then replace x by x^a in the expansion $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ and multiply both sides of the relation by x^{b-1}.

(ii) Now integrate both sides w.r.t. x between limits 0 to x and put x = 1 , –1 or i according to the given series.

(iii) If in the denominator of binomial coefficient product of two numerals appear, then integrate twice. On first time take limits between 0 to x and on second time take suitable limits.

(iv) If the sum contains $C_0, C_1, C_2, ..., C_n$ with all positive signs, then integrate between limits 0 to 1.

(v) If the sum contains alternate signs (i.e., +, –) then integrate between limits –1 to 0

(vi) If the sum contains odd coefficients (i.e., C_0, C_2, C_4,...) then integrate between –1 to + 1

(vii) If the sum contains even coefficients (i.e., C_1, C_3, C_5,...) then subtract (ii) from (i) and then divide by 2.

➤ **Example 1.** Prove that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

➤ ***Solution*** Consider

$$(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n,$$

Integrating both sides within limits 0 to 1, we get

$$\int_0^1 (1+x)^n dx = \int_0^1 (C_0 + C_1x + C_2x^2 + ... + C_nx^n)dx$$

$$\frac{(1+x)^{n+1}}{n+1}\Big]_0^1 = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}\Big]_0^1$$

$$\Rightarrow \frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1}$$

Hence $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

➤ **Example 2.** Prove that

$$\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + ... = \frac{2^n}{n+1}$$

➤ ***Solution*** Consider

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ... + C_nx^n,$$

Integrating both sides within limits –1 to 1, we get

$$\int_{-1}^1 (1+x)^n dx$$

$$= \int_{-1}^1 (C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 .... + C_nx^n)dx$$

$$= \int_{-1}^1 (C_0 + C_2x^2 + C_4x^4 ....)dx$$

$$+ \int_{-1}^{-1} (C_1x + C_3x^3 + ...)dx$$

$$= 2\int_0^1 (C_0 + C_2x^2 + C_4x^4 + ...)dx + 0$$

(By Property of definite integral on odd functions)

$$\frac{(1+x)^{n+1}}{n+1}\Big]_{-1}^1 = 2\left(C_0x + \frac{C_2x^3}{3} + \frac{C_4x^5}{5} + ...\right)\Big]_0^1$$

$$\frac{2^{n+1}}{n+1} = 2\left(C_0 + \frac{C_2}{3} + \frac{C_4}{5} + ...\right)$$

Hence $\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + ... = \frac{2^n}{n+1}$

www.pdfworld.net

www.pdfworld.net

➤ **Example 3.** Prove that

$$3C_0 + 3^2\frac{C_1}{2} + 3^3\frac{C_2}{3} + 3^4\frac{C_3}{4} + \ldots\ldots + \frac{3^{n+1}C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$$

➤ ***Solution*** Consider

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_nx^n$,

Integrating both sides from 0 to 3, we get :

$$\int_0^3 (1+x)^n dx = \int_0^3 (C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n)dx$$

$$\Rightarrow \left.\frac{(1+x)^{n+1}}{n+1}\right]_0^3 = \left.C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \frac{C_3x^4}{4} \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_0^3$$

$$\frac{4^{n+1}-1}{n+1} = 3C_0 + 3^2\frac{C_1}{2} + 3^3\frac{C_2}{3} + 3^4\frac{C_3}{4} + \ldots\ldots + \frac{3^{n+1}C_n}{n+1}$$

Hence $$3C_0 + 3^2\frac{C_1}{2} + 3^3\frac{C_2}{3} + 3^4\frac{C_3}{4} + \ldots\ldots + \frac{3^{n+1}C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$$

➤ **Example 4.** Prove

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \ldots\ldots + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1}.$$

➤ ***Solution*** **Algebraic Method**

$$\text{L.H.S.} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \ldots\ldots + (-1)^n.\frac{C_n}{n+1}$$

$$= \sum_{r=0}^{n}(-1)^r.\frac{{}^nC_r}{r+1}$$

$$= \frac{1}{n+1}\sum_{r=0}^{n}(-1)^r.{}^{n+1}C_{r+1} \quad \left\{\frac{n+1}{r+1}.{}^nC_r = {}^{n+1}C_{r+1}\right\}$$

$$= \frac{1}{n+1}[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \ldots\ldots + (-1)^n.{}^{n+1}C_{n+1}]$$

$$= \frac{1}{n+1}[-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \ldots\ldots + (-1)^n.{}^{n+1}C_{n+1} + {}^{n+1}C_0]$$

$$= \frac{1}{n+1} = \text{R.H.S.}$$

$\{\text{since} -{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \ldots + (-1)^n{}^{n+1}C_{n+1} = 0\}$

Integration Method

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$.

Integrating both sides, within the limits – 1 to 0.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0} = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots + C_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\frac{1}{n+1} - 0 = 0 - \left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots\ldots + (-1)^{n+1}\frac{C_n}{n+1}\right]$$

$$C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots\ldots + (-1)^{n+1}\frac{C_n}{n+1} = \frac{1}{n+1}.$$

➤ **Example 5.** Evaluate

$$\frac{C_0}{4} - \frac{C_1}{5} + \frac{C_2}{6} - \frac{C_3}{7} + \ldots\ldots + \frac{(-1)^nC_n}{n+4}$$

where $C_r = {}^nC_r$.

➤ ***Solution***

STUDY TIP

(1) Since the series is alternating, we consider the expansion of $(1-x)^n$.

(2) In the expansion C_p occurs with x^p and we want p + 4 in the denominator therefore we will integrate by multiplying first by x^3.

(3) As there are no extra numbers whose powers are ascending, we integrate between 0 and 1. If the problem is $\frac{C_0}{4} - \frac{C_1.2}{5} + \frac{C_2}{6}.2^2 - \frac{C_3}{7}.2^3 + \ldots$, we need to integrate between 0 and 2.

Thus the method is as follows:

$(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + \ldots + (-1)^nC_nx^n$

On multiplying with x^3 we get

$x^3(1-x)^n = C_0x^3 - C_1x^4 + C_2x^5 - \ldots + (-1)^n C_nx^{n+3}$

We integrate both sides w.r.t. x

$$\Rightarrow \int_0^1 x^3(1-x)^n dx$$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

$= \int_0^1 C_0x^3 - C_1x^4 + C_2x^5 - .... + (-1)^n C_n x^{n+3})dx$

The RHS is simply equal to the expression in the question, which is now equal to

$$\int_0^1 x^3(1-x)^n dx = \int_0^1 (1-x)^3(1-(1-x)^n dx$$

$$= \int_0^1 x^n(1-x)^3 dx = \int_0^1 x^n(1-3x+3x^2-x^3)dx$$

$$= \int_0^1 x^n - 3x^{n+1} + 3x^{n+2} - x^{n+3} dx$$

$$= \frac{1}{n+1} - \frac{3}{n+2} + \frac{3}{n+3} - \frac{1}{n+4}$$

$$= \left(\frac{1}{n+1} - \frac{1}{n+4}\right) + 3\left(\frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{3}{(n+1)(n+4)} - \frac{3}{(n+2)(n+3)}$$

$$= \frac{6}{(n+1)(n+2)(n+3)(n+4)}$$

➤ **Example 6.** Prove that $\frac{2^2}{1.2} C_0 + \frac{2^3}{2.3} C_1 + \frac{2^4}{3.4} C_2 + ... + \frac{2^{n+2} C_n}{(n+1)(n+2)} = \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}$

➤ ***Solution*** Consider $(1 + x)^n$

$= C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$ ...(1)

Integrating both sides of (1) within limits 0 to x we get

$$\int_0^x (1+x)^n dx = \int_0^x (C_0 + C_1 x + C_2 x^2 + ... C_n x^n) dx$$

$$\frac{(1+x)^{n+1}}{(n+1)}\Big]_0^1 = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1}\Big]_0^1$$

$$\Rightarrow \frac{(1+x)^{n+1} - 1}{(n+1)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1} ...(2)$$

Integrating again both sides of (2) within limits 0 to 2, we get

$$\int_0^2 \frac{(1+x)^{n+1} - 1}{(n+1)} dx = \int_0^2 \left(C_0 + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1}\right) dx$$

$$\Rightarrow \frac{1}{(n+1)}\left(\frac{(1+x)^{n+2}}{n+2} - x\right)\Big]_0^2$$

$$= \frac{C_0x^2}{1.2} + \frac{C_1x^3}{2.3} + \frac{C_3x^4}{3.4} + ... + \frac{C_0x^{n+2}}{(n+1)(n+2)}\Big]_0^2$$

$$\Rightarrow \frac{1}{(n+1)}\left\{\frac{3^{n+2}}{n+2} - 2 - \frac{1}{n+2}\right\}$$

$$= \frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_1 + \frac{2^4 C_2}{3.4} + ... + \frac{C_n x^{n+2}}{(n+1)(n+2)}$$

Hence, $\frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_1 + \frac{2^4 C_2}{3.4} + ... + \frac{2^{n+2} C_n}{(n+1)(n+2)}$

$$= \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}$$

➤ **Example 7.** Evaluate $\frac{C_0}{1.2} - \frac{C_1}{2.3} + \frac{C_2}{3.4} - \frac{C_3}{4.5} + ...$ to (n + 1) terms.

➤ ***Solution*** Consider

$(1 + x)^n = C_0 + C_1x + C_2x^n + C_3x^3 + ...$ ...(1)

Integrating equation (1) w.r.t. x, taking limits from 0 to x and interchanging sides

$$\Rightarrow \int_0^x (C_0 + C_1x + C_2x^2 + C_3x^3 + ...) dx = \int_0^x (1+x)^n dx$$

$$\Rightarrow \left(C_0.x + C_1.\frac{x^2}{2} + C_2.\frac{x^3}{3} + C_3.\frac{x^4}{4} + ...\right) = \frac{(1+x)^{n+1} - 1}{n+1} ...(2)$$

Integrating equation (2), taking limits from –1 to 0.

$$\Rightarrow \int_{-1}^0 \left(C_0x + C_1.\frac{x^2}{2} + C_2.\frac{x^3}{3} + C_3.\frac{x^4}{4} + ...\right) dx = \int_{-1}^0 \frac{(1+x)^{n+1} - 1}{n+1}$$

$$\Rightarrow \left(\frac{C_o.x^2}{2} + \frac{C_1.x^3}{2.3} + \frac{C_2x^4}{3.4} + \frac{C_3.x^5}{4.5} + ...\right)_{-1}^0 = \left(\frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{n+1}\right)\Big|_{-1}^0$$

www.pdfworld.net

www.pdfworld.net

$$\Rightarrow\left(\frac{C_0}{1.2}-\frac{C_1}{2.3}+\frac{C_2}{3.4}-\frac{C_3}{4.5}+....\right)$$

$$=\frac{1}{(n+1)(n+2)}-\frac{1}{n+1}=\frac{1-(n+2)}{(n+1)(n+2)}=\frac{-1}{n+2}$$

Practice Problems [J]

1. Prove that $4C_0+\frac{4^2}{1}.C_1+\frac{4^3}{3}C_2+.....+\frac{4^{n+1}}{n+1}C_n=\frac{5^{n+1}-1}{n+1}$.

2. If $(1+x)^n=\sum_{r=0}^{n}C_r.x^r$ then prove that

$$\frac{2^2.C_0}{1.2}+\frac{2^3.C_1}{2.3}+\frac{2^4.C_2}{3.4}+.....+\frac{2^{n+2}.C_n}{(n+1)(n+2)}$$

$$=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

3. Prove that $\frac{C_0}{1}+\frac{C_2}{3}+\frac{C_4}{5}+.....=\frac{2^n}{n+1}$

4. Prove that

$$\frac{C_0}{2}+\frac{C_1}{3}+\frac{C_2}{4}+\frac{C_3}{5}+.....+\frac{C_n}{n+2}$$

$$=\frac{1+n.2^{n+1}}{(n+1)(n+2)}$$

5. Prove that

$$\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\frac{C_3}{5}+.....+(-1)^n.\frac{C_n}{n+2}$$

$$=\frac{1}{(n+1)(n+2)}$$

6. Prove that

$$\frac{C_1}{1}-\frac{C_2}{2}+\frac{C_3}{3}-\frac{C_4}{4}+.....+(-1)^{n-1}.\frac{C_n}{n}$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....+\frac{1}{n}$$

5.14 SERIES INVOLVING PRODUCT OF BINOMIAL COEFFICIENTS

Such series are added by multiplying two expansions, one involving the first factors as coefficients and the other involving the second factors as coefficients and finally equating coefficients of a suitable power of x on both sides.

1. If difference of the lower suffices of binomial coefficients in each term is same.

➤ **Example 1.** Evaluate the sum ${}^nC_0{}^nC_2+{}^nC_1{}^nC_3+{}^nC_2{}^nC_4+{}^nC_3{}^nC_5+.....+{}^nC_{n-2}{}^nC_n$

➤ ***Solution*** Here $3-1=4-2=8-6=....=2$

Since each term of series is positive, consider

$(1+x)^n=C_0+C_1x+C_2x^2+.....+C_nx^n$ ...(1)

Interchanging 1 and x,

$(x+1)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+....+C_n$ ...(2)

Then multiply (1) and (2)

$(1+x)^n(x+1)^n=(C_0+C_1x+C_2x^2+.....+C_nx^n)\times(C_0x^n+C_1x^{n-1}+C_2x^{n-2}+....+C_n)$

and equate the coefficient of x^{n-2} on both sides.

Thus, the required sum is equal to coefficient of x^{n-2} in $(1+x)^{2n}$ which is equal to ${}^{2n}C_{n-2}$.

STUDY TIP

(i) ${}^nC_0{}^nC_r+{}^nC_1{}^nC_{r+1}+{}^nC_2{}^nC_{r+2}+.....+{}^nC_{n-r}{}^nC_n$
$=$ coefficient of x^{n-r} in $(1+x)^{2n}={}^{2n}C_{n-r}$

(ii) $({}^nC_0)^2+({}^nC_1)^2+({}^nC_2)^2+....+({}^nC_n)^2={}^{2n}C_n$

➤ **Example 2.** Prove that $({}^{2n}C_0)^2-({}^{2n}C_1)^2-({}^{2n}C_2)^2-....+(-1)^{2n}({}^{2n}C_{2n})^2=(-1)^n.{}^{2n}C_n$

➤ ***Solution*** Since the terms of the series are alternately positive and negative we consider one of the expansions with altenate signs.

$(1-x)^{2n}={}^{2n}C_0-{}^{2n}C_1x+{}^{2n}C_2x^2-...+(-1)^{2n}.{}^{2n}C_{2n}x^{2n}$, ...(1)

and $(x+1)^n={}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+{}^{2n}C_2x^{2n-2}+...+{}^{2n}C_{2n}$, ...(2)

Multiplying (1) and (2), we get

$(x^2-1)^{2n}=({}^{2n}C_0-{}^{2n}C_1x+{}^{2n}C_2x^2-...+(-1)^{2n}.{}^{2n}C_{2n}x^{2n})\times({}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+{}^{2n}C_2x^{2n-2}+...+{}^{2n}C_{2n}$, ...(3)

Now, coefficient of x^{2n} in R.H.S.

$=({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-.....+(-1)^{2n}({}^{2n}C_{2n})^2$,

www.pdfworld.net

www.pdfworld.net

$\because$ General term in L.H.S., $T_{r+1} = {}^{2n}C_r (x^2)^{2n-r} (-1)^r$
Putting $2(2n - r) = 2n$
$\therefore \quad r = n \quad T_{n+1} = {}^{2n}C_n x^{2n} (-1)^n$
Hence coefficient of x^{2n} in L.H.S. $= (-1)^n \cdot {}^{2n}C_n$
But (3) is an identity, therefore coefficient of x^{2n} in R.H.S. = coefficient of x^{2n} in L.H.S.

$$\Rightarrow \quad ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^{2n} ({}^{2n}C_{2n})^2 = (-1)^n . {}^{2n}C_n$$

➤ **Example 3.** Evaluate the sum
${}^{2n+1}C_0^2 + {}^{2n+1}C_1^2 + {}^{2n+1}C_2^2 + {}^{2n+1}C_n^2$.

➤ ***Solution*** Let $S = ({}^{2n+1}C_0^2 + {}^{2n+1}C_1^2 + ... + {}^{2n+1}C_n^2) + ({}^{2n+1}C_{n+1}^2 + ... + {}^{2n+1}C_{2n+1}^2)$
$S = x + y$

Note that $x = y \Rightarrow \quad x = \frac{1}{2} S$

Consider the expansion of $(1 + x)^n$ and $(x + 1)^n$. Multiply and equate the co-efficient of x^{2n+1} to get $S = {}^{4n+2}C_{2n+1}$

$$\Rightarrow \quad x = \frac{1}{2} \cdot \frac{(4n+1)!}{(2n+1)!\,(2n+1)!}$$

$$\text{or} \quad x = \frac{2^{2n+1} \cdot (2n+1)! \left[1 . 3 . 5 ..... (4n+1)\right]}{(2n+1)!\,(2n+1)!}$$

$$= \frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} ..... \frac{(8n+2)}{2n+1}$$

2. If sum of the lower suffices of binomial coefficients in each term is same.

➤ **Example 4.** Evaluate the sum
$C_0C_n + C_1C_{n-1} + C_2C_{n-2} + ... + C_nC_0$

➤ ***Solution*** Here $0 + n = 1 + (n - 1) = 2 + (n - 2)$
$= .... = n + 0 = n.$
Since each term of series is positive, consider
$(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n, \quad ...(1)$
and $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n, ...(2)$
Then multiply (1) and (2)
$(1 + x)^{2n} = (C_0 + C_1x + C_2x^2 + ... + C_nx^n) \times (C_0 + C_1x + C_2x^2 + ... + C_nx^n)$
and equate the coefficient of x^n on both sides.
Thus, the required sum is equal to coefficient of x^n in $(1 + x)^{2n}$ which is equal to ${}^{2n}C_n$.

➤ **Example 5.** Prove that
${}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + ... + {}^nC_r = {}^{m+n}C_r$
if $r < m$, $r < n$ and m, n, r are postive integers.
This result is known as Vandermonde's Theorem.

➤ ***Solution*** Here sum of lower suffices of binomial coefficients in each term is r.
i.e., $r = r - 1 + 1 = r - 2 + 2 = ... = r = r$

$$(1 + x)^m = {}^mC_0 + {}^mC_1x + ... + {}^mC_{r-2}x^{r-2} + {}^mC_{r-1}x^{r-1} + {}^mC_rx^r ...+ ..... + {}^mC_mx^m \quad ...(1)$$

$$\text{and } (1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ...+ {}^nC_rx^r + ... + {}^nC_nx^n \quad ...(2)$$

Multiplying (1) and (2), we get

$$(1 + x)^{m+n} = ({}^mC_0 + {}^mC_1x + ... + {}^mC_{r-2}x^{r-2} + {}^mC_{r-1}x^{r-1} + {}^mC_rx^r ...+ ..... + {}^mC_mx^m) \times ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ...+{}^nC_rx^r + ...+ {}^nC_nx^n) \quad ...(3)$$

Now coefficient of x^r in R.H.S.
$= {}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + ... + {}^mC_0 \cdot {}^nC_r$
$= {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + ... + {}^nC_r$
= coefficient of x^r in $(1 + x)^{m+n} = {}^{m+n}C_r$

➤ **Example 6.** Prove that $C_0C_n - C_1C_{n-1} + C_2C_{n-2} - ..... + (-1)^nC_n = 0$ or $(-1)^{n/2} \frac{n!}{(n/2)!(n/2)!}$ according as n is odd or even.

➤ ***Solution*** Since the terms of the series are alternately positive and negative we consider one of the expansions with alternate signs.

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_{n-2}x^{n-2} + C_{n-1}x^{n-1} + C_nx^n \quad ... (1)$$

$$(1 - x)^n = C_0 - C_1x + C_2x^2 - .... + (-1)^n C_nx^n \quad ... (2)$$

Multiplying (1) and (2), we get

$$(1 - x^2)^n = (C_0 + C_1x + ...C_{n-2}x^{n-2} + C_{n-1}x^{n-1} + C_nx^n) \times (C_0 - C_1x + C_2x^2 - .... + (-1)^n C_nx^n) \quad ...(3)$$

Now, coefficient of x^n in R.H.S.
$= C_0C_n - C_1C_{n-1} + C_2C_{n-2} - .... + (-1)^nC_nC_0$
Now general term in L.H.S.
$T_{r+1} = {}^nC_r(-x^2)^r = (-1)^r \cdot {}^nC_rx^{2r}$
$2r = n$
Putting $r = n/2, \quad T_{n/2+1} = (-1)^{n/2} \cdot {}^nC_{n/2}x^n$
$\therefore$ The coefficient of x^n in L.H.S.
$= (-1)^{n/2} \cdot {}^nC_{n/2}$
$= (-1)^{n/2} \cdot \frac{n!}{(n/2)!(n/2)!}$

www.pdfworld.net

www.pdfworld.net

$$= \begin{cases} 0, & \text{if n is odd} \\ (-1)^{n/2} \dfrac{n!}{(n/2)!(n/2)!} & \text{if n is even} \end{cases}$$

But (3) is an identity, therefore the coefficient of x^n in R.H.S. = coefficient of x^n in L.H.S.

$C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \ldots + (-1)^nC_nC_0$

$$= \begin{cases} 0, & \text{if n is odd} \\ (-1)^{n/2} \dfrac{n!}{(n/2)!(n/2)!} & \text{if n is even} \end{cases}$$

3. If each term is the product of two binomial coefficients divided or multiplied by an integer then integrate or differentiate one of the expansions. Then multiply the two expansions and equate the coefficients of suitable power of x on both sides.

➤ **Example 7.** Prove that

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$$

➤ ***Solution*** Consider

$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$

Differentiating both sides w.r.t x, we get.

$$\Rightarrow \quad n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1} \quad \ldots(1)$$

and $(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots + C_n$ ...(2)

Multiplying (1) and (2), we get

$n(1 + x)^{2n-1} = (C_1 + 2C_2x + 3C_3x^2 + \ldots nC_nx^{n-1})$

$\times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots + C_n)$ ...(3)

Now, coefficient of x^{n-1} on R.H.S.

$= C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2$

and coefficient of x^{n-1} on L.H.S. $= n.\ {}^{2n-1}C_{n-1}$

$$= n \cdot \frac{(2n-1)!}{(n-1)!n!} = \frac{(2n-1)!}{(n-1)!(n-1)!} = \frac{(2n-1)!}{((n-1)!)^2}$$

but (3) is an identity, therefore the coefficient of x^{n-1} in R.H.S. = coefficient of x^{n-1} in L.H.S.

$$\Rightarrow \quad C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$$

➤ **Example 8.** Prove that $C_0^2 + \dfrac{C_1^2}{2} + \dfrac{C_2^2}{3} + \ldots + \dfrac{C_n^2}{n+1}$

$$= \frac{(2n+1)!}{\{(n+1)!\}^2}$$

➤ ***Solution*** Consider

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Integrating both sides w.r.t x within limits 0 to x,

$$\int_0^x (1+x)^n dx = \int_0^x (C_0 + C_1x + C_2x + \ldots + C_nx^n)dx$$

$$\frac{(1+x)^{n+1} - 1}{(1+n)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{n+1} \quad \ldots(1)$$

and $(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n$...(2)

Multiplying (1) and (2), we get

$$\frac{1}{(n+1)}\left\{(1+x)^{2n+1} - (1+x)^n\right\}$$

$$= \left(C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{n+1}\right) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n) \quad \ldots(3)$$

Now coefficient of x^{n+1} in R.H.S. of (3)

$$= C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \ldots + \frac{C_n^2}{n+1}$$

and coefficient of x^{n+1} in L.H.S. of (3)

$$= \frac{1}{(n+1)}\left\{{}^{2n+1}C_{n+1} - 0\right\} = \frac{1}{(n+1)} \cdot \frac{(2n+1)!}{(n+1)!n!}$$

$$= \frac{(2n+1)!}{(n+1)!(n+1)!} = \frac{(2n+1)!}{\{(n+1)!\}^2}$$

But (3) is an identity, therefore coefficient of x^{n+1} in R.H.S. of (3) = coefficient of x^{n+1} in L.H.S. of (3).

Hence, $C_0^2 + \dfrac{C_1^2}{2} + \dfrac{C_2^2}{3} + \ldots + \dfrac{C_n^2}{n+1} = \dfrac{(2n+1)!}{\{(n+1)!\}^2}$

➤ **Example 9.** Prove that $1.C_0^2 + 3.C_1^2 + 5.C_2^2 + \ldots + (2n+1).C_n^2 = 2n.{}^{2n-1}C_n + {}^{2n}C_n$

➤ ***Solution*** **Algebraic Method**

L.H.S. $= 1.C_0^2 + 3.C_1^2 + 5.C_2^2 + \ldots + (2n+1)C_n^2$.

$$= \sum_{r=0}^{n} (2r+1)^nC_r^2 = \sum_{r=0}^{n} 2.r({}^nC_r)^2 + \sum_{r=0}^{n} ({}^nC_r)^2$$

$$= 2\sum_{r=0}^{n}.n.{}^{n-1}C_{r-1}\ {}^nC_r + {}^{2n}C_n$$

www.pdfworld.net

www.pdfworld.net

$(1+x)^n = {}^nC_1 + {}^nC_4 + {}^nC_2 x^2 + \ldots\ldots {}^nC_n x^n$ ...(1)

$(x+1)^{n-1} = {}^{n-1}C_0 x^{n-1} + {}^{n-1}C_1 x^{n-2} + \ldots + {}^{n-1}C_{n-1} x_0$ ...(2)

Multiplying (1) and (2) and comparing coefficients of x^n.

${}^{n-1}C_0 \cdot {}^nC_1 + {}^{n-1}C_1 \cdot {}^nC_2 + \ldots + {}^{n-1}C_{n-1} \cdot {}^nC_n = {}^{2n-1}C_n$

$\sum_{r=0}^{n} {}^{n-1}C_{r-1}\ {}^nC_r = {}^{2n-1}C_n$

Hence, required summation is $2n \cdot {}^{2n-1}C_n + {}^{2n}C_n = \text{R.H.S.}$

Differentiation Method

$(1+x^2)^n = C_0 + C_1 x^2 + C_2 x^4 + C_3 x^6 + \ldots\ldots\ldots\ldots + C_n x^{2n}$

Multiplying both sides by x

$x(1+x^2)^n = C_0 x + C_1 x^3 + C_2 x^5 + \ldots\ldots + C_n x^{2n+1}.$

Differentiating both sides

$x \cdot n(1+x^2)^{n-1} \cdot 2x + (1+x^2)^n$

$= C_0 + 3 \cdot C_1 x^2 + 5 \cdot C_2 x^4 + \ldots\ldots\ldots + (2n+1) C_n x^{2n}$ ...(1)

$(x^2+1)^n = C_0 x^{2n} + C_1 x^{2n-2} + C_2 x^{2n-4} + \ldots\ldots\ldots + C_n$ ...(2)

Multiplying (1) and (2)

$(C_0 + 3C_1 x^2 + 5C_2 x^4 + \ldots\ldots + (2n+1) C_n x^{2n})$

$\times (C_0 x^{2n} + C_1 x^{2n-2} + \ldots\ldots + C_n)$

$= 2n x^2 (1+x^2)^{2n-1} + (1+x^2)^{2n}$

Comparing coefficient of x^{2n},

$C_0^2 + 3C_1^2 + 5C_2^2 + \ldots\ldots (2n+1) C_n^2 = 2n \cdot {}^{2n-1}C_{n-1} + {}^{2n}C_n.$

➤ **Example 10.** Prove that for $m \geq 2$,

$C_0 - C_1 + C_2 - \ldots + (-1)^{m-1} C_{m-1}$

$= \dfrac{(n-1)(n-2)\ldots\ldots(n-m+1)}{(m-1)!}(-1)^{m-1}$

➤ ***Solution***

Let $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$

$\therefore \quad (1-x)^n = C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \ldots$

$+ (-1)^{m-1} C_{m-1} x^{m-1} + \ldots + (-1)^n C_n x^n$

Now $(1-x)^n (1-x)^{-1}$

$= (C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^{m-1} C_{m-1} x^{m-1} + \ldots +$

$(-1)^n C_n x^n) \times (1 + x + x^2 + \ldots + x^{m-2} + x^{m-1} + x^m + \ldots)$ ...(1)

Coefficient of x^{m-1} in R.H.S. of (1)

$= C_0 - C_1 + C_2 - \ldots\ldots + (-1)^{m-1} C_{m-1}$ ...(2)

Coefficient of x^{m-1} in L.H.S. of (1) $= {}^{n-1}C_{m-1}(-1)^{m-1}$

$= \dfrac{(n-1)(n-2)\ldots\ldots(n-m+1)}{(m-1)!}(-1)^{m-1}$ ...(3)

Hence, from (2) and (3)

$C_0 - C_1 + C_2 - \ldots + (-1)^{m-1} C_{m-1}$

$= \dfrac{(n-1)(n-2)\ldots\ldots(n-m+1)}{(m-1)!}(-1)^{m-1}$

➤ **Example 11.** Find the coefficient of x^n in the expansion of

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!}\right)^2.$$

➤ ***Solution*** Here, $\left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots + \dfrac{x^n}{n!}\right)^2$

$= \left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots + \dfrac{x^n}{n!}\right) \times$

$\left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots + \dfrac{x^n}{n!}\right)$

coefficient of x^n in $\left(1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots + \dfrac{x^n}{n!}\right)^2$

$= 1.\dfrac{1}{n!} + \dfrac{1}{1!}.\dfrac{1}{(n-1)!} + \dfrac{1}{2!}.\dfrac{1}{(n-2)!} +$

$\dfrac{1}{3!}.\dfrac{1}{(n-3)!} + \ldots\ldots + \dfrac{1}{n!}.1$

$= \dfrac{1}{n!}\left(\dfrac{n!}{0!\,n!} + \dfrac{n!}{1!(n-1)!} + \dfrac{n!}{2!(n-2)!} + \ldots\ldots + \dfrac{n!}{n!\,0!}\right)$

$= \dfrac{1}{n!}({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n) = \dfrac{1}{n!} \cdot 2^n.$

➤ **Example 12.** If $a_0, a_1, a_2, \ldots a_{2n}$ be the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x, then show that $a_0^2 - a_1^2 + a_2^2 - \ldots\ldots + a_{2n}^2 = a_n$.

➤ ***Solution*** We have

$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_{2n} x^{2n}$ ...(1)

Replace x by $-\dfrac{1}{x}$ in (1) $\qquad \left(1 - \dfrac{1}{x} + \dfrac{1}{x^2}\right)^n$

$= a_0 - \dfrac{a_1}{x} + \dfrac{a_2}{x^2} - \ldots\ldots + \dfrac{a_{2n}}{x^{2n}}$ ...(2)

Multiplying (1) and (2), we get

www.pdfworld.net

www.pdfworld.net

$$(1+x+x^2)^n \times \left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$$

$$= (a_0 + a_1x + a_2x^2 + a_{2x}x^{2n}) \times \left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - .... + \frac{a_{2n}}{x^{2n}}\right)$$

$$\Rightarrow \quad \frac{(1+x^2+x^4)^n}{x^{2n}} = (a_0 + a_1x + a_2x^2 + a_{2x}x^{2n}) \times \left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - .... + \frac{a_{2n}}{x^{2n}}\right) \quad ...(3)$$

The constant term in R.H.S.

$$= a_0^2 - a_1^2 + a_2^2 - ..... + a_{2n}^2$$

Now, constant term in $\frac{(1+x^2+x^4)^n}{x^{2n}}$

= coefficient of x^{2n} in $(1 + x^2 + x^4)^n \quad = a_n$

[Replace x by x^2 in (1)]

Since (3) is an identity, therefore constant term in R.H.S. = constant term in L.H.S.

$a_0^2 - a_1^2 + a_2^2 - .... + a_{2n}^2 = a_n$.

➤ **Example 13.** Prove that

$a_0 \cdot {}^nC_0 - a_1 {}^nC_1 + a_2 {}^nC_2 - ..... + (-1)^n a_n {}^nC_n$

$$= \begin{cases} (-1)^{4n/3} \ {}^nC_{n/3} & \text{if n is multiple of 3} \\ 0 & \text{otherwise} \end{cases}$$

➤ ***Solution*** $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$ ...(1)

The series appears if the expansion of $(1 + x + x^2)^n$ is multiplied with the expansion $(x - 1)^n$.

$(x-1)^n = [C_0x^n - C_1x^{n-1} + C_2x^{n-2} - .... + (-1)^nC_n\ ]$ ...(2)

The coefficient of x^n in the multiplication of RHS of (1) and (2)

$= [a_0C_0 - a_1C_1 + a_2C_2 - a_3C_3 + ...+(-1)^n a_0C_0 -]$

which must be equal to coeff. of x^n in E where

$E = (x - 1)^n(1 + x + x^2)^n = (-1)^n(1 - x)^n(1 + x+ x^2)^n$

$= (-1)^n(1 - x^3)^n$

$\Rightarrow$ coeff of x^n

$$= \begin{cases} (-1)^n(-1)^{n/3} \ {}^nC_n & \text{if n is multiple of 3} \\ 0 & \text{otherwise} \end{cases}$$

➤ **Example 14.** If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time represented by

$$\sum\sum_{0\le i<j\le n} C_iC_j \text{ is equal to } 2^{n-1} - \frac{2n!}{2.n!n!}$$

➤ ***Solution*** Since $(C_0 + C_1 + C_2 + ... + C_{n-1} + C_n)^2$

$= (C_0^2 + C_1^2 + C_2^2 + ...C_{n-1}^2 + C_n^2) + 2(C_0C_1 + C_0C_2 + C_0C_3 + ... + C_0C_n + C_1C_2 + C_1C_3 + ... + C_1C_n + C_2C_3 + C_2C_4 + ... + C_2C_n + ... + C_{n-1}C_n)$

$$(2^n)^2 = {}^{2n}C_n + 2\sum\sum_{0\le i<j\le n} C_iC_j$$

Hence $$\sum\sum_{0\le i<j\le n} C_iC_j = 2^{n-1} - \frac{2n!}{2.n!n!}$$

➤ **Example 15.** Prove that

$$\sum\sum_{0\le i<j\le n} (i+j)\,C_iC_j = n\left(2^{2n-1} - \frac{1}{2}{}^{2n}C_n\right)$$

➤ ***Solution*** Let $P = \sum\sum_{0\le i<j\le n} (i+j)\,C_iC_j$ ...(1)

Replacing i by n – i and j by n – j then we get

$$P = \sum\sum_{0\le i<j\le n} (n-i+n-j)C_{n-i}C_{n-j}$$

$$= \sum\sum_{0\le i<j\le n} \{2n-(i+j)\}C_iC_j \qquad \{\because {}^nC_r = {}^nC_{n-r}\}$$

$$= 2n\sum\sum_{0\le i<j\le n} C_iC_j - \sum\sum_{0\le i<j\le n} (i+j)C_iC_j$$

$$= 2n.\left(2^{2n-1} - \frac{{}^{2n}C_n}{2}\right) - P$$

$\Rightarrow 2P = n \cdot 2^{2n} - n \cdot {}^{2n}C_n$

$\therefore \quad P = n\left(2^{2n-1} - \frac{1}{2}{}^{2n}C_n\right)$

Hence $$\sum\sum_{0\le i<j\le n} (i+j)\,C_iC_j = n\left(2^{2n-1} - \frac{1}{2}{}^{2n}C_n\right)$$

➤ **Example 16.** If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...$ $+ C_nx^n$, find the sum $\sum_{0\le i<j\le n} \Sigma\ (C_i - C_j)^2$.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** $E = \sum_{j=1}^{n} (C_0 - C_j)^2 + \sum_{j=2}^{n} (C_1 - C_j)^2 + \ldots + (C_{n-1} - C_j)^2$

$= n\left(C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2\right) - 2$(sum of the products of binomial co-efficients taken two at a time)

$= n \cdot {}^{2n}C_n - 2S$

But $(C_0 + C_1 + C_2 + \ldots\ldots C_n)^2$

$= C_0^2 + C_1^2 + \ldots\ldots + C_n^2 + 2S$

$\Rightarrow 2S = 2^{2n} - {}^{2n}C_n$

$\Rightarrow E = n \cdot {}^{2n}C_n - (2^{2n} - {}^{2n}C_n) = (n+1)\,{}^{2n}C_n - 2^{2n}$

➤ **Example 17.** Evaluate $\sum_{i=0}^{n} \sum_{j=1}^{n} {}^nC_j\,{}^jC_i, \quad i \le j$

➤ ***Solution*** $\sum_{i=0}^{n} \sum_{j=1}^{n} {}^nC_j\ {}^jC_i,$

$= {}^nC_1\,({}^1C_0 + {}^1C_1) + {}^nC_2\,({}^2C_0 + {}^2C_1 + {}^2C_2)$
$+ {}^nC_3\,({}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3)$
$+ {}^nC_4\,({}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) + \ldots\ldots\ldots$
$+ {}^nC_n\,({}^nC_0 + {}^nC_1 + {}^nC_2 \ldots\ldots + {}^nC_n)$

$= {}^nC_1(2) + {}^nC_2(2)^2 + {}^nC_3(2)^3 + {}^nC_4(2)^4 + \ldots\ldots + {}^nC_n(2)^n$

$= (1+2)^n - 1 = 3^n - 1.$

Practice Problems [K]

1. Prove that $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n$
2. Prove that $C_0C_2 + C_1C_3 + C_2C_4 + \ldots\ldots + C_{n-2}\,C_n = {}^{2n}C_{n-2}$
3. If $S_n = {}^nC_0\,{}^nC_1 + {}^nC_1\,{}^nC_2 + \ldots\ldots + {}^nC_{n-1}\,{}^nC_n$, and if $\frac{S_{n+1}}{S_n} = \frac{15}{4}$, find n
4. Prove that $\sum_{r=0}^{n-2}\left({}^nC_r \cdot {}^nC_{r+2}\right) = \frac{(2n)!}{(n-2)!\,(n+2)!}$
5. Prove that $C_o^2 + 3 \cdot C_1^2 + 5 \cdot C_2^2 + \ldots\ldots + (2n+1)\,C_n^2 = \frac{(n+1)\,(2n)!}{n!\,n!}$
6. Find the sum ${}^{2n+1}C_0^2 - {}^{2n+1}C_1^2 + {}^{2n+1}C_2^2 - {}^{2n+1}C_3^2 + \ldots\ldots - {}^{2n+1}C_{2n+1}^2$.
7. Prove that $({}^{2n}C_1)^2 + 2 \cdot ({}^{2n}C_2)^2 + 3 \cdot ({}^{2n}C_3)^2 + \ldots + 2n \cdot ({}^{2n}C_{2n})^2 = \frac{(4n-1)!}{\left[(2n-1)!\right]^2}$

5.15 SERIES INVOLVING PRODUCT OF BINOMIAL COEFFICIENTS WITH VARIABLE UPPER INDICES

➤ **Example 1.** Show that (i) $C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-1}C_n + C_2 \cdot {}^{2n-2}C_n - C_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n\,C_n \cdot {}^nC_n = 1.$

(ii) $C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-2}C_n + C_2 \cdot {}^{2n-4}C_n - \ldots\ldots = 2^n$

➤ ***Solution*** (i) Here, all coefficients have lower suffix same.

${}^{2n}C_n$ = coefficient of x^n in $(1+x)^{2n}$

${}^{2n-1}C_n$ = coefficient of x^n in $(1+x)^{2n-1}$

${}^{2n-2}C_n$ = coefficient of x^n in $(1+x)^{2n-2}$

${}^{2n-3}C_n$ = coefficient of x^n in $(1+x)^{2n-3}$

.....

${}^nC_n$ = coefficient of x^n in $(1+x)^n$

L.H.S. $= C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-1}C_n + C_2 \cdot {}^{2n-2}C_n - C_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n\,C_n \cdot {}^nC_n$

= coefficient of x^n in $[C_0(1+x)^{2n} - C_1(1+x)^{2n-1} + C_2(1+x)^{2n-2} - C_3(1+x)^{2n-3} + \ldots\ldots + (-1)^n\,C_n(1+x)^n]$

= coefficient of x^n in $(1+x)^n\,[C_0(1+x)^n - C_1(1+x)^{n-1} + C_2\,(1+x)^{n-2} - C_3(1+x)^{n-3} + \ldots\ldots + (-1)^n\,C_n]$

= coefficient of x^n in $(1+x)^n\,[(1+x)-1]^n$

= coefficient of x^n in $(1+x)^n\,x^n = 1$ = R.H.S.

(ii) L.H.S. $= C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-2}C_n + C_2 \cdot {}^{2n-4}C_n - \ldots\ldots$

= coefficient of x^n in $[C_0\,(1+x)^{2n} - C_1\,(1+x)^{2n-2} + C_2\,(1+x)^{2n-4} - \ldots\ldots]$

= coefficient of x^n in $[(1+x)^2 - 1]^n$

= coefficient of x^n in $(2x + x^2)^n$

= coefficient of x^n in $(2+x)^n$

$= 2^n$ = R.H.S.

www.pdfworld.net

www.pdfworld.net

➤ **Example 2.** If C_r stands for ${}^nC_r$ then simplify ${}^nC_0 \cdot {}^{2n}C_r - {}^nC_1 \cdot {}^{2n-2}C_r + ...$

➤ ***Solution*** ${}^nC_0 \cdot {}^{2n}C_r - {}^nC_1 \cdot {}^{2n-2}C_r + ...$

= coefficient of x^r in $[C_0 (1+x)^{2n} - C_1 (1+x)^{2n-2} + C_2 (1+x)^{2n-4} - ..... + (-1)^n C_n (1+x)^0]$

= coefficient of x^r in $[(1+x)^2 - 1]^n$

= $[2x + x^2]^n = x^n (2+x)^n$

= coefficient of x^r in $x^n (2+x)^n$

= coefficient of x^{r-n} in $(2+x)^n$

$$= \begin{cases} {}^nC_{r-n}\, 2^{2n-r} & \text{if } r > n \\ 0 & \text{if } r < n \end{cases}$$

➤ **Example 3.** Find the sum of the series,

${}^nC_0 \cdot {}^{n+1}C_1 + {}^nC_1 \cdot {}^nC_{n-1} + {}^nC_2 \cdot {}^{n-1}C_{n-2} + {}^nC_3 \cdot {}^{n-2}C_{n-3} + ... + {}^nC_n \cdot {}^1C_0$

➤ ***Solution*** Let $S = {}^nC_0 \cdot {}^{n+1}C_1 + {}^nC_1 \cdot {}^nC_1 + {}^nC_2 \cdot {}^{n-1}C_1 + {}^nC_3 \cdot {}^{n-2}C_1 + ... + {}^nC_n \cdot {}^1C_0$

Also $S = {}^nC_0 \cdot {}^1C_0 + {}^nC_{n-1} \cdot {}^2C_1 + {}^nC_{n-2} \cdot {}^3C_1 + ... + {}^nC_0 \cdot {}^{n+1}C_1$

Adding the two series

$2S = {}^nC_0 ({}^{n+1}C_1 + {}^1C_0) + {}^nC_1 ({}^nC_1 + {}^2C_1) + ... + {}^nC_n ({}^1C_0 + {}^{n+1}C_1)$

$= (n+2)[{}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n]$

$S = (n+2)\, 2^{n-1}$

5.16 BINOMIAL THEOREM FOR RATIONAL INDEX

If $n \in Q$ and $|x| < 1$ i.e. $-1 < x < 1$, then

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + ... \text{ to } \infty$$

(i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. $(n+1)$ and the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3 ..... {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term. For instance, $\frac{n}{1!}$ cannot be written as ${}^nC_1$ since ${}^nC_1$ is meaningless when n is a negative integer or a fraction.

Also the coefficient of the general term can never vanish unless one of the factors of its numerator is zero; the series will therefore stop at the r^{th} term, when $n - r + 1$ is zero; that is when $r = n + 1$; but since r is a positive integer this equality can never hold except when the index n is positive and integral. Thus the expansion by the Binomial Theorem extends to $n + 1$ terms when n is a positive integer and to an infinite number of terms in all other cases.

(iii) The expansion of $(1+x)^n$, where n is a positive integer is valid for all values of x where as the expansion of $(1+x)^n$, where n is a negative integer or fraction is valid only when $|x| < 1$ i.e. $-1 < x < 1$.

The expansion in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of 1/x, which then will be small.

(iv) The first terms must be unity in the expansion, when index 'n' is a negative integer or fraction $(x+y)^n =$

$$\begin{bmatrix} x^n\left(1+\frac{y}{x}\right)^n = x^n\left\{1 + n.\frac{y}{x} + \frac{n(n-1)}{2!}\left(\frac{y}{x}\right)^2 + ...\right\} \text{ if } \left|\frac{y}{x}\right| < 1 \\ y^n\left(1+\frac{x}{y}\right)^n = y^n\left\{1 + n.\frac{x}{y} + \frac{n(n-1)}{2!}\left(\frac{x}{y}\right)^2 + ...\right\} \text{ if } \left|\frac{x}{y}\right| < 1 \end{bmatrix}$$

(v) The $(r+1)$th term is called the general term and is denoted by T_{r+1}.

The $(r+1)$th term in the expansion of $(1+x)^n$ is given by

www.pdfworld.net

www.pdfworld.net

$$T_{r+1} = \frac{n(n-1)(n-2).....(n-r+1)}{r!}.x^r$$

(vi) The following expansions should be remembered.

In each case it is assumed that $|x| < 1$.

(a) $(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + ...$

$= 1 + x + x^2 + x^3 + x^4 + ... \infty$

(b) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... \infty$

(c) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - .... \infty$

(d) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \infty$

(e) $(1-x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{3.4.5}{1.2.3}x^3 + \frac{3.4.5.6}{1.2.3.4}x^4 + ...$

$= \frac{1}{2}[1.2 + 2.3x + 3.4x^2 + 4.5x^3 + 5.6x^4 + ...]$

(f) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ...$

(g) If $n \in N$, $(1-x)^{-n} = 1 + {}^nC_1 x + {}^{n+1}C_2 x^2 + {}^{n+2}C_3 x^3 + ... + {}^{n+r-1}C_r x^r + ...$

Thus, the coefficient of x^r in $(1-x)^{-n}$ is ${}^{n+r-1}C_r$.

(h) $(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + ...$

(vii) Approximations

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 .....$$

If $|x| < 1$, the terms of the above expansion go on decreasing and if x is very small a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its squares and higher powers may be neglected then $(1+x)^n = 1 + nx$, approximately.

This is an approximate value of $(1+x)^n$.

➤ **Example 1.** If $-1 < x < 1$, show that $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... \text{ to } \infty$

➤ ***Solution*** We know that if n is a negative integer or fraction $(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + ...$ to ∞ provided $-1 < x < 1$

Putting n = –2 and –x in place of x, we get

$$(1-x)^{-2} = 1 + \frac{(-2)}{1!}(-x) + \frac{(-2)(-2-1)}{2!}(-x)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3 + ... \text{ to } \infty$$

$= 1 + 2x + 3x^2 + 4x^3 + ... \text{ to } \infty$

➤ **Example 2.** If $|x| < 1$, show that $(1-x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - ..... \text{ to } \infty$

➤ ***Solution*** Putting $n = \frac{1}{2}$ and x as –x we get

$$(1-x)^{\frac{1}{2}} = 1 + \frac{\left(\frac{1}{2}\right)}{1!}(-x) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + ... \text{ to } \infty$$

$$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - ..... \text{ to } \infty$$

➤ **Example 3.** Expand $1/(3+2x)^2$ to three terms (i) in ascending powers of x, (ii) in descending powers of x, (iii) for what values of x each expansion is valid.

➤ ***Solution*** (i) $\frac{1}{(3+2x)^2}$

$$= \frac{1}{9}\left(1 + \frac{2}{3}x\right)^{-2} = \frac{1}{9}\left\{1 - 2\left(\frac{2}{3}x\right) + 3\left(\frac{2}{3}x\right)^2 - ....\right\};$$

$\therefore\ 1/(3+2x)^2 = \frac{1}{9} - \frac{4}{27} + \frac{4}{27}x^2$ to three terms.

(ii) $\frac{1}{(2x+3)^2} = \frac{1}{4x^2}\left(1 + \frac{3}{2x}\right)^{-2}$

www.pdfworld.net

www.pdfworld.net

$$= \frac{1}{4x^2}\left\{1 - 2\left(\frac{3}{2x}\right) + 3\left(\frac{3}{2x}\right)^2 - \ldots.\right\};$$

$\therefore \quad \frac{1}{(2x+3)^2} = \frac{1}{4x^2} - \frac{3}{4x^3} + \frac{27}{16x^4}$ to three terms.

(iii) The first expansion is valid if $\left|\frac{2}{3}x\right| < 1$, that is if $|x| < \frac{3}{2}$.

The second is valid if $\left|\frac{3}{2}\right| < 1$, that is if $|x| > \frac{3}{2}$

➤ **Example 4.** Find the general term in the expansion of $(1+x)^{\frac{1}{2}}$

➤ ***Solution*** The $(r+1)^{th}$ term

$$= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots.\left(\frac{1}{2}-r+1\right)}{r\,!}x^r$$

$$= \frac{1(-1)(-3)(-5)\ldots.(-2r+3)}{2^r\,r\,!}x^r$$

The number of factors in the numerator is r, and $r-1$ of these are negative; therefore, by taking -1 out of each of these negative factors, we may write the above expression

$$= (-1)^{r-1}\frac{1.3.5\ldots.(2r-3)}{2^r\,r\,!}x^r$$

➤ **Example 5.** Find the $(r+1)$th term and the coefficient of x^r in the expansion of $(1-2x)^{-\frac{3}{2}}$, where $|2x| < 1$.

➤ ***Solution*** $\because$ $(r+1)$th term in $(1+x)^n$

$$= \frac{n(n-1)\ldots.(n-r+1)}{r\,!}x^r$$

$\therefore$ $(r+1)$th term in $(1-2x)^{-\frac{3}{2}}$ is given by

$$t_{r+1} = \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)\ldots..\left(-\frac{3}{2}-r+1\right)}{r\,!}(-2x)^r$$

$$= \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\ldots..\left(-\frac{1+2r}{2}\right)}{r\,!}(-2)^r x^r$$

$$= \frac{(-1)^r\,3.5.7\ldots...(2r+1)}{2^r.\,r\,!}(-1)^r.2^r.x^r$$

$$= \frac{(-1)^{2r}\,3.5.7\ldots...(2r+1)}{r\,!}x^r = \frac{3.5.7\ldots...(2r+1)}{r\,!}x^r$$

The coefficient of $x^r = \frac{3.5.7\ldots...(2r+1)}{r\,!}$

➤ **Example 6.** Find the coefficient of x^5 in the expansion of $\frac{1+x}{1-x}$ if $|x| < 1$.

➤ ***Solution*** $\frac{1+x}{1-x} = (1+x)(1-x)^{-1}$

$$= (1+x)\left[1 + \frac{(-1)}{1\,!}(-x) + \frac{(-1)(-1-1)}{2\,!}(-x)^2 + \frac{(-1)(-1-1)(-1-2)}{3\,!}(-x)^3 + \ldots.\text{to}\ \infty\right]$$

$$= (1+x)(1+x+x^2+x^3+x^4+\ldots.\ \text{to}\ \infty)$$

$$= [1+x+x^2+x^3+x^4+\ldots.\ \text{to}\ \infty] + [x+x^2+x^3+x^4+x^5+\ldots.\ \text{to}\ \infty]$$

$$= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \ldots.\ \text{to}\ \infty$$

Hence coefficient of $x^5 = 2$

➤ **Example 7.** Prove that the coefficient of x^r in $(1-x)^{-n}$ is ${}^{n+r-1}C_r$

➤ ***Solution*** $(r+1)$th term in the expansion of $(1-x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\ldots.(-n-r+1)}{r\,!}(-x)^r$$

$$= (-1)^r\frac{n(n+1)(n+2)\ldots.(n+r+1)}{r\,!}(-x)^r$$

$$= \frac{n(n+1)(n+2)\ldots.(n+r+1)}{r\,!}x^r$$

$$= \frac{(n-1)!\,n(n+1)\ldots.(n+r-1)}{(n-1)!\,r\,!}x^r$$

Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\,r\,!} = {}^{n+r-1}C_r$

➤ **Example 8.** Find the first three terms in the expansion of $(1+3x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{3}}$

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Expanding the two binomials as far as the term containing x^2, we have

$$\left(1+\frac{3}{2}x-\frac{9}{8}x^2-.....\right)\left(1+\frac{2}{3}x+\frac{8}{9}x^2+....\right)$$

$$=1+x\left(\frac{3}{2}+\frac{2}{3}\right)+x^2\left(\frac{8}{9}+\frac{3}{2}.\frac{2}{3}-\frac{9}{8}\right)....$$

$$=1+\frac{13}{6}x+\frac{55}{72}x^2....$$

➤ **Example 9.** If x is so small such that its square and higher powers may be neglected then find the value of $\dfrac{(1-3x)^{1/2}+(1+x)^{5/3}}{(4+x)^{1/2}}$

➤ ***Solution*** $\dfrac{(1-3x)^{1/2}+(1+x)^{5/3}}{(4+x)^{1/2}}=\dfrac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}}$

$$=\frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}=\frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$$

$$=\frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right)=1-\frac{x}{8}-\frac{19}{2}x=1-\frac{41}{24}x$$

➤ **Example 10.** Sum the series

$$1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+..... \text{ to } \infty$$

➤ ***Solution*** $(1+x)^n=1+\dfrac{n}{1!}x+\dfrac{n(n-1)}{2!}x^2+..... \text{ to } \infty$

Comparing the given series with (1), we get $nx=\dfrac{1}{3}$ ...(1)

and $\dfrac{n(n-1)}{2!}x^2=\dfrac{1.3}{3.6}$ ...(2)

Squaring (1) and dividing it by (2), we get

$$\frac{n^2x^2}{n(n-1)}\times\frac{2!}{x^2}=\frac{1}{9}\times\left(\frac{3.6}{1.3}\right)$$

or $\dfrac{2n}{n-1}=\dfrac{2}{3}$ or $n-1=3n$ or $n=-1/2$

From (1), $x=\dfrac{1}{3n}=\dfrac{1}{3\left(-\frac{1}{2}\right)}=-\dfrac{2}{3}$

Hence sum of the given series $=(1+x)^n$

$$=\left(1-\frac{2}{3}\right)^{-\frac{1}{2}}=\left(\frac{1}{3}\right)^{-\frac{1}{2}}=\frac{1}{\sqrt{\frac{1}{3}}}=\sqrt{3}$$

Alternative : The given series is

$$1+\frac{\frac{1}{2}}{1!}\left(\frac{2}{3}\right)+\frac{\frac{1}{2}\frac{3}{2}}{2!}\frac{\frac{1}{2}\frac{3}{2}}{2!}\left(\frac{2}{3}\right)^2+\frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{3!}\left(\frac{2}{3}\right)^3+....$$

$$\text{to } \infty=\left(1-\frac{2}{3}\right)^{-\frac{1}{2}}=\sqrt{3}$$

➤ **Example 11.** Given $n\geq 4$, $n\in N$ and $|x|<1$.

Let $f(x)=\dfrac{1+x^3}{1-x}$ & $g(x)=\dfrac{1-x+x^2}{1-x^2}$, find the co-efficient of x^n in the product $f(x).g(x)$.

➤ ***Solution*** $f(x).g(x)$

$$=\frac{(x^2-x+1)(x^2-x+1)}{(1-x)^{-2}}$$

$$=\left[(x^2+1)-x\right]^2(1-x)^{-2}$$

= co-efficient of x^n in $(x^4-2x^3+3x^2-2x+1)(1-x)^{-2}$

$=$ {co-efficient of x^{n-4} in $(1-x)^{-2}$}
-2 {co-efficient of x^{n-3} in $(1-x)^{-2}$}
$+3$ {coefficient of x^{n-2} in $(1-x)^{-2}$}
-2 { co-efficient of x^{n-1} in $(1-x)^{-2}$ }
$+$ {co-efficient of x^n in $(1-x)^{-2}$}

$={}^{n-3}C_1-2\,{}^{n-2}C_1+2\,{}^{n-1}C_1-2\,{}^{n}C_1+{}^{n+1}C_1$

$=n-1$

Practice Problems [L]

1. If $|x|<1$, show that $(1-x)^{-3}=1+3x+6x^2+10x^3+....$ to ∞
2. If $|x|<1$, show that $(1+x)^{-\frac{2}{5}}$

$$=1-\frac{x}{5}+\frac{3x^2}{25}-\frac{11x^3}{125}+..... \text{ to } \infty$$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

3. Expand the following expressions to four terms :

(i) $(1+x)^{\frac{3}{2}}$ (ii) $(1+2x)^{\frac{-1}{2}}$

(iii) $\left(1+\frac{x}{3}\right)^{-3}$

4. Find the 8th term of $(1+2x)^{\frac{-1}{2}}$

5. Find the $(r+1)^{th}$ term of $(1+x)^{\frac{1}{2}}$

6. Find the coefficient of x^6 in the expansion of $(1-2x)^{\frac{-5}{2}}$

7. If x be so small that its square and higher powers may be neglected, find the value of

(i) $(1-7x)^{\frac{1}{3}}(1+2x)^{\frac{-3}{4}}$

(ii) $\sqrt{4-x}.\left(3-\frac{x}{2}\right)^{-1}$

8. Find the first three terms in the expansion of $\frac{1}{(1+x)^2\sqrt{1+4x}}$

9. Show that the n^{th} coefficient in the expansion of $(1-x)^{-n}$ is double of the $(n-1)^{th}$.

10. If x is positive, which is the first negative term in the expansion of $(1+x)^{\frac{27}{5}}$?

11. If x is positive, the terms in the expansion of $(1-x)^{\frac{19}{2}}$, beginning with a certain term, are all positive. Prove this and find which is the first of this series of positive terms.

12. Sum the series $1+\frac{1}{1}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+....$ to ∞

5.17 MULTINOMIAL THEOREM

Multinomial Theorem for a Positive Integral Index

The idea of binomial theorem carries over from binomials to sums of more than two elements.

As an example consider the expression $(x + y + z + w)^{16}$; this, by definition, is a product of sixteen identical factors $x + y + z + w$:

$(x + y + z + w)(x + y + z + w).....(x + y + z + w)$.

The expansion of this has a term, for example, of the form $x^3y^5z^6w^2$, because the sum of the exponents is $3 + 5 + 6 + 2 = 16$. This particular term ocurs in the expansion as often as x can be chosen from three of the sixteen factors, y from five of the remaining thirteen factors, z from six of the remaining eight factors, and w then taken automatically from the other two factors. Paralleling the argument made in the case of the binomial expansion, we see that this is simply

$$^{16}C_3 \cdot {}^{13}C_5 \cdot {}^{8}C_6 \cdot 1 = \frac{16!}{13!3!}.\frac{13!}{8!5!}.\frac{8!}{6!2!} = \frac{16!}{3!5!6!2!}$$

The expansion of $(x + y + z + w)^{16}$ has been shown to contain the term $\frac{16!}{3!5!6!2!}x^3y^5z^6w^2$.

This coefficient is equivalent to calculating the number of ways of ordering the following sixteen letters :

x x x y y y y y z z z z z z w w.

More generally, we can say that the expansion of $(x + y + z + w)^{16}$ is the sum of all terms of the form

$$\frac{16!}{a!b!c!d!}x^ay^bz^cw^d.$$

where a, b, c, d range over all possible sets of non-negative integers satisfying $a + b + c + d = 16$. As a simple case we note the solution $a = 16, b = 0, c = 0, d = 0$, belonging to the term

$$\frac{16!}{16!0!0!0!}x^{16}y^0z^0w^0.$$

or more simply x^{16} in the expansion. Other solutions of $a + b + c + d = 16$ are for example, $a = 4, b = 5, c = 5, d = 2$ and $a = 3, b = 5, c = 2, d = 6$ and belong to the terms

$$\frac{16!}{4!5!5!2!}x^4y^5z^5w^2 \text{ and } \frac{16!}{3!5!2!6!}x^3y^5z^2w^6,$$

respectively.

www.pdfworld.net

www.pdfworld.net

Further generalization is apparent. For any positive integer
(in place of the special number 16) we see that the expansion of $(x + y + z + w)^n$ is the sum of all terms of the form

$$\frac{n!}{a!b!c!d!} x^a y^b z^c w^d,$$

where a, b, c, d range over all solutions of $a + b + c + d = n$ in non-negative integers.
There is no reason to restrict attention to a sum of four elements x, y, z, w.

For any positive integer n, the multinomial expansion of $(x + y + z + w + ...)^n$ is the sum of all terms of the form

$$\frac{n!}{a!b!c!d!.....} x^a y^b z^c w^d ....., $$

where a, b, c, d,.... range over all solutions of $a + b + c + d + ... = n$ in non-negative integers.

The number of terms in the expansion of $(x_1 + x_2 + ... + x_r)^n$ = number of whole number solutions of
$a_1 + a_2 + ... + a_r = n$
$= {}^{r+n-1}C_n$ or ${}^{r+n-1}C_{r-1}$
Examples :
(i) The binomial expansion can also be written as

$$(x + y)^n = \sum_{\alpha+\beta=n} \frac{n!}{\alpha!\,\beta!} x^\alpha y^\beta$$

Here the summation runs over all those non negative integers whose sum is n
e.g. $\alpha = 0, \beta = n$; $\alpha = 1. \beta = n - 1$ and so on.

$$(ii)\ (x + y + z)^n = \sum_{\alpha+\beta+\gamma=n} \frac{n!}{\alpha!\,\beta!\,\gamma!} x^\alpha y^\beta z^\gamma$$

(iii) In the expansion of $(a - b - c + d)^6$ the coefficient of a^3b^2c is $(-1)^3 \frac{6!}{3!\,2!\,1!\,0!} = -60$

➤ **Example 1.** Find the coefficient of x^8 in the expansion of $(1 + x^2 - x^3)^9$.

➤ ***Solution*** We have

$$(1 + x^2 - x^3)^9 = \sum_{\alpha+\beta+\gamma=9} \frac{9!}{\alpha!\,\beta!\,\gamma!} 1^\alpha (x^2)^\beta (-x^3)^\gamma$$

$$= \sum_{\alpha+\beta+\gamma=9} \frac{9!}{\alpha!\,\beta!\,\gamma!} (-1)^\gamma \times x^{2\beta+3\gamma}$$

Now we must have $2\beta + 3\gamma = 8$ also $\alpha + \beta + \gamma = 9$.
It is easy to see that there are only two triplets of values of α, β, γ which can satisfy both equations namely (6, 1, 2) and (5, 4, 0). Thus the coefficient of x^8 is

$$\frac{9!}{6!\,1!\,2!}(-1)^2 + \frac{9!}{5!\,4!\,0!}(-1)^0 = 378.$$

Alternative

$$(1 + x^2 - x^3)^9 = (1 + x^2)^9 + {}^9C_1(1 + x^2)^8 (-x^3) + {}^9C_2(1 + x^2)^7 (-x^3)^2 + {}^9C_3(1 + x^2)^6 (-x^3)^3 + ...$$

Now, there is no need to consider fourth term and terms after that, since the smallest power of x in these will be 9. There will not be a term containing x^8.
Therefore, coefficient of x^8 = coefficient of x^8 in
$(1 + x^2)^9 + {}^9C_2$ {coefficient of x^2 in $(1 + x^2)^8$}
$= {}^9C_4 + {}^9C_2 . {}^7C_1 = 378$

➤ **Example 2.** Find the coefficient of $a^3 . b^4 . c^7$ in the expansion of $(bc + ca + ab)^8$

➤ ***Solution*** Here $a^3 . b^4 . c^7 = (ab)^x (bc)^y (ca)^z$
$\Rightarrow$ $z + x = 3, x + y = 4, y + z = 7$
$\Rightarrow$ $y = 4, z = 3, x = 0$
$\therefore$ coefficient of $a^3 . b^4 . c^7$ in the expansion of $(bc + ca + ab)^8$
= the coefficient of $(ab)^0.(bc)^4.(ca)^3$ in the expansion of $(bc + ca + ab)^8$

$$= \frac{8!}{0!.\,4!.\,3!} = 280$$

➤ **Example 3.** Find sum of rational terms in $(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{5})^{10}$

➤ ***Solution*** The general term of the given expression

$$= \frac{10!}{\alpha!\,\beta!\,\gamma!} 2^{\alpha/2} . 3^{\beta/3} . (5)^{\gamma/6}$$

where α, β, γ satisfy the following properties :
$0 \le \alpha, \beta, \gamma \le 10$; $\alpha + \beta + \gamma = 10$
$\therefore$ $\alpha = 0, 2, 4, 6, 8, 10$; $\beta = 0, 3, 6$; $\gamma = 0, 6$
Hence possible pairs of (α, β, γ)
$= (4, 6, 0)$; $(4, 0, 6)$; $(10, 0, 0)$
$\therefore$ There exists three rational terms.
So, sum of rational terms

$$= \frac{10!}{4!.\,6!} . 2^2 . 3^2 + \frac{10!}{4!.\,6!} . 2^2 . 5 + \frac{10!}{10!} . 2^5 = 12632$$

www.pdfworld.net

www.pdfworld.net

➤ **Example 4.** Find the coefficient of $a^2 b^3 c^4 d$ in the expansion of $(1 - b - c + d)^{10}$

➤ ***Solution*** $(a - b - c + d)^{10}$

$$= \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!r_2!r_3!r_4!}(a)^{r_1}(-b)^{r_2}(-c)^{r_3}(d)^{r_4}$$

Since we want to get $a^2 b^3 c^4 d$ this implies that

$r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 1$

$\therefore$ coefficient of $a^2 b^3 c^4 d$ is

$$\frac{(10)!}{2!\,3!\,4!\,1!}(9-1)^3(-1)^4 = -12600$$

➤ **Example 5.** Find the coefficients of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$.

➤ ***Solution*** Coefficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$ is $\sum \frac{10!}{n!n_2!n_3!}(1)^{n_1}(3)^{n_2}(-1)^{n_3}$

where $n_1 + n_2 + n_3 = 10$ and $n_2 + 3n_3 = 7$, the possible values of n_1, n_2 and n_3 are shown here

n_1	n_2	n_3
3	7	0
5	4	1
7	1	2

$\therefore$ The coefficient of $x^7 = \frac{10!}{3!7!0!}(1)^3(3)^7(-2)^0$

$$+ \frac{10!}{5!4!1!}(1)^5(3)^4(-2)^1 + \frac{10!}{7!1!2!}(1)^7(3)^1(-2)^2$$

$$= 262440 - 20120 + 4320 = 62640$$

The greatest coefficient in the expansion of $(a_1 + a_2 + a_3 + ... + a_m)^n$ is $\frac{n!}{(q!)^{m-r}((q+1)!)^r}$

where q is the quotient and r is the remainder when n is divided by m.

➤ **Example 6.** Find the greatest coefficient in the expansion of $(a + b + c + d)^{15}$

➤ ***Solution*** Here n = 15 and m = 4

($\because$ a, b, c, d are four terms)

$15 = 4 \times 3 + 3$ $\quad\therefore$ q = 3 and r = 3

Hence the greatest coefficient = $\frac{15!}{(3!)^1(4!)^3}$

Practice Problems [M]

1. Find the coefficient of the term $x^2y^3z^2$ in the expansion of $(x + y + z)^7$.
2. What is the coefficient of xyzwuv in the expansion of $(x + y + z + w + u + v)^6$?
3. Find the coefficient of (i) a^3b^3c in the expansion of $(2a + b + 3c)^7$ (ii) x^6 in the expansion of $(1 + 2x - x^2)^5$.
4. Verify that the expansion of $(x + y)^8$ can be expressed in this way : the sum of all terms of the form $\frac{8!}{a!\,b!}x^a y^b$ where a and b range over all pairs of non-negative integers a and b such that a + b = 8.
5. How many terms are there in each of the expansions :
   (a) $(x + y + z)^6$ (b) $(a + 2b + 5c + d)^4$?

5.18 EXPONENTIAL SERIES

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \infty$; where x may be any real or complex

(ii) If a > 0 and x is any number, then $a^x = e^{x \log a}$

$$= 1 + \frac{x}{1!}\ln a + \frac{x^2}{2!}\ln^2 a + \frac{x^3}{3!}\ln^3 a + ... \infty$$

(a) e is an irrational number lying between 2.7 & 2.8.

www.pdfworld.net

www.pdfworld.net

The sum of the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....$ to ∞ lies between 2 and 3 and is denoted by the symbol e [read as exponential number]. Its value correct upto 10 places of decimal is 2.7182818284.

Thus $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+..... \text{ to } \infty$

(b) $e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+..... \text{ to } \infty$, where x is any number.

(c) $e+e^{-1} = 2\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+..... \infty\right)$

(d) $e-e^{-1} = 2\left(1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+..... \infty\right)$

➤ **Example 1.** Show that

$$\frac{1}{e} = 2\left[\frac{1}{3!}+\frac{2}{5!}+\frac{3}{7!}+.....\text{to } \infty\right]$$

➤ ***Solution*** The nth term of the series $2\left[\frac{1}{3!}+\frac{2}{5!}+\frac{3}{7!}+.....\right]$ $t_n = \frac{2n}{(2n+1)!} = \frac{2n+1-1}{(2n+1)!}$

$$= \frac{2n+1}{(2n+1)!} - \frac{1}{(2n+1)!} = \frac{1}{2n!} - \frac{1}{(2n+1)!}$$

$\therefore t_1 = \frac{1}{2!}-\frac{1}{3!}, t_2 = \frac{1}{4!}-\frac{1}{5!}, t_3 = \frac{1}{6!}-\frac{1}{7!}$ etc.

Sum of the given series

$$= \frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+.....$$

$$= 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+..... = e^{-1} = \frac{1}{e}$$

➤ **Example 2.** Find the sum of the series $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+..... \text{ to } \infty$

➤ ***Solution*** $t_n = \frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{n-1+1}{(n-1)!}$

$$= \frac{n-1}{(n-1)!} + \frac{1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$$

$\therefore t_1 = 0+\frac{1}{0!}, \quad t_2 = \frac{1}{0!}+\frac{1}{1!}, t_3 = \frac{1}{1!}+\frac{1}{2!}, ...$

Sum of the series

$$= \left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+.....\right)+\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+.....\right)$$

$= e+e = 2e$

➤ **Example 3.** Find the sum of the series $1+\frac{1+2}{2!}+\frac{1+2+2^2}{3!}+\frac{1+2+2^2+2^3}{4!}+....\text{to } \infty$

➤ ***Solution*** $t_n = \frac{1+2+2^2+....+2^{n-1}}{n!} = \frac{1(2^n-1)}{(2-1)n!}$

$$= \frac{2^n-1}{n!} = \frac{2^n}{n!}-\frac{1}{n!}\ t_1 = \frac{2}{1!}-\frac{1}{1!}, t_2$$

$$= \frac{2^2}{2!}-\frac{1}{2!}, \quad t_3 = \frac{2^3}{3!}-\frac{1}{3!}, \quad .....$$

$\therefore$ Sum of the given series

$$= \left[\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+....\right]-\left[\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+....\right]$$

$= (e^2-1)-(e-1) = e^2-e = e(e-1)$

➤ **Example 4.** Find the sum of

$$(x+y)(x-y)+\frac{1}{2!}(x+y)(x-y)(x^2+y^2)$$

$$+\frac{1}{3!}(x+y)(x-y)(x^4+x^2y^2+y^4)+.....$$

➤ ***Solution*** The given series

$$= (x^2-y^2)+\frac{1}{2!}(x^4-y^4)+\frac{1}{3!}(x^6-y^6)+.....$$

$$= \left(1+\frac{x^2}{1!}+\frac{x^4}{2!}+.....\right)-\left(1+\frac{y^2}{1!}+\frac{y^4}{2!}+....\right)$$

$= e^{x^2}-e^{y^2}$

Summation of Series $\Sigma\, u_n.\frac{x^n}{n!}$, where u_n is a polynomial in n

We can find $a_0, a_1, ... a_r$, independent of n, so that $u_n = a_0+a_1n+a_2n(n-1)+....+a_rn(n-1)...(n-r+1)$, and then $\Sigma_{n=0}^{\infty} u_n.\frac{x^n}{n!} = a_0\Sigma_{n=0}^{\infty}\frac{x^n}{n!}+a_1x\Sigma_{n=1}^{\infty}\frac{x^{n-1}}{n-1!}$

www.pdfworld.net

www.pdfworld.net

$$+ a_2x^2 \Sigma_{n=2}^{\infty} \frac{x^{n-2}}{n-2!} + ... + a_rx^r \Sigma_{n=r}^{\infty} \frac{x^{n-r}}{n-r!}$$

$$= (a_0 + a_1x + a_2x^2 + ... + a_rx^r)\, e^x$$

➤ **Example 5.** Find the value of $\Sigma\,(n^3+5)/n!$

➤ ***Solution*** Here $n^3+5 = n(n-1)(n-2)+3n(n-1)+n+5$; hence, putting $a_0=5, a_1=1, a_2=3, a_3=1$ and $x=1$ in the result above, we have

$$\Sigma_{n=0}^{\infty}\,(n^3+5) = (5+1+3+1)e = 10e$$

Summation of Series $\Sigma\, u_n x^n / (n+a)(n+b)...(n+k)$ n! , where u_n is a polynomial in n

and a, b, ... k are unequal positive integers.

Series of this kind can be summed as in the following example :

➤ **Example 6.** Find the value of $\Sigma_{n=1}^{\infty} \frac{n^2+1}{n+2} \cdot \frac{x^n}{n!}$

➤ ***Solution*** Denote the sum by S, then

$$S = \Sigma_{n=1}^{\infty} \frac{(n^2+1)(n+1)}{(n+2)} . x^n$$

Now $(n^2+1)(n+1) = n^3+n^2+n+1$

$= a + b(n+2) + c(n+2)(n+1) + d(n+2)(n+1)n$,

where a, b, c, d, can be found by division, their values being a = –5, b = 5, c = –2, d = 1.

$$\text{Thus } S = -5\Sigma_1^{\infty} \frac{x^n}{(n+2)!} + 5\Sigma_1^{\infty} \frac{x^n}{(n+1)!} - 2\Sigma \frac{x^n}{n!} + \Sigma \frac{x^n}{(n-1)!}$$

$$\text{Now } \Sigma_1^{\infty} \frac{x^n}{(n+2)!} = \frac{1}{x^2}\left(e^x - 1 - x - \frac{x^2}{2}\right)$$

$$\Sigma_1^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x}\left(e^x - 1 - x\right), \quad \Sigma_1^{\infty} \frac{x^n}{n!} = e^x - 1.$$

$$\Sigma_1^{\infty} \frac{x^n}{(n-1)!} = xe^x$$

$$\therefore S = \frac{1}{2x^2}(10 - x^2) - \frac{e^x}{x^2}(5 - 5x + 2x^2 - x^3)$$

Practice Problems [N]

1. Prove that $\frac{e^x - e^{-x}}{x} = 2\left[1 + \frac{x^3}{3!} + \frac{x^4}{5!} + ...\text{to } \infty\right]$

Find the sum to infinity of the series :

2. $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \frac{4}{9!} + ... \text{ to } \infty$
3. $\frac{1}{2!} + \frac{3}{4!} + \frac{5}{6!} + \frac{7}{8!} + ...\text{to } \infty$
4. $1 + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + ...$
5. $\frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \frac{9}{5!} + ...$
6. $\frac{3}{2!} + \frac{5}{4!} + \frac{7}{6!} + \frac{9}{8!} + ...$
7. $1 + \frac{1+2}{2!} + \frac{1+2+3}{3!} + ...$
8. $\frac{1}{2} - \frac{1}{3(1)!} + \frac{1}{4(2)!} - \frac{1}{5(3)!} + ...$
9. $1^3 + \frac{2^3}{1!} + \frac{3^3}{2!} + \frac{4^3}{3!} + ...$
10. $1^3 - \frac{2^3}{1!} + \frac{3^3}{2!} - \frac{4^3}{3!} + ...$
11. $1 + \frac{x}{2!} + \frac{2x^2}{3!} + \frac{3x^3}{4!} + ...$
12. Prove that

$$\Sigma_0^{\infty} \frac{(n+1)^3}{n!} x^n = (1 + 7x + 6x^2 + x^3)e^x$$

5.19 LOGARITHMIC SERIES

1. $l\text{n}\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \text{ to } \infty$, where $-1 < x \le 1$
2. $l\text{n}\,(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ..... \text{ to } \infty$, where $-1 \le x < 1$
3. $\ell\text{n}\left(\frac{1+x}{1-x}\right) = l\text{n}\,(1+x) - l\text{n}\,(1-x)$

www.pdfworld.net

www.pdfworld.net

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\text{to } \infty\right]$$

$$- \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots\text{to } \infty\right]$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right], \text{ where } |x| < 1$$

4. $l\text{n } 2 = 1 - \frac{1}{2} + \frac{1}{3} - \ldots. + (-1)^{n-1}\frac{1}{n} + \ldots.$ to ∞ = 0.693...

5. $l\text{n}10 = 2.303$

➤ **Example 1.** Show that $l\text{n } 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots.$

➤ ***Solution*** Since $l\text{n } (1 + x)$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots..$$

$$\therefore \quad l\text{n } (1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \ldots\ldots \text{ to } \infty$$

or $$l\text{n } 2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots\ldots$$

$$= \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots\ldots.$$

Series which can be summed by the logarithmic series

If k is a positive integer and $|x| < 1$, then

$$\Sigma_{n=1}^{n=\infty} \frac{x^n}{n+k} = -\frac{1}{x^k}\left\{x + \frac{x^2}{2} + \ldots. + \frac{x^k}{k} + \log(1-x)\right\}$$

➤ **Example 2.** If $|x| < 1$. Find the value of $\Sigma_{n=1}^{n=\infty} \frac{n^3 + n^2 + 1}{n(n+2)} x^n$

➤ ***Solution*** The series is convergent when $|x| < 1$

Let S be the sum, then since

$$\frac{n^3 + n^2 + 1}{n(n+2)} = n - 1 + \frac{1}{2}.\frac{1}{n} + \frac{3}{2}.\frac{1}{n+2},$$

we have $S = \Sigma_1^{\infty}(n-1)x^n + \frac{1}{2}\Sigma_1^{\infty}\frac{x^n}{n} + \frac{3}{2}\frac{x^n}{n+2}$

$$\therefore \quad S = \frac{x^2}{(1-x)^2} - \frac{1}{2}\log(1-x)$$

$$- \frac{3}{2x^2}\left\{x + \frac{x^2}{2} + \log(1-x)\right\}$$

This method can be applied to sum any series of which the general term is $P x^n/(n + 1)n + b)\ldots (n + k)$, where P is a polynomial in n and a, b, ... k are unequal positive integers.

➤ **Example 3** Find the value of

$$\frac{5}{1.2.3} + \frac{7}{3.4.5} + \frac{9}{5.6.7} + \ldots\ldots \text{ to } \infty.$$

➤ ***Solution*** We have log 2

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots..$$

$$= \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots..$$

Again, log 2

$$= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \ldots.. = 1 - \frac{1}{2.3} - \frac{1}{4.5} - \ldots.$$

$$\therefore \quad 2\log 2 = 1 + \left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{3.4} - \frac{1}{4.5}\right) + \ldots..$$

$$= 1 + \frac{2}{1.2.3} + \frac{2}{3.4.5} + \frac{2}{5.6.7.} + \ldots.$$

The given series $= \frac{3+2}{1.2.3} + \frac{5+2}{3.4.5} + \frac{7+2}{5.6.7} + \ldots..$

$$= \left(\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots.\right)$$

$$+ \left(\frac{2}{1.2.3} + \frac{2}{3.4.5} + \frac{2}{5.6.7}\right) + \ldots..$$

$$= \log 2 + 2\log 2 - 1 = 3\log 2 - 1$$

Practice Problems [O]

1. Prove that $\frac{\log(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\ldots\ldots$ to ∞

2. Prove that $\log\frac{1}{1+x} = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \ldots..$ to ∞

www.pdfworld.net

www.pdfworld.net

3. In the expansion of $(1 + bx + cx^2) \log_e (1 + x)$, find the coefficients of x^3 and x^4, and determine the values of b and c so that each of these coefficients may vanish.

Sum to infinity the series :

4. $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots.$

5. $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \ldots.$

6. $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + \ldots.$

7. $\frac{1}{1.2} + \frac{x}{3.4} + \frac{x^2}{5.6} + \frac{x^3}{7.8} + \ldots.$

8. $1 + \frac{x^2}{3} + \frac{x^4}{15} + \ldots + \frac{x^{2n}}{4n^2 - 1} + \ldots\ldots$

Target Problems for JEE Advanced

➤ **Problem 1.** In the expansion, by the binomial formula, of the expression $\left(x\sqrt{x} + \frac{1}{x^4}\right)^n$ the binomial coefficient in the third term is by 44 larger than that in the second term. Find the term not containing x.

➤ ***Solution*** From the conditions of the problem it follows that ${}^nC_2 = {}^nC_1 + 44$, or $\frac{n(n-1)}{2} = n + 44$.

Solving this equation for n we find n = 11.

The general term of the expansion of $\left(x\sqrt{x} + \frac{1}{x^4}\right)^{11}$ is ${}^{11}C_m x^{\frac{3}{2}(11-m)-4m}$

By the hypothesis we have $\frac{3}{2}(11 - m) - 4m = 0$ which yields m = 3.

Hence, the sought for term is equal to ${}^{11}C_3$

➤ **Problem 2.** In the expansion of the expression $\left(1 + x + \frac{6}{x}\right)^{10}$ find the term not containing x.

➤ ***Solution*** Putting $x + \frac{6}{x} = u$ we can write

$\left(1 + x + \frac{6}{x}\right)^{10} = (1 + u)^{10} = 1 + {}^{10}C_1 u + {}^{10}C_2 u^2 + \ldots\ldots + {}^{10}C_{10} u^{10}$,

where $u^k = \left(x + \frac{6}{x}\right)^k = x^k + {}^kC_1 x^{k-2} 6 + \ldots.$

$+ {}^kC_s x^{k-2s} 6^s + \ldots.. + \frac{6^k}{x^k}$ ...(1)

for every summand in expression (1) which does not contain x we have the condition k – 2s = 0. Consequently, this summand is equal to ${}^sC_{2s}.6^s$. Collecting all these terms we conclude that a summand not containing x in the original expression is equal to

$1 + {}^{10}C_2 \cdot {}^2C_1 \cdot 6 + {}^{10}C_4 \cdot {}^4C_2 \cdot 6^2 + {}^{10}C_6 \cdot {}^6C_3 \cdot 6^3 + {}^{10}C_8 \cdot {}^8C_4 \cdot 6^4 + {}^{10}C_{10} \cdot {}^{10}C_5 \cdot 6^5$.

➤ **Problem 3.** Find the value of x for which the sixth term of $\left(\sqrt{(2^{\log(10-3^x)})} + 5\sqrt{2^{(x-2)\log 3}}\right)^m$ is equal to 21 and coefficients of second, third and fourth terms are the first, third and fifth terms of an arithmetic progression.

➤ ***Solution*** The sixth term of given binomial expansion = 21

$\therefore \quad {}^mC_5 \left(\sqrt{(2^{\log(10-3^x)})}\right)^{m-5} \cdot (5\sqrt{2^{(x-2)\log 3}})^5 = 21$

The other given condition is, ${}^mC_1 + {}^mC_3 = 2 \cdot {}^mC_2$

$\because \quad m \neq 0$ so $m^2 - 9m + 14 = 0$

$\Rightarrow \quad (m - 7)(m - 2) = 0$

$\because \quad m = 2$ is not possible $\quad \because \quad m = 7$

Putting m = 7 in equation (1), we get x = 0, 2.

➤ **Problem 4.** In the binomial expansion of $\left(\sqrt{x} + \frac{1}{2\sqrt[4]{x}}\right)^n$ the first three coefficients form an arithmetic progression. Find all the rational terms in the expansion assuming x as prime.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Since the coefficients 1, $\frac{n}{2}, \frac{n(n-1)}{8}$ form an arithmetic progression, we can form an equation

$$\frac{n(n-1)}{8} + 1 = n,$$

whose roots are n = 8 and n = 1 respectively ; n = 1 is an extraneous root. For n = 8 the general term of the expansion has the form

$$T_k = \binom{8}{k} x^{\frac{k}{2}} \left(\frac{1}{2}\right)^{8-k} x^{\frac{8-k}{4}} = 2^{k-8} \binom{8}{k} x^{\frac{k+8}{4}}$$

This term is rational if k + 8 a multiple of 4, where $0 \le k \le 8$. This condition is fulfilled for k = 0, 4, 8.
The first, the fifth and the ninth terms of the expansion are rational.

➤ **Problem 5.** Some three consecutive coefficients in the expansion of $(1 + x)^n$ are in AP. Find two suitable positive integral values of n and the three coefficients.

➤ ***Solution*** Let the (r + 1)th, (r + 2)th and (r + 3)th coeffcients be in AP.

$\therefore$ ${}^nC_r, {}^nC_{r+1}, {}^nC_{r+2}$ are in AP

$\Rightarrow$ $2.\ {}^nC_{r+1} = {}^nC_r + {}^nC_{r+2}$

$\Rightarrow$ $2 . \frac{n!}{(r+1)!(n-r-1)!}$

$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+2)!(n-r-2)!}$

$\Rightarrow$ $\frac{2}{(r+1)(n-r-1)}$

$= \frac{1}{(n-r)(n-r-1)} + \frac{1}{(r+2)(r+1)}$

$\Rightarrow 2(r+2)(n-r) = (r+1)(r+2) + (n-r)(n-r-1)$

$\Rightarrow (n-r)^2 - (2r+5)(n-r) + (r^2+3r+2) = 0$

$\Rightarrow n - r = \frac{2r+5 \pm \sqrt{(2r+5)^2 - 4(r^2+3r+2)}}{2}$

$\Rightarrow n = r + \frac{2r+5 \pm \sqrt{8r+17}}{2}$

Here n and r are positive integers. So 8r + 17 must be a perfect square for positive integral value of r.

When r = 1, $n = 1 + \frac{7 \pm \sqrt{25}}{2} = 1 + \frac{7 \pm 5}{2} = 7, 2.$

r = 1, n = 2 is not possible because if n = 2, there will be only three terms in the expansion
and for r = 1, r + 3 = 4.
r = 1, n = 7 is possible. In this case the coefficients are ${}^7C_1, {}^7C_2, {}^7C_3$.

Again, when r = 4, $n = 4 + \frac{13 \pm \sqrt{49}}{2} = 4 + \frac{13 \pm 7}{2}$
= 14, 7. n = 7 has already been taken.
So r = 4, n = 14 and the coefficients are ${}^{14}C_4, {}^{14}C_5, {}^{14}C_6$, respectively.

➤ **Problem 6.** Find the condition under which the expansion of $(1 + x)^n$ in powers of a (where n is an integer and $a \neq 0$) contains two equal consecutive terms. Can this expansion contain three equal consecutive terms ?

➤ ***Solution*** The general term T_{k+1} of the expansion is equal to ${}^nC_k a^k$.
If $T_k = T_{k+1}$, then ${}^nC_{k-1}\ a^{k-1} = {}^nC_k a^k$, that is

$$\frac{n!a^{k-1}}{(k-1)!(n-k+1)!} = \frac{n!a^k}{k!(n-k)!}$$

We obtain $k = \frac{n+1}{1+\frac{1}{a}}$. We have thus established the required condition :

the number $1 + \frac{1}{a}$ must be the divisor for the number n + 1.

Furthermore, the relation $T_k = T_{k+1} = T_{k+2}$ is equivalent to the equalities

$$\frac{1}{(n-k+1)(n-k)} = \frac{a}{k(n-k)} = \frac{a^2}{k(k+1)}$$

that is $\frac{k}{n-k+1} = a$, $\frac{k+1}{n-k} = a$

From the latter relations we obtain the equality n + 1 = 0 which is impossible.

➤ **Problem 7.** Let $(1 + x^2)^2 (1 + x)^n = \sum_{k=0}^{n+4} a_k x^4$. If a_1, a_2 and a_3 are in arithmetic progression, find n.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** We have $(1+x^2)^2(1+x)^n = (1+2x^2+x^4)\times\left\{1+nx+\frac{n(n-1)}{2!}x^2\frac{n(n-1)(n-2)}{3!}x^3+\ldots\right\}$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$ (say)

Now, comparing the coefficients of x, x^2 and x^3, we get

$a_1 = n$, $a_2 = 2 + \frac{n(n-1)}{2} = \frac{n^2-n+4}{2}$, and

$a_3 = 2n + \frac{n(n-1)(n-2)}{6} = \frac{n^3-3n^2+14n}{6}$

But a_1, a_2, a_3 are in A.P.,

Therefore $2a_2 = a_1 + a_3$

$\Rightarrow \quad n^2 - n + 4 = n + \frac{n^3-3n^2+14n}{6}$

$\Rightarrow \quad 6n^2 - 6n + 24 = 6n + n^3 - 3n^2 + 14n$

$\Rightarrow \quad n^3 - 9n^2 + 26n - 24 = 0$

$\Rightarrow \quad (n-2)(n-3)(n-4) = 0$

$\therefore \quad n = 2, 3, 4.$

➤ **Problem 8.** Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

➤ ***Solution*** We have $2222^{5555} + 5555^{2222} =$

$= (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222})$ ...(1)

The number $2222^{5555} + 4^{5555}$ is divisible by $2222 + 4 = 2226 = 7.318$ which is divisible by 7. (because a sum of two odd powers is always divisible by the sum of the base of the powers),

The difference $5555^{2222} - 4^{2222}$ is also divisible by 7 since it is divisible by $5555 - 4 = 5551 = 7.793$ (because the difference of any integral powers with equal exponents is divisible by the difference of the base)

The difference $4^{5555} - 4^{2222}$

$= 4^{2222}(4^{3333} - 1) = 4^{2222}(2^{6666} - 1) = 4^{2222}(64^{1111} - 1)$

This expression is divisible by $64 - 1 = 63 = 7.9$. Hence $4^{5555} - 4^{2222}$ is also divisible by 7. Hence each brackets of (1) are divisible by 7.

Thus $2222^{5555} + 5555^{2222}$ is divisible by 7.

➤ **Problem 9.** Find the remainder which is obtained when the number

$10^{10} + 10^{(10^2)} + \ldots.. + 10^{(10^{10})}$ is divided by 7.

➤ ***Solution*** First of all we note that the number $10^6 - 1 = 999999$ is divisible by 7 (because $999\,999 = 7.142857$). It readily follows that the division of 10^N by 7 (where N is an arbitrary whole number) leaves the same remainder as the division by 7 of the number 10^r where r is the remainder resulting from the division of N by 6. Indeed, if $N = 6k + r$ then the number

$10^N - 10^r = 10^{6k+r} - 10^r = 10^r(10^{6k} - 1)$

$= 10^r . (10^6 - 1)(10^{6k-6} + 10^{6k-12} + \ldots. + 10^6 + 1)$ is divisible by 7.

Further, the division of any integral power of 10 by 6 leaves a remainder equal to 4; indeed, according to the tests for divisibility by 2 and by 3, the difference $\underbrace{10^n - 4 = 999\ldots96}_{(n-1)\text{ times}} = 6$. Thus the remainders resulting from the division by 6 of all the exponents of the powers in the addends of the above sum are equal to 4. Consequently, when each of these 10 addends is divided by 7 we obtain the same remainder as in the case when 10^4 is divided by 7, and the division of the whole sum by 7 leaves the same remainder as the division by 7 of the number

$10^4 + 10^4 + 10^4 + 10^4 + 10^4 + 10^4 + 10^4 + 10^4 + 10^4 + 10^4 = 10^5 = 100\,000 = 7.14\,285 + 5$

Thus, the sought for remainder is equal to 5.

➤ **Problem 10.** Prove that the decimal representations of the numbers 9^{9^9} and $9^{9^{9^9}}$ (where for instance, 9^{9^9} means $9^{(9^9)}$) have the same last two digits.

➤ ***Solution*** It is clear that both numbers 9^{9^9} and $9^{9^{9^9}}$ end with 9, that is their last digits coincide.

Further, by binomial theorem, we have

$A = 9^{9^9} = (10-1)^{9^9}$

$= 10^a - {}^aC_1.10^{a-1} + {}^aC_2.10^{a-2} - \ldots + {}^aC_{n-1}.10 - 1$ and

$B = 9^{9^{9^9}} = (10-1)^{9^{9^9}}$

$= 10^b - {}^bC_1.10^{b-1} + {}^bC_2.10^{b-2)} - \ldots. + {}^bC_{n-1}.10 - 1$

where $a = 9^9$ and $b = 9^{9^9}$.

www.pdfworld.net

www.pdfworld.net

Thus, the last two digits of the numbers under consideration coincide with the last two digits of the numbers.

${}^{a}C_1.10 - 1 = 10a - 1$ and ${}^{b}C_1.10 - 1 = 10b - 1$ respectively.

Further, both numbers $a = 9^9$ and $b = 9^{9^9}$ end with the digit 9. Therefore the last two digits of the numbers 10a and 10b are 90 and those of the numbers A and B are 89.

➤ **Problem 11.** Find the last two digits of $7^{(14^{14})}$.

➤ ***Solution*** The number $7^4 - 1 = 2401 - 1 = 2400$ is divisible by 100. It follows that if $n = 4k$ (i.e. if n is divisible by 4) then $7^n - 1$ is divisible by 100 (because $7^{4k} - 1 = (7^4)^k - (1)^k$ is divisible by $7^4 - 1$).

Further, $14^{14} = 2^{14}.\ 7^{14}$ is divisible by 4, and consequently $7^{(14^{14})} - 1$ is divisible by 100.

It follows that the last two digits of the number $7^{(14^{14})}$ are 01.

➤ **Problem 12.** Let I denotes the integral part and F the proper fractional part of $\left(3+\sqrt{5}\right)^n$ where $n \in N$ and if ρ denotes the rational part and σ the irrational part of the same, show that

$$\rho = \frac{1}{2}(I+1) \text{ and } \sigma = \frac{1}{2}(I+2F-1).$$

➤ ***Solution*** Given $\left(3+\sqrt{5}\right)^n = I + F$

Consider $(3-\sqrt{5})^n = F'$ (a proper fraction)

Now $\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n = 2\,[{}^{n}C_0 . 3^n + {}^{n}C_2 . 3^{n-2} . 5 + {}^{n}C_4 . 3^{n-4} . 5^2 + \ldots\ldots]$

or $I + F + F' = 2\,[3^n + {}^{n}C_2\, 3^{n-2} . 5 + \ldots\ldots]$

= even integer

Hence $F + F'$ is also an integer but $0 < F + F' < 2$

$\Rightarrow F + F' = 1$

Hence $I = 2\,[3^n + {}^{n}C_2 . 3^{n-2} . 5 + \ldots\ldots] - 1$ ...(1)

and $F = 1 - \left(3-\sqrt{5}\right)^n$

Now $\rho = {}^{n}C_0 . 3^n + {}^{n}C_2 . 3^{n-2} . 5 + {}^{n}C_4 . 3^{n-4} . 5^2 + \ldots\ldots$

$\therefore \quad \rho = \dfrac{I+1}{2}$ from (1)

Also $I + F = \rho + \sigma$

$\therefore \quad \sigma = I + F - \rho = I + F - \left(\dfrac{I+1}{2}\right)$

$= \dfrac{2I+2F-I-1}{2} \quad \sigma = \dfrac{I+2F-1}{2}$

➤ **Problem 13.** Prove that if p is an odd prime, it divides the difference $[(2+\sqrt{5})^p] - 2^{p+1}$

➤ ***Solution*** Since $(2+\sqrt{5})^P + (2-\sqrt{5})^P$ is an integer and since $-1 < (2-\sqrt{5})^P < 0$.

It follows that

$[(2+\sqrt{5})^P] = (2+\sqrt{5})^P + (2-\sqrt{5})^P$

$(2+\sqrt{5})^P + (2-\sqrt{5})^P$

$= 2(2^p + {}^{p}C_2 2^{p-2}5 + {}^{p}C_4 2^{p-4}5^2 + \ldots\ldots + {}^{p}C_{p-1} 2 . 5^{(p-1)/2})$,

so that $[(2+\sqrt{5})^p] - 2^{p+1}$

$= 2({}^{p}C_2 2^{p-2}5 + {}^{p}C_4 2^{p-4}5^2 + \ldots\ldots + {}^{p}C_{p-1} 2 . 5^{(p-1)/2})$,

All the binomial coefficients

$${}^{p}C_2 = \frac{p(p-1)}{1.2}$$

$${}^{p}C_4 = \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$${}^{p}C_{p-1} = p$$

are divisible by the prime p (they are integers and the numerators are divisible by the prime p, but the denominators are not). This means that the difference $[(2+\sqrt{5})^P] - 2^{p+1}$ is divisible by p.

➤ **Problem 14.** If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots + C_n x^n$. prove that

(i) $C_0 + C_3 + C_6 + C_9 + \ldots\ldots = \dfrac{1}{3}\left(2^n + 2\cos\dfrac{n\pi}{3}\right)$

(ii) $C_1 + C_4 + C_7 + C_{10} + \ldots\ldots$

www.pdfworld.net

www.pdfworld.net

$$= \frac{1}{3}\left(2^n + 2\cos(n-2)\frac{\pi}{3}\right)$$

(iii) $C_0 + C_6 + C_{12} + \ldots\ldots$

$$= \frac{1}{6}\left[2^n + 2\cos\frac{n\pi}{3} + 2(\sqrt{3})^n.\cos\frac{n\pi}{6}\right]$$

➤ ***Solution*** In the given series difference of suffices is 3 or 6

(i) On putting x = 1, ω and ω^2 in succession (ω, ω^2 are complex cube roots of unity) we get

$(1+1)^n = C_0 + C_1 + C_2 + C_3 + C_4 \ldots\ldots$

$(1+\omega)^2 = C_0 + C_1\omega + C_2\omega^2 + C_3\omega^3 + C_4\omega^4 + \ldots\ldots$

$(1+\omega^2)^n = C_0 + C_1\omega^2 + C_2\omega^4 + C_3\omega^6 + C_4\omega^8 + \ldots\ldots$

On adding we get

$2^n + (1+\omega)^n + (1+\omega^2)^n = 3C_0 + C_1(1+\omega+\omega^2) + C_2(1+\omega^2+\omega^4) + 3C_3 + C_4(1+\omega^4+\omega^8) + \ldots\ldots$

We easily note that terms involving $C_1, C_4, C_7, \ldots$ vanish since $1+\omega+\omega^2 = 0$, $1+\omega^2+\omega^4 = 1+\omega^2+\omega = 0$ etc.

Thus, $3[C_0 + C_6 + C_9 + \ldots\ldots]$

$$= 2^n + (1+\omega)^n + (1+\omega^2)^n.$$

Now $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$$\Rightarrow \omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} \Rightarrow 1+\omega = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \; 1+\omega^2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$\Rightarrow 1+\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, \; 1+\omega^2 = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$

$$\Rightarrow (1+\omega)^n + (1+\omega^2)^n = 2\cos\frac{n\pi}{3}$$

Therefore, $C_0 + C_6 + C_9 + \ldots$

$$= \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

(ii) Now in order to prove the second result we note that we have to retain C_1, C_4, C_7, C_{10} and we must get rid of other binomial coefficients.

$(1+1)^n = C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + \ldots\ldots$

$$\frac{(1+\omega)^n}{\omega} = \frac{C_0}{\omega} + C_1 + C_2\omega + C_3\omega^2 + C_4\omega^3 + \ldots\ldots$$

$$\frac{(1+\omega^2)^n}{\omega^2} = \frac{C_0}{\omega^2} + C_1 + C_2\omega^2 + C_3\omega^4 + C_4\omega^6 + \ldots\ldots$$

(We have divided the expansions of $(1+\omega)^n$, $(1+\omega^2)^n$ by ω and ω^2 respectively)

As earlier, we get, on adding

$3[C_1 + C_4 + C_7 + \ldots\ldots]$

$$= 2^n + \frac{1}{\omega}(1+\omega)^n + \frac{1}{\omega^2}(1+\omega^2)^n$$

$$= 2^n + \omega^2(1+\omega)^n + \omega(1+\omega^2)^n \quad \left(\because \frac{1}{\omega} = \omega^2, \frac{1}{\omega^2} = \omega\right)$$

$$= 2^n + \omega^2\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + \omega\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$

$$= 2^n + \cos\frac{n\pi}{3}(\omega^2+\omega) + i\sin\frac{n\pi}{3}(\omega^2-\omega)$$

$$= 2^n + \cos\frac{n\pi}{3}(-1) + i\sin\frac{n\pi}{3}(-i\sqrt{3})$$

$$= 2^n - \cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}$$

$$= 2^n + 2\left(-\frac{1}{2}\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{2}\sin\frac{n\pi}{3}\right)$$

$$= 2^n + 2\cos\left(\frac{n\pi}{3} - \frac{2\pi}{3}\right)$$

$$= 2^n + 2\cos(n-2)\frac{\pi}{3}$$

Thus $C_1 + C_4 + C_7 + C_{10} + \ldots\ldots$

$$= \frac{1}{3}\left[2^n + 2\cos(n-2)\frac{\pi}{3}\right]$$

(iii) The third result will be obtained by noting that

$= (1+1)^n + (1+\omega)^n + (1+\omega^2)^n + (1+1)^n + (1-\omega)^n + (1-\omega^2)^n$

www.pdfworld.net

www.pdfworld.net

$= (1+1)^n + (1+1)^n + (1+\omega)^n + (1-\omega)^n + (1+\omega^2)^n + (1-\omega^2)^n$

$= 2[C_0 + C_2 + C_4 + C_6 + C_8 + \ldots\ldots] + 2\left[C_0 + C_2\omega + C_4\omega^2 + C_6\omega^6 + \ldots\ldots\right] + 2\left[C_0 + C_2\omega^2 + C_4\omega^4 + C_6\omega^{12} + \ldots\ldots\right]$

$= 6[C_0 + C_6 + C_{12} + \ldots\ldots]$

Now as earlier $(1+\omega)^n + (1+\omega^2)^n = 2\cos\frac{n\pi}{3}$

Now $1 - \omega = \frac{3}{2} - \frac{i\sqrt{3}}{2}$

$1 - \omega^2 = \frac{3}{2} + \frac{i\sqrt{3}}{2}$

Put $\frac{3}{2} = r\cos\theta$,

$\frac{\sqrt{3}}{2} = r\sin\theta, \left(r^2 = 3, \tan\theta = \frac{1}{\sqrt{3}} \theta = \frac{\pi}{3}\right)$

We have $(1-\omega)^n + (1-\omega^2)^n$

$= (r\cos\theta - i\, r\sin\theta)^n + (r\cos\theta + i\, r\sin\theta)^n$

$= r^n \cdot 2\cos n\theta = \left(\sqrt{3}\right)^n .2\cos\frac{n\pi}{6}$

Thus we arrive at $C_0 + C_6 + C_{12} + \ldots\ldots$

$= \frac{1}{6}\left[2^n + 2\cos\frac{n\pi}{3} + 2(\sqrt{3})^n \cos\frac{n\pi}{6}\right]$

➤ **Problem 15.** Show that, ${}^nC_1 - 3.{}^nC_3 + 3^2.{}^nC_5 - 3^3.{}^nC_7 + \ldots\ldots = \begin{cases} 0 & \text{if } n = 6k,\ 6k+3 \\ 2^{n-1} & \text{if } n = 6k+1, 6k+2 \\ -2^{n-1} & \text{if } n = 6k+4,\ 6k+5 \end{cases}$

➤ ***Solution*** LHS $= {}^nC_1 - (\sqrt{3})^2\ {}^nC_3 + (\sqrt{3})^4\ {}^nC_5 - \ldots\ldots$

$= \frac{1}{\sqrt{3}}\left[\sqrt{3}.{}^nC_1 - (\sqrt{3})^3\ {}^nC_3 + (\sqrt{3})^5\ {}^nC_5\right]$

$= \frac{1}{\sqrt{3}}$.(Imaginary part of E) where $E = (1 + i\sqrt{3})^n$

Now $E = 2^n\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^n = 2^n\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n$

$= 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$

$\Rightarrow$ LHS $= \frac{1}{\sqrt{3}}.2^n\left(\sin\frac{n\pi}{3}\right)$

Now if n = 6k or 6k + 3, $\sin\frac{n\pi}{3} = 0$

if n = 6k + 1, $\sin\frac{n\pi}{3} = \sin\frac{\pi}{3}(6k+1)$

$= \sin\left(2k\pi + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

if n = 6k + 2, $\sin\frac{n\pi}{3} = \sin\frac{\pi}{3}(6k+2)$

$= \sin\left(2k\pi + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Thus in both cases, LHS $= \frac{1}{\sqrt{3}}.2^n.\frac{\sqrt{3}}{2} = 2^{n-1}$

We similarly sort out the cases n = 6k + 4, 6k + 5 by using $\sin\frac{4\pi}{3} = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

➤ **Problem 16.** Prove that ${}^nC_1 x(1-x)^{n-1} + 2\,{}^nC_2 x^2 (1-x)^{n-2} + \ldots + k\,{}^nC_k x^k (1-x)^{n-k} + \ldots + n\,{}^nC_k x^n = nx$

➤ ***Solution*** The general term on the left hand side of the equality can be transformed in the following way :

$k\,{}^nC_k x^k (1-x)^{n-k} = n\,{}^{n-1}C_{k-1} x^k (1-x)^{n-k}$

$= nx\,{}^{n-1}C_{k-1} x^{k-1} (1-x)^{n-1-(k-1)}$.

Therefore the left member of the equality can be written in the form

$nx[{}^{n-1}C_0(1-x)^{n-1}x + {}^{n-1}C_1 x(1-x)^{n-2} + \ldots + {}^{n-1}C_{n-1}x^{n-1}]$

$= nx[x + 1 - x]^{n-1} = nx.$

Let us consider the summation of the series $\sum_{r=0}^{n} \frac{{}^nC_r}{f(r)}$, where f(r) is a polynomial.

We need to eliminate r in the multiplier of binomial coefficient using $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$

➤ **Problem 17.** Find the sum of the series $\sum_{r=0}^{n}\left(\frac{n-3r+1}{n-r+1}\right).\frac{{}^nC_r}{2^r}$.

www.pdfworld.net

www.pdfworld.net

➤ ***Solution*** Let $S = \sum_{r=0}^{n}\left(\frac{n-3r+1}{n-r+1}\right)\cdot\frac{{}^{n}C_r}{2^r}$

$= \sum_{r=0}^{n}\left(1-\frac{2r}{n-r+1}\right)\cdot\frac{{}^{n}C_r}{2^r}$

$= \sum_{r=0}^{n}\frac{{}^{n}C_r}{2^r} - \sum_{r=0}^{n}\frac{2r}{(n-r+1)}\cdot\frac{n!}{r!(n-r)!.2^r}$

$= \sum_{r=0}^{n}\frac{{}^{n}C_r}{2^r} - \sum_{r=0}^{n}\frac{{}^{n}C_{r-1}}{2^{r-1}} = \frac{1}{2^n}$.

➤ **Problem 18.** Prove that $\sum_{r=0}^{n}(-1)^r\ \frac{{}^{n}C_r}{{}^{r+3}C_r}$

$= \frac{3!}{2(n+3)}$

➤ ***Solution*** LHS

$= \sum_{r=0}^{n}\left[(-1)^r\ \frac{n!}{r!(n-r)!}\ \frac{r!\ 3\ 1}{(r+3)!}\ \frac{(n+3)!}{(n+3)!}\right]$

$= \frac{3!\ n!}{(n+3)!}\ \sum_{r=0}^{n}(-1)^r\ .\ {}^{n+3}C_{r+3}$

$= \frac{3!\ n!}{(n+3)!}\ [{}^{n+3}C_3 - {}^{n+3}C_4 + {}^{n+3}C_5$

$- \ldots\ldots (-1)^n\ .{}^{n+3}C_{n+3}]$ ...(1)

We know that, $\sum_{k=0}^{n+3}(-1)^k\ {}^{n+3}C_k = 0$

or $\sum_{k=0}^{2}(-1)^k\ {}^{n+3}C_k + \sum_{k=3}^{n+3}(-1)^k\ {}^{n+3}C_k = 0$

or $({}^{n+3}C_0 - {}^{n+3}C_1 + {}^{n+3}C_2)$

$= [{}^{n+3}C_3 - {}^{n+3}C_4 + {}^{n+3}C_5 - \ldots. (-1)^n\ {}^{n+3}C_{n+3}]$ ...(2)

But RHS of (2) is the same as that of the terms in bracket of (1) .

Hence LHS $= \frac{3!\ n!}{(n+3)!}\left[1 - n - 3 + \frac{(n+3)(n+2)}{2}\right]$

$= \frac{3!\ n!}{(n+3)!}\ \frac{(n+3)(n+2) - 2(n+2)}{2} = \frac{3!\ n!}{(n+3)!}$

$\frac{(n+2)(n+1)}{2} = \frac{(n+2)!\ 3!}{2(n+3)!} = \frac{3!}{2(n+3)} = \text{RHS}$

Alternative : Consider the expansion of $(1 + x)^n$. Integrate thrice both the sides between 0 and x and put $x = -1$ to get the result.

➤ **Problem 19.** Prove that $\sum_{r=1}^{m}\frac{(m+1)(r-1)m^{r-1}}{r\ {}^{m}C_r}$

$= m^m - 1$

➤ ***Solution*** L.H.S. $= \sum_{r=1}^{m}\left(\frac{[mr-(m-r+1)]m^{r-1}}{r\ {}^{m}C_r}\right)$

$= \sum_{r=1}^{m}\left(\frac{m^r}{{}^{m}C_r} - \frac{m^{r-1}(m-r+1)r!(m-r)!}{r\,.\,m!}\right)$

$= \sum_{r=1}^{m}\left(\frac{m^r}{{}^{m}C_r} - \frac{m^{r-1}(m-r+1)!(r-1)!}{m!}\right)$

$= \sum_{r=1}^{m}\left(\frac{m^r}{{}^{m}C_r} - \frac{m^{r-1}}{{}^{m}C_{r-1}}\right) = m^m - 1.$

➤ **Problem 20.** If $a_n = \sum_{r=0}^{n}\frac{1}{{}^{n}C_r}$, then prove that

$\sum_{r=0}^{n}\frac{r^2}{{}^{n}C_r} = (n+1)(n+2)\,a_{n+2} - 3(n+1)a_{n+1}$

$+ a_n - (n+1)(n+6).$

➤ ***Solution*** $\sum_{r=0}^{n}\frac{r^2}{{}^{n}C_r} = \sum_{r=0}^{n}\left[\frac{(r+1)^2}{{}^{n}C_r} - \frac{2r+1}{{}^{n}C_r}\right]$

$= \sum_{r=0}^{n}(r+1)\frac{n+1}{{}^{n+1}C_{r+1}} - \sum_{r=0}^{n}\frac{2r+1}{{}^{n}C_r} = (n+1)$

$\sum_{r=0}^{n}\frac{r+2}{{}^{n+1}C_{r+1}} - (n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{r+1}} - 2\sum_{r=0}^{n}\frac{r+1}{{}^{n}C_r} + \sum_{r=0}^{n}\frac{1}{{}^{n}C_r}$

$= (n+1)(n+2)\ \sum_{r=0}^{n}\frac{1}{{}^{n+2}C_{r+2}} - (n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{r+1}}$

$= 2(n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{r+1}} + \sum_{r=0}^{n}\frac{1}{{}^{n}C_r}$

$= (n+1)(n+2)$

$\left[a_{n+2} - \frac{1}{{}^{n+2}C_0} - \frac{1}{{}^{n+2}C_1}\right] - 3(n+1)\left[a_{n+1} - \frac{1}{{}^{n-1}C_0}\right] + a_n$

www.pdfworld.net

www.pdfworld.net

$= (n+1)(n+2)a_{n+2} - (n+1)(n+2) - (n+1)$
$\quad - 3(n+1)a_{n+1} - 3(n+1) + a_n$
$= (n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + a_n - (n+1)(n+6).$

➤ **Problem 21.** Prove that $\sum_{r=0}^{2n} \frac{r}{a_r} = n\sum_{r=0}^{2n} \frac{1}{a_r}$, where $\left(\sum_{r=0}^{2} x^r\right)^n = \sum_{r=0}^{2n} a_r x^r$.

➤ ***Solution*** We are given that $(1 + x + x^2)^n$
$= a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}.$ ...(1)

Replacing x by $\frac{1}{x}$, we have

$$\frac{(x^2+x+1)^n}{x^{2n}} = \frac{a_0x^{2n} + a_1x^{2n-1} + ... + a_{2n-1}x + a_{2n}}{x^{2n}} \quad ....(2)$$

From (1) and (2), we find that $a_0 = a_{2n}$

$a_1 = a_{2n-1}$
: :
: :
$a_r = a_{2n-r}.$

Now $\sum_{r=0}^{2n} \frac{r}{a_r} = \sum_{r=0}^{2n} \frac{2n-(2n-r)}{a_r} = \sum_{r=0}^{2n} \frac{2n}{a_r} - \sum_{r=0}^{2n} \frac{2n-r}{a_r}$

$= \sum_{r=0}^{2n} \frac{2n}{a_r} - \sum_{r=0}^{2n} \frac{2n-r}{a_{2n-r}} = 2n\sum_{r=0}^{2n} \frac{1}{a_i} - \sum_{r=0}^{2n} \frac{r}{a_r}$

$\Rightarrow \quad 2\sum_{r=0}^{2n} \frac{r}{a_r} = 2n\sum_{r=0}^{2n} \frac{1}{a_r} \Rightarrow \sum_{r=0}^{2n} \frac{r}{a_r} = n\sum_{r=0}^{2n} \frac{1}{a_r}$

➤ **Problem 22.** If n is positive integer, show that

$$\frac{n!}{(x+1)(x+2)....(x+n)}$$
$$= \frac{{}^nC_1}{x+1} + \frac{2.({}^nC_2)}{x+2} + ... + (-1)^{n+1}.\frac{n({}^nC_n)}{x+n}$$

Also deduce that

$$\frac{1}{n+1} = \frac{{}^nC_1}{2} - \frac{2({}^nC_2)}{3} + \frac{3({}^nC_3)}{4} - ... + (1)^{n+1}.\frac{n({}^nC_n)}{x+n}.$$

➤ ***Solution*** Let $\frac{n!}{(x+1)(x+2)....(x+n)}$

$$= \frac{A_1}{x+1} + \frac{A_2}{x+2} + ... + \frac{A_n}{x+n} \quad ...(1)$$

On comparing both sides, we get

$$\Rightarrow \quad A_1 = \frac{n!}{1.2.3.....(n-1)} = n = {}^nC_1$$

$$\Rightarrow \quad A_2 = \frac{n!}{(-1)1.2.3....(n-2)} = -n(n-1) = -2{}^nC_2$$

$$\Rightarrow \quad A_n = \frac{n!}{(-n+1)(-n+2)....(-n+n-1)}$$

$$= \frac{n!}{(-1)^{n-1}(n-1)!}$$
$$= (-1)^{n+1}\, n\, .\, {}^nC_n$$

From (1) we get,

$$\frac{n!}{(x+1)(x+2)....(x+n)}$$
$$= \frac{{}^nC_1}{x+1} - \frac{2({}^nC_2)}{x+2} + ... + \frac{(-1)^{n+1}n({}^nC_n)}{(x+n)}.$$

Again putting x = 1 in both sides we get

$$\frac{n!}{2.3.4.....n(1+n)} = \frac{{}^nC_1}{2} - \frac{2({}^nC_2)}{3} + ... + \frac{(-1)^{n+1}n({}^nC_n)}{(n+1)}.$$

➤ **Problem 23.** Prove $\sum_{r=0}^{n} r^2.{}^nC_r p^r q^{n-r} = npq + n^2p^2$ if $p + q = 1$

➤ ***Solution*** Consider $(1 + x)^n$
$= {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ...+ {}^nC_nx^n$
Differentiating both sides w.r.t. x, we have
$n(1 + x)^{n-1} = {}^nC_1\,(1) + {}^nC_2\,(2x) + {}^nC_3(3x^2) + ...$
$\quad + {}^nC_n(nx^{n-1})$
Multiplying both sides by x, we get
$nx(1 + x)^{n-1} = {}^nC_1\,(x) + {}^nC_2\,(2x^2) + {}^nC_3(3x^3) + ...$
$\quad + {}^nC_n(nx^n)$
Again differentiating both sides w.r.t. x we get
$nx(n-1)(1 + x)^{n-2} + (1 + x)^{n-1}\, n$
$= {}^nC_1(1)^2 + {}^nC_2(2)^2x + {}^nC_3(3)^2x^2 + ...$
$\quad + {}^nC_n(n)^2x^{n-1}$

putting $x = \frac{p}{q}$ on both sides.

$$\Rightarrow \quad \frac{np}{q}(n-1)\left(1+\frac{p}{q}\right)^{n-2} + \left(1+\frac{p}{q}\right)^{n-1} n$$

www.pdfworld.net

www.pdfworld.net

$$= {}^nC_1(1)^2 + {}^nC_2(2)^2\left(\frac{p}{q}\right) + {}^nC_3(3)^2\left(\frac{p^2}{q^2}\right) + ...$$

$$+ {}^nC_n(n)^2\left(\frac{p^{n-1}}{q^{n-1}}\right)$$

$$\Rightarrow \frac{np(n-1)}{q^{n-1}} + \frac{n}{q^{n-1}} = \frac{1}{q^{n-1}}[{}^nC_1(1)^2q^{n-1} + {}^nC_2(2)^2pq^{n-2}$$

$$+ {}^nC_3(3)^2p^2q^{n-3} + ... + {}^nC_n(n)^2p^{n-1}]$$

$(\because$ given $p + q = 1)$

$$\Rightarrow \quad np(n-1) + n = [{}^nC_1(1)^2q^{n-1} + {}^nC_2(2)^2pq^{n-2}$$

$$+ {}^nC_3(3)^2p^2q^{n-3} + ... + {}^nC_n(n)^2p^{n-1}]$$

Multiplying both sides by p

$$\Rightarrow \quad np^2(n-1) + np = {}^nC_1(1)^2pq^{n-1} + {}^nC_2(2)^2p^2q^{n-2}$$

$$+ {}^nC_3(3)^2p^3q^{n-3} + ... + {}^nC_n(n)^2p^n]$$

$$\Rightarrow \quad n^2p^2 + np(1-p) = \sum_{r=0}^{n} r^2\,{}^nC_r p^r q^{n-r}$$

$$n^2p^2 + npq = \sum_{r=0}^{n} r^2\,{}^nC_r\,p^r\,q^{n-r}$$

Hence $\sum_{r=0}^{n} r^2\,{}^nC_r\,p^r\,q^{n-r} = n^2p^2 + npq$

➤ **Problem 24.** Given that $S_n = 1 + q + q^2 + ... + q^n$ and σ_n

$$= 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n;\ q \neq 1$$

prove that ${}^{n+1}C_1 + {}^{n+1}C_2 \cdot S_1 + {}^{n+1}C_3 \cdot S_2 + ...$

$$+ {}^{n+1}C_{n+1} \cdot S_n = 2^n \cdot \sigma_n$$

➤ ***Solution*** We have $S_n = 1 + q + ...+ q^n$

$$= \left(\frac{1-q^{n+1}}{1-q}\right)\sigma_n$$

$$= 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n$$

$$= \frac{1-\left(\frac{q+1}{2}\right)^{n+1}}{1-\left(\frac{q+1}{2}\right)} = \frac{2^{n+1} - (q+1)^{n+1}}{2^n(1-q)}$$

$\therefore$ L.H.S.

$$= {}^{n+1}C_1 + {}^{n+1}C_2 \cdot S_1 + {}^{n+1}C_3 \cdot S_2 + ... + {}^{n+1}C_{n+1} \cdot S_n$$

$$= {}^{n+1}C_1 + {}^{n+1}C_2 \cdot \left(\frac{1-q^2}{1-q}\right) + {}^{n+1}C_3 \cdot \left(\frac{1-q^3}{1-q}\right) + ...$$

$$+ {}^{n+1}C_{n+1} \cdot \left(\frac{1-q^{n+1}}{1-q}\right) \qquad [\text{using (1)}]$$

$$= \frac{1}{(1-q)}\{({}^{n+1}C_1(1-q) + {}^{n+1}C_2(1-q^2) + {}^{n+1}C_3$$

$$(1-q^3) + ... + {}^{n+1}C_{n+1}(1-q^{n+1})\}$$

$$= \frac{1}{(1-q)}\{({}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + ... + {}^{n+1}C_{n+1})\}$$

$$- \{({}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + {}^{n+1}C_3 q^3 + ...$$

$$+ {}^{n+1}C_{n+1} q^{n+1})\}$$

$$= \frac{1}{(1-q)}\{(2^{n+1} - 1) - ((1+q)^{n+1} - 1)\}$$

$$= \frac{1}{(1-q)}(2^{n+1} - (1+q)^{n+1})\}$$

$$= 2^n\ \sigma_n = \text{R.H.S.}$$

➤ **Problem 25.** If $(1+x)^n = \sum_{r=0}^{n} C_r x^r$, then evaluate

$$\frac{C_0}{m!} + \frac{n.C_1}{(m+1)!} + \frac{n.(n-1).C_2}{(m+2)!} + ...$$

$$+ \frac{n(n-1)...2.1.C_n}{(n+m)!}$$

➤ ***Solution*** Consider the expansion

$$\left(1+\frac{1}{x}\right)^n . (1+x)^{m+n}$$

$$= \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ... + \frac{C_n}{x^n}\right) \times ({}^{m+n}C_0 + {}^{m+n}C_1 x + ...$$

$$+ {}^{m+n}C_m x^m + {}^{m+n}C_{m+1} . x^{m+1} + ... + {}^{m+n}C_{m+n} x^{m+n})$$

Equating the coefficients of x^m in both sides, we get

$$\Rightarrow \quad C_0 \cdot {}^{m+n}C_m + C_1 \cdot {}^{m+n}C_{m+1} + C_2 \cdot {}^{m+n}C_{m+2}$$

$$+ ..... + C_n \cdot {}^{m+n}C_{m+n}$$

$$= \text{coeff. of } x^m \text{ in } \frac{(1+x)^n(1+x)^{m+n}}{x^n}$$

www.pdfworld.net

www.pdfworld.net

$$+\frac{C_2(m+n)!}{(m+2)!.(n-2)!}+.....+\frac{C_n.(m+n)!}{(m+n)!.1}$$

$$={}^{m+2n}C_{m+n}=\frac{(m+2n)!}{(m+n)!.\,n!}$$

$$\Rightarrow (m+n)!\left[\frac{C_0}{m!.\,n!}+\frac{C_1}{m+1!.n-1!}+.....+\frac{C_n}{m+n!}\right]$$

$$=\frac{(m+2n)!}{(m+n)!.n!}$$

$$\Rightarrow \frac{C_0}{m!}+\frac{n.C_1}{(m+1)!}+\frac{n(n-1)C_2}{(m+2)!}+.....+\frac{n(n-1)....21.C_n}{(m+n)!}$$

$$=\frac{(m+n+1)(m+n+2).....(m+2n)}{m+n!}$$

➤ **Problem 26.** For $m, n \in N$ & $m > n$, show that, $\frac{{}^{m}C_n}{2}+\frac{{}^{m+1}C_n}{2^2}+\frac{{}^{m+2}C_n}{2^3}+..... \infty$ $= 1 + {}^{m}C_1 + {}^{m}C_2 + ...+ {}^{m}C_n$.

➤ ***Solution*** L.H.S. = coefficient of x^n in

$$\frac{(1+x)^m}{2}+\frac{(1+x)^{m+1}}{2^2}+\frac{(1+x)^{m+2}}{2^3}+..... + \infty$$

$$=\frac{(1+x)^m}{2\left[1-\frac{1+x}{2}\right]}=(1+x)^m\,(1-x)^{-1}$$

coefficient of x^n in

$$\left[{}^{m}C_0+{}^{m}C_1x+{}^{m}C_2x^2+....+{}^{m}C_n x^n+....\right](1+x)^{-1}$$

$$= {}^{m}C_0 + {}^{m}C_1 + {}^{m}C_2 + {}^{m}C_3 + ...+ {}^{m}C_n = \text{L.H.S}$$

➤ **Problem 27.** In the expansion of $\left(1+x+\frac{7}{x}\right)^{11}$ find the term independent of x.

➤ ***Solution***

$$\left(1+x+\frac{7}{x}\right)^{11}=\sum_{r_1+r_2+r_3=11}\frac{(11)!}{r_1!\,r_2!\,r_3!}(1)^{r_1}(x)^{r_2}\left(\frac{7}{x}\right)^{r_3}$$

The exponent 11 is to be divided among the base variables 1, x and 7/x.

in such a way so that we get x^0. Therefore possible set of values of (r_1, r_2, r_3) are

(11, 0, 0), (9, 1, 1), (7, 2, 2), (5, 3, 3), (3, 4, 4), (1, 5, 5)

Hence the required term is

$$=\frac{(11)!}{(11)!}(7^0)+\frac{(11)!}{9!\,1!\,1!}7^1+\frac{(11)!}{7!\,2!\,2!}7^2$$

$$+\frac{(11)!}{5!\,3!\,3!}7^3+\frac{(11)!}{3!\,4!\,4!}7^4+\frac{(11)!}{1!\,5!\,5!}7^5$$

$$=1+\frac{(11)!}{9!\,2!}\cdot\frac{2!}{1!\,1!}7^1+\frac{(11)!}{7!\,4!}\cdot\frac{4!}{2!\,2!}7^2+$$

$$\frac{(11)!}{5!\,6!}\cdot\frac{6!}{3!\,3!}7^3+\frac{(11)!}{3!\,8!}\cdot\frac{8!}{4!\,4!}7^4+\frac{(11)!}{1!\,10!}\cdot\frac{(10)!}{5!\,5!}7^5$$

$$= 1 + {}^{11}C_2\,.\,{}^{2}C_1\,.\,7^1 + {}^{11}C_4\,.\,{}^{4}C_2\,.\,7^2 + {}^{11}C_6\,.\,{}^{6}C_3\,.\,7^3$$
$$+ {}^{11}C_8\,.\,{}^{8}C_4\,.\,7^4 + {}^{11}C_{10}\,.\,{}^{10}C_5\,.\,7^5$$

$$=1+\sum_{r=1}^{5}{}^{11}C_{2r}\,.\,{}^{2r}C_r\,.\,7^r$$

Things to Remember

1. Binomial Theorem for Positive Integral Index
   If n is a positive integer and x, a ∈ C then
   $(x+a)^n = {}^{n}C_0x^n + {}^{n}C_1x^{n-1}a^1 + {}^{n}C_2x^{n-2}a^2 + ...$
   $+ {}^{n}C_rx^{n-r}a^r + ... + ... + {}^{n}C_{n-1}xa^{n-1} + {}^{n}C_na^n$

2. The general term or the $(r+1)^{th}$ term in the expansion of $(x+a)^n$ is given by
   $T_{r+1} = {}^{n}C_r\,x^{n-r}\,.\,a^r$
   $$=\frac{n(n-1)....(n-r+1)}{r!}x^{n-r}a^r$$

3. The pth term from the end in the expansion of $(x+a)^n$ is same as the $(n+1-p+1)$ i.e. $(n-p+2)$th term from the beginning of the expansion.

4. The middle term(s) is the expansion of $(x+a)^n$ is (are) :

www.pdfworld.net

www.pdfworld.net

(a) If n is even, there is only one middle term since the number of terms is odd.
It is given by $T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot a^{n/2}$

(b) If n is odd, there are two middle terms since the number of terms is even.
They are given by $T_{(n+1)/2}$ and $T_{[(n+1)/2]+1}$

5. The ratio of two consecutive terms in the expansion of $(x + a)^n$ is

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^{n-r} a^r}{{}^nC_{r-1} x^{n-r+1} a^{r-1}} = \frac{(n-r+1)}{r}\frac{a}{x}$$

6. To find the numerically greatest term in the expansion of $(x + a)^n$

(a) Calculate $\frac{n+1}{1+\left|\frac{x}{a}\right|} = m$

(b) If m is an integer, then T_m and T_{m+1} are numerically equal and greatest terms.

(c) If m is not integer, then $T_{[m]+1}$ is the numerically greatest term, where [.] denotes the greatest integer function.

7. Divisibility of $x^n \pm y^n$ by $x \pm y$, $x, y, n \in N$

(i) If n is even, then $x^n - y^n$ is divisible by $x \pm y$.

(ii) If n is odd, then $x^n - y^n$ is divisible by $x - y$.

(iii) If $n \in N$, then $x^n - y^n$ is divisible by $x - y$.

(iv) If n is odd, then $x^n + y^n$ is divisible by $x + y$.

8. (i) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(ii) $C_0 - C_1 + C_2 - .... + (-1)^n C_n = 0$

(iii) $C_0 + C_2 + ... = 2^{n-1}$

(iv) $C_1 + C_3 + ... = 2^{n-1}$

(v) $C_0 - C_2 + C_4 - .... = 2^{n/2} \cos\frac{n\pi}{4}$

(vi) $C_1 - C_3 + C_5 - .... = 2^{n/2} \sin\frac{n\pi}{4}$

(vii) $C_0 + C_4 + C_8 + ... = \frac{1}{2}(2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4})$

(viii) $C_2 + C_6 + C_{10} + ... = \frac{1}{2}(2^{n-1} - 2^{n/2}\cos\frac{n\pi}{4})$

(ix) $C_1 + C_5 + C_9 + ... = \frac{1}{2}(2^{n-1} + 2^{n/2}\sin\frac{n\pi}{4})$

(x) $(C_0 - C_2 + C_4 - ....)^2 + (C_1 + C_5 - .....)^2 = 2^n$

9. (i) ${}^mC_m + {}^{m+1}C_m + {}^{n+2}C_m + ... + {}^nC_m = {}^{n+1}C_{m+1}$.

(ii) $C_1 + 2C_2 + 3C_3 + ... + nC_n = n.2^{n-1}$

(iii) $C_1 + 2^2 C_2 + 3^2 C_3 + ... + n^2 C_n = n(n+1)2^{n-2}$

(iv) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(v) ${}^nC_0\,{}^nC_r + {}^nC_1\,{}^nC_{r+1} + {}^nC_2\,{}^nC_{r+2} + ... + {}^nC_{n-r}\,{}^nC_n = {}^{2n}C_{n-r}$

(vi) $({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + ... + ({}^nC_n)^2 = {}^{2n}C_n$

10. $\sum_{0 \le i < j \le n}\sum C_i C_j = 2^{n-1} - \frac{2n!}{2.n!n!}$

11. If $n \in Q$ and $|x| < 1$ i.e. $-1 < x < 1$, then

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + ... \text{ to } \infty$$

12. If $n \in N$, $(1-x)^{-n} = 1 + {}^nC_1 x + {}^{n+1}C_2 x^2 + {}^{n+2}C_3 x^3 + ... + {}^{n+r-1}C_r x^r + ...$
Thus, the coefficient of x^r in $(1-x)^{-n}$ is ${}^{n+r-1}C_r$.

13. For any positive integer n, the multinomial expansion of $(x + y + z + w + ...)^n$ is the sum of all terms of the form

$$\frac{n!}{a!b!c!d!.....}x^a y^b z^c w^d .....,$$

where a, b, c, d,.... range over all solutions of $a + b + c + d + ... = n$ in non-negative integers.

www.pdfworld.net

www.pdfworld.net

[Objective Exercises]

Single Correct Answer Type

1. Which one of the following is true?

(A) The binomial expansion for $(a+b)^n$ contains n terms.

(B) The Binomial Theorem can be written in condensed form as $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

(C) The sum of the binomial coefficients in $(a+b)^n$ cannot be 2^n.

(D) There are no values of a and b such that $(a+b)^4 = a^4 + b^4$.

2. If $(1+ax)^n = 1 + 8x + 24x^2 + ...$ then is $\frac{a-n}{a+n}$ is equal to

(A) 3 (B) –3

(C) –1/3 (D) 1/3

3. The coefficients of x^{20} in the expansion of $(1+x^2)^{40}.\left(x^2+2+\frac{1}{x^2}\right)^{-5}$ is

(A) $^{30}C_{10}$ (B) $^{30}C_{25}$

(C) 1 (D) None of these

4. Let $F(x) = 1 + 9x + 36x^2 + 84x^3 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$. Then $f(-0.99) =$

(A) 10^{-9} (B) 10^{-18}

(C) 10^{-99} (D) 10^{99}

5. If 2nd, 3rd, and 4th terms in the expansion of $(x+y)^n$ be 240, 720 and 1080, then n is equal to

(A) 4 (B) 5

(C) 6 (D) 7

6. If the coefficients of $(r+1)^{th}$ term in the expansion of $(1+x)^{n+1}$ is equal to ℓ, and coefficients of r^{th} and $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is equal to m, n respectively, then ℓ is equal to

(A) $m^2 + n^2$ (B) $m^2 - n^2$

(C) $\frac{(m-n)}{(m+n)}$ (D) None of these

7. The sum of all rational terms in the expansion of $(3^{1/5} + 2^{1/3})^{15}$ is

(A) 39 (B) 49

(C) 59 (D) 69

8. The ratio of the coefficient of x^{10} in $(1-x^2)^{10}$ and the term independent of x in $\left(x-\frac{2}{x}\right)^{10}$ is

(A) 1 : 16 (B) 1 : 32

(C) 32 : 1 (D) 16 : 1

9. The number of integral terms in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$ is

(A) 128 (B) 129

(C) 130 (D) 131

10. If in the expansion of $\left(2a-\frac{a^2}{4}\right)^9$ the sum of middle terms is S, then

(A) $S = \left(\frac{63}{32}\right) a^{14}(1+8)$

(B) $S = \left(\frac{63}{32}\right) a^{14}(a-8)$

(C) $S = \left(\frac{63}{32}\right) a^{13}(a-8)$

(D) $S = \left(\frac{63}{32}\right) a^{13}(18-a)$

11. The coefficient of x^4 in the expansion of $(2-x+3x^2)^6$ is

(A) 3680 (B) 3660

(C) 3880 (D) None of these

12. If the r^{th} term is the middle term in the expansion of $\left(x^2-\frac{1}{2x}\right)^{20}$, then the (r + 3)th term is

(A) $^{20}C_{14} \cdot \frac{1}{2^{14}} \cdot x$

(B) $^{20}C_{12} \cdot \frac{1}{2^{12}} \cdot x^2$

www.pdfworld.net

www.pdfworld.net

(C) $-\frac{1}{2^{13}} \cdot {}^{20}C_7 \cdot x$

(D) None of these

13. The numerically greatest term in the expansion of $(8 - 5x)^{18}$ (where $x = 2/5$) is

(A) 1632×2^{24} (B) 1632×2^{32}

(C) 1632×2^{23} (D) None of these

14. If the sum of the coefficients in the expansion of $(1 + 2x)^n$ is 6561. Then the greatest coefficient in the expansion is

(A) 2000 (B) 3561

(C) 1120 (D) None of these

15. The greatest term, in the expansion of $(2 + 3x)^{10}$ where $x = \frac{11}{8}$ is

(A) ${}^{10}C_4 \cdot \left(\frac{18}{5}\right)^4$ (B) ${}^{10}C_4 \cdot \left(\frac{33}{8}\right)^4$

(C) ${}^{10}C_5 \cdot \left(\frac{18}{5}\right)^4$ (D) None of these

16. The coefficients of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$ is

(A) 62640 (B) 62600

(C) 62400 (D) None of these

17. The number of rational terms in $(\sqrt{2} + 3\sqrt{2} + 6\sqrt{5})^{10}$

(A) 2 (B) 3

(C) 4 (D) None of these

18. The integral part of $(7 + 2\sqrt{5})^{2n+1}$ is $(n \in N)$

(A) an even number

(B) an odd number

(C) an even or an odd number depending upon the value of n

(D) None of these

19. Let $f(n) = 10^n + 3 \cdot 4^{n+2} + 5$, $n \in N$. The greatest value of the integer which divides f(n) for all n is

(A) 27 (B) 9

(C) 3 (D) None of these

20. 3^{51} when divided by 8, leaves the remainder

(A) 1 (B) 6

(C) 5 (D) 3

21. The fractional part of $\frac{32^{32}}{3}$, is equal to

(A) 0 (B) 1/3

(C) 1/2 (D) 1/6

22. The number of terms in the expansion of $\left(x^2 + \left(\frac{2x^2+1}{x^2}\right)\right)^{60}$ is

(A) 120 (B) 61

(C) 160 (D) 121

23. If $n \in N$ and $n \geq 5$, then the value of $[(3 + \sqrt{3})^n] + [(3 - \sqrt{3})^n]$ is where [.] represents greatest integer function :

(A) an even integer (B) an odd integer

(C) may be even or odd (D) data insufficient

24. The interval in which x must lie so that the numerically greatest term in the expansion of $(1 - x)^{21}$ has the greatest binomial coefficient is $(x > 0)$

(A) $\left[\frac{5}{6}, \frac{6}{5}\right]$ (B) $\left(\frac{5}{6}, \frac{6}{5}\right)$

(C) $\left(\frac{4}{5}, \frac{5}{4}\right)$ (D) $\left[\frac{4}{5}, \frac{5}{4}\right]$

25. The unit digit in the product $(7^{71} \times 6^{59} \times 3^{65})$ is

(A) 6 (B) 2

(C) 4 (D) 1

26. If the co-efficients of two consecutive terms in the expansion of $(3 + 2x)^{74}$ are equal, then counting from the end, these terms are

(A) 30^{th}, 31^{st} (B) 11^{th}, 12^{th}

(C) 45^{th}, 46^{th} (D) 44^{th}, 45^{th}

27. The coefficients of x^6 in $[(1 + x)^6 + (1 + x)^7 + ... + (1 + x)^{15}]$ is

(A) ${}^{16}C_9$ (B) ${}^{16}C_5 - {}^6C_5$

(C) ${}^{16}C_6 - 1$ (D) None of these

www.pdfworld.net

www.pdfworld.net

28. The sum of coefficients of the polynomial $(p^x + 2q^y - 5r^z + 10x^u)^{1001}$ is
(where p, q, r, s, x, y, z and u are variables)
(A) 2^{8008} (B) 2^{1001}
(C) 2^{1001+8} (D) 2^{3003}

29. The number of irrational terms in the expansion of $(\sqrt[8]{5} + \sqrt[6]{2})^{100}$ is
(A) 97 (B) 98
(C) 96 (D) 99

30. If P_n denotes the product of all the coefficients in the expansion of $(1+x)^n$ and $9! \cdot P_{n+1} = 10^9 \cdot P_n$ then n is equal to
(A) 10 (B) 9
(C) 19 (D) None of these

31. If $f(x) = \sum_{r=0}^{100} a_r x^r$ and f(0) and f(1) are odd numbers
(A) f(x) is odd or even according as x is odd or even
(B) f(x) is even or odd according as x is odd or even
(C) f(x) is even for all integral values of x
(D) f(x) is odd for all integeral values of x

32. In the expansion of $(1+x)^{2m}\left(\frac{x}{1-x}\right)^{-2m}$ the term independent of x is
(A) ${}^{2m}C_m$ (B) ${}^{2m}C_0$
(C) $(-1)^m \cdot {}^{2m}C_m$ (D) None of these

33. The value of the expression ${}^{n+1}C_2 + 2({}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots\ldots + {}^nC_2)$ is
(A) Σn (B) Σn^2
(C) Σn^3 (D) None of these

34. The sum of the last ten coefficients in the expansion of $(1+x)^{19}$ when expanded in increasing powers of x is
(A) 2^{18} (B) 2^{19}
(C) $2^{18} - {}^{19}C_{10}$ (D) None of these

35. If $n > 2$, then ${}^2C_1 (x-1)^2 - {}^nC_2 (x-2)^2 + {}^nC_3 (x-3)^2 + \ldots\ldots + (-1)^{n-1}(x-n)^2$ is equal to
(A) $\{x^2 - 5)(x-1)$ (B) $x^2 - 1$
(C) $x^2 + 1$ (D) x^2

36. The sum of series ${}^{20}C_0 + {}^{20}C_2 2^2 + {}^{20}C_4 2^4 + {}^{20}C_6 2^6 + \ldots\ldots + {}^{20}C_{20} 2^{20}$ is
(A) $2^{20} - 1$ (B) $2^{19} - \frac{1}{2}$
(C) 0 (D) ${}^{20}C_{10}$

37. The sum $({}^{10}C_3 + {}^{11}C_3 + {}^{12}C_3 + \ldots\ldots + {}^{20}C_3)$ is equal to
(A) ${}^{21}C_4$
(B) ${}^{21}C_4 + {}^{10}C_4$
(C) ${}^{21}C_{17} - {}^{10}C_6$
(D) None of these

38. The sum $(\frac{1}{2}\,{}^{10}C_0 - {}^{10}C_1 + 2.{}^{10}C_2 - 2^2 \cdot {}^{10}C_3 + \ldots\ldots + 2^9 \cdot {}^{10}C_{10})$ is equal to
(A) $3^{10}/2$ (B) 1/2
(C) 0 (D) None of these

39. ${}^3k \cdot {}^nC_0 \cdot {}^nC_k - 3^{k-1} \cdot {}^nC_1 \cdot {}^{n-1}C_{k-1} + 3^{k-2}\,{}^nC_{k-2} + \ldots\ldots\ldots + (-1)^k \cdot {}^nC_k \cdot {}^{n-k}C_0$ is equal to:
(A) ${}^nC_k$ (B) ${}^nC_k \cdot 2^k$
(C) ${}^nC_k \cdot 3^k$ (D) $2 \cdot {}^nC_k$

40. The value of ${}^nC_1 x(1-x)^{n-1} + 2.{}^nC_2 x^2 (1-x)^{n-2} + 3.{}^nC_3 x^3 (1-x)^{n-3} + \ldots\ldots + n \cdot {}^nC_n x^n$ is
(A) n (B) x
(C) nx (D) none of these

41. If $(1 + x + x^2)^n = a_0 + a_1 x + \ldots\ldots + a_{2n} x^{2n}$ then $a_0 - a_2 + a_4 - a_6 + \ldots\ldots = (\text{if } n = 4p+2)$
(A) 1 (B) –1
(C) i^n (D) None of these

42. If $(1 + x - 2x^2)^8 = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_{16} x^{16}$ then the sum $(a_1 + a_3 + a_5 + \ldots\ldots + a_{15})$ is equal to
(A) -2^7 (B) 2^7
(C) 2^8 (D) None of these

www.pdfworld.net

www.pdfworld.net

43. $2(1 + x^3)^{10} = \sum_{k=0}^{100}\left(a_k x^k - \cos\frac{\pi}{2}(x+k)\right)$ then the value of $a_0 + a_2 + a_4 + ... + a_{100}$ is
(A) 2^{99} (B) 2^{100}
(C) 2^{101} (D) None of these

44. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... a_{20}x^{2n}$ then $(a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + ... - a_{2n-1}a_{2n})$ is
(A) 0 (B) $(-1)^{2n+1}\ {}^{2n}C_n$
(C) 1 (D) None of these

45. $\sum\sum\sum_{0\le i<j<k\le n} 1$ is equal to
(A) ${}^{n+1}C_3$ (B) ${}^{n+1}C_2$
(C) ${}^{n}C_3$ (D) ${}^{n}C_2$

46. If the coefficient of x, x^2 and x^3 in the expansion $\frac{(1+4x^2+x^4)}{(1-x)^4}$ are p, q, r, respectively, then the value of p + 2q – r is
(A) 0 (B) –4
(C) –18 (D) 6

47. The coefficient of $a^8b^8c^9d^9$ in $(abc + abd + acd + bcd)^{10}$ is
(A) 10 ! (B) $\frac{10!}{8!4!9!9!}$
(C) 2520 (D) None of these

48. The coefficient of x^3y^4z in the expansion of $(1 + x + y - z)^9$ is
(A) $2 . {}^9C_7 . {}^7C_4$ (B) $-2 . {}^9C_2 . {}^7C_3$
(C) ${}^9C_7 . {}^7C_4$ (D) None of these

49. The number of terms in the expansion of $(a + b + c + d + e + f)^n$ is
(A) ${}^mC_5$ (B) ${}^nC_6$
(C) ${}^{n+4}C_6$ (D) None of these

50. If x is so small that its two and higher powers can be neglected and if $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{5}{2}} = 1 + kx$, then k is equal to
(A) 1 (B) – 2
(C)10 (D) 11

Multiple Correct Answer Type for JEE Advanced

51. Coefficients of x^{103} in $(1 + x + x^2 + x^3 + x^4)^{199}$ $(x - 1)^{201}$ is less than
(A) 10 (B) 9
(C) 8 (D) 7

52. If n is a positive integer and $(3\sqrt{3} + 5)^{2n+1} = \alpha + \beta$, where α is an integer and $0 < \beta < 1$, then
(A) α is an even integer
(B) $(\alpha + \beta)^2$ is divisible by 2^{2n+1} is
(C) the integer just below $(3\sqrt{3} + 5)^{2n+1}$ is divisible by 3.
(D) α is divisible by 10

53. The co–efficient of the middle term in the expansion of $(1 + x)^{2n}$ is
(A) $\frac{1.3.5.7.....(2n-1)}{n!}\ 2^n$
(B) ${}^{2n}C_n$
(C) $\frac{(n+1)(n+2)(n+3)....(2n-1)(2n)}{1.2.3.(n-1)\ n}$
(D) $\frac{2.6.10.14..... (4n-6)(4n-2)}{1.2.3.4.....(n-1) . n}$

54. If it is known that the third term of the binomial expansion $\left(x + x^{\log_{10} x}\right)^5$ is 10^6 then x is equal to
(A) 10 (B) $10^{-5/2}$
(C) 100 (D) 5

55. Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + ...$ If A_0, A_1, A_2 are in A.P. then the value of n is
(A) 2 (B) 3
(C) 5 (D) 7

56. If the 6th term in the expansion of $\left(\frac{3}{2} + \frac{x}{3}\right)^n$ when x = 3 is numerically greatest then the possible integral value(s) of n can be

www.pdfworld.net

www.pdfworld.net

(A) 11 (B) 12
(C) 13 (D) 14

57. Consider the expansion of

$$\left(x^3 + 3(2)^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^3$$

(A) coefficient of x^2 is non-zero
(B) coefficient of x^2 is zero
(C) coefficient of x^3 is non-zero
(D) ratio of coefficient of x^3 to that of x^{-3} is 1 : 3

58. In the expansion of $\left(\sqrt[3]{4} + \frac{1}{\sqrt[4]{6}}\right)^{20}$

(A) the number of irrational terms = 19
(B) middle term is irrational
(C) the number of rational term = 2
(D) 9th term is rational

59. The square of any odd number is of the form :
(A) $8n+1$ (B) $6n+1$
(C) $4n+1$ (D) $2n+1$

60. Which of the following is correct ?
(A) $(101)^{100} > (100)^{101}$
(B) $(26)^{25} < (25)^{26}$
(C) $(300)^{299} < (299)^{300}$
(D) $(199)^{198} = (198)^{199}$

61. $7^9 + 9^7$ is divisible by
(A) 16 (B) 24
(C) 64 (D) 72

62. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1 x + a_2x^2 + \ldots.. + a_{20}x^{20}$, then
(A) $a_1 = 20$ (B) $a_2 = 210$
(C) $a_4 = 8085$ (D) $a_{20} = 2^2 . 3^7 . 7$

63. Which of the following expression(s), is(are) divisible by 1225?
(A) $6^{2n} - 35n - 1$ (B) $6^{2n} - 35n + 1$
(C) $6^{2n} - 35n$ (D) $6^{2n} - 35n + 1224$

64. The sum of the co–efficients in the expansion of $(1 - 2x + 5x^2)^n$ is a and the sum of the co–efficients in the expansion of $(1+x)^{2n}$ is b. Then
(A) $a = b$
(B) $(x - a)^2 - (x - b)^2 = 0$
(C) $\sin^2 a + \cos^2 b = 1$
(D) $ab = 1$

65. The series $\frac{{}^nC_0}{n} + \frac{{}^nC_1}{n+1} + \frac{{}^nC_2}{n+2} + \ldots.. + \frac{{}^nC_n}{2n}$ is equal to

(A) $\int_0^1 x^{n-1}(1-x)^n\,dx$

(B) $\int_0^1 x^{n-1}(1+x)^n\,dx$

(C) $\int_1^2 x^{n-1}(1+x)^n\,dx$

(D) $\int_1^2 x^n(1-x)^{n-1}\,dx$

66. If the sum of the coefficients in the expansion $(2 + 3cx + c^2 x^2)^{12}$ vanishes then c equals
(A) –2 (B) 2
(C) 1 (D) –1

67. Let $R = (8+3\sqrt{7})^{20}$ and [R] = the greatest integer less than or equal to R. Then
(A) [R] is even
(B) [R] is odd
(C) $R - [R] = 1 - \frac{1}{(8+3\sqrt{7})^{20}}$
(D) $R + R[R] = 1 + R^2$

68. If the expansion of $(3x + 2)^{-1/2}$ is valid in ascending powers of x the x lies in the interval.
(A) (0, 2/3)
(B) (–3/2, 3/2)
(C) (–2/3, 2/3)
(D) $(-\infty, -3/2) \cup (3/2, \infty)$

pdfworld.net

www.pdfworld.net

www.pdfworld.net

69. If $C_0, C_1, C_2, ...C_n$ have their usual meaning, then $\frac{C_0}{n(n+1)} - \frac{C_1}{(n+1)(n+2)} + \frac{C_2}{(n+2)(n+3)} - .....$ to (n + 1) terms is equal to

(A) $\int_0^1 x^{n+1}(1-x)^{n+1}\,dx$

(B) $\int_0^1 x^n(1+x)^{n+1}\,dx$

(C) $\int_0^1 x^{n-1}(1-x)^{n+1}\,dx$

(D) $\int_0^1 x^{n+1}(1-x)^{n-1}\,dx$

70. In the expansion of $(x + y + z)^{20}$

(A) coefficient of $x^7 y^8 z^7$ is zero

(B) total numer of distinct terms is 231

(C) every term is of the form $\frac{20!x^{20-r}y^{r-k}z^k}{(20-r)!(r-k)!k!}$

(D) sum of coefficients is 3^{20}

Comprehension - 1

Consider $f_n(x) = (1+x)(1+2x)(1+4x)$ .....
$...(1+2^n x) = a_{n,0} + a_{n,1}x + ...+ a_{n,n}x^n$

71. $a_{100,1}$ is

(A) $2^{101} - 1$ (B) $2^{100} - 1$

(C) $2^{101} + 1$ (D) $2^{100} + 1$

72. $a_{100,2} - a_{99,2}$ equals

(A) $2^{200} - 2^{99}$ (B) $2^{199} - 2^{99}$

(C) $2^{200} - 2^{99}$ (D) $2^{199} - 2^{100}$

73. $a_{100,2}$ equals

(A) $\frac{(2^{101}-1)(2^{101}-2)}{3}$

(B) $\frac{(2^{100}-1)(2^{101}-2)}{3}$

(C) $\frac{(2^{101}-1)(2^{101}-2)}{2}$

(D) $\frac{(2^{100}-1)(2^{101}-2)}{2}$

Comprehension - 2

Consider the identity $(1+ x + x^2 + x^3)^n$
$= a_0 + a_1x + a_2x^2 + a_3x^3 + ...+a_{3n}x^{3n}$
The coeffcients of various powers of x can be found in two ways. One method is to differentiate the identity certain number of times and then put x = 0 on both sides of the identity.
For example, a_2 is obtained by differentiating both sides twice and put x = 0 i.e., $a_2 = \frac{n(n+1)}{2}$.
Another method is to expand $(1 + x + x^2 + x^3)^n$ as $(1 + x)^n (1 + x^2)^n$ and collect the coefficients of the required power of x.

74. The value of a_{3n-2} is

(A) $\frac{(n+1)}{2}$ (B) $\frac{n(n+1)}{2}$

(C) $\frac{n^2-n+2}{2}$ (D) None of these

75. The coefficient of x^4 in the expression of $(1 + x + x^2 + x^3)^4$ is

(A) 31 (B) 32

(C) 33 (D) None of these

76. If n = 4, then the value of ${}^4C_0 - {}^4C_1 . a_1 + {}^4C_2 . a_2 - {}^4C_3 . a_3 + a_4$ is

(A) 4 (B) –4

(C) 0 (D) None of these

Comprehension - 3

Consider the identity $(1 + x)^{6m} = \sum_{r=0}^{6m} {}^{6m}C_r . x^r$. By putting different values of x on both sides, we can get summation of several series involving binomial coefficients. For example, by putting $x = \frac{1}{2}$ we get

$\sum_{r=0}^{6m} {}^{6m}C_r \frac{1}{2^r} = \left(\frac{3}{2}\right)^{6m}$.

www.pdfworld.net

www.pdfworld.net

77. The value of $\sum_{r=0}^{6m} {}^{6m}C_r \, 2^{r/2}$ is equal to

(A) $\frac{3^{6m}}{2}$ (B) $(1+\sqrt{2})^{3m}$

(C) $(3+2\sqrt{2})^{3m}$ (D) None of these

78. The value of $\sum_{r=0}^{3m} (-1)^r \, {}^{6m}C_{2r}$ is

(A) 2^{3m}

(B) 0 if m is odd

(C) -2^{3m} if m is even

(D) None of these

79. The value of $\sum_{r=0}^{3m} (-3)^{r-1} \, {}^{6m}C_{2r-1}$ is

(A) 0 (B) 1

(C) depends on m (D) None of these

Comprehension - 4

If m, n, r are positive itnegers and if r < m, r < n, then

${}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots\ldots + {}^nC_r$

= Coefficient of x^r in $(1+x)^m (1+x)^n$

= Coefficient of x^r in $(1+x)^{m+n} = {}^{m+n}C_r$

80. The value of

${}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + \ldots\ldots + {}^nC_n \cdot {}^nC_0$ is

(A) ${}^{2n}C_{n-1}$ (B) ${}^{2n}C_n$

(C) ${}^{2n}C_{n+1}$ (D) ${}^{2n}C_2$

81. The value of r for which

${}^{30}C_r \cdot {}^{20}C_0 + {}^{30}C_{r-1} \cdot {}^{20}C_1 + \ldots\ldots + {}^{30}C_0 \cdot {}^{20}C_r$ is maximum, is

(A) 10 (B) 15

(C) 20 (D) 25

82. The value of r $(0 \le r \le 30)$ for which

${}^{20}C_r \cdot {}^{10}C_0 + {}^{20}C_{r-1} \cdot {}^{10}C_1 + \ldots\ldots + {}^{20}C_0 \cdot {}^{10}C_r$ is minimum, is

(A) 0 (B) 1

(C) 5 (D) 15

Comprehension - 5

Let P be a product given by $P = (x+a_1)(x+a_2)\ldots\ldots (x+a_n)$ and Let $S_1 = a_1 + a_2 + \ldots\ldots + a_n = \sum_{i=1}^{n} a_i$,

$S_2 = \sum\sum_{i<j} a_i \cdot a_j$, $S_3 = \sum\sum\sum_{i<j<k} a_i \cdot a_j \cdot a_k$ and so on then it can be shown that $P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots + S_n$.

83. The coefficient of x^8 in the expression $(2+x)^2 (3+x)^3 (4+x)^4$ must be

(A) 26 (B) 27

(C) 28 (D) 29

84. The coefficient of x^{203} in the expression $(x-1)(x^2-2)(x^2-3)\ldots\ldots(x^{29}-20)$ must be

(A) 11 (B) 12

(C) 13 (D) 15

85. The coefficient of x^{98} in the expression of $(x-1)(x-2)\ldots\ldots(x-100)$ must be

(A) $1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2$

(B) $(1+2+3+\ldots\ldots+100)^2 - (1^2+2^2+3^2+\ldots\ldots+100^2)$

(C) $\frac{1}{2}[(1+2+3+\ldots\ldots+100)^2 - (1^2+2^2+3^2+\ldots\ldots+100^2)]$

(D) None of these

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true, R is false.

(D) A is false, R is true.

86. **Assertion (A) :** The sum of the coefficients of p(x) if $(20x^{27} + 2x^2 + 1) P(x) = 2001x^{2001}$ is 87

Reason (R) : The sum of coefficient of f(x) is the value of f(1) for any polynomial f.

www.pdfworld.net

www.pdfworld.net

87. Assertion (A) : The term independent of x in the expansion of $\left(x+\frac{1}{x}+2\right)^m$ is $\frac{(2m)!}{(m!)^2}$.

Reasons (R) : The coefficient of x^b in the expansion of $(1+x)^n$ is ${}^nC_b$.

Reasons (R) : Any positive integral power of $\sqrt{2}-1$ can be expressed as $A+B\sqrt{2}$ where A and B are integers.

88. Assertion (A): If n is an odd prime then $[(\sqrt{5}+2)^n]-2^{n+1}$ is not divisible by 20 n, where [.] denotes greatest integer function

Reasons (R): If n is prime then ${}^nC_1$, ${}^nC_2$, ... ${}^nC_{n-1}$ must be divisible by n.

89. Assertion (A): If n is even then ${}^{2n}C_1+{}^{2n}C_3+{}^{2n}C_5+.....+{}^{2n}C_{n-1}=2^{2n-1}$.

Reasons (R): ${}^{2n}C_1+{}^{2n}C_3+{}^{2n}C_5+...+{}^{2n}C_{2n-1}=2^{2n-1}$.

90. Assertion (A): The number $3^{512}-1$ is divisible by 2^{10} and not by 2^{11}.

Reason (R): $(1+2)^{512}-1={}^{512}C_1.2+{}^{512}C_2\,2^2+.....+{}^{512}C_{512}\,2^{512}=2^{10}$ x integer.

91. Assertion (A): The value of $\sqrt{2\sqrt[3]{2\sqrt[4]{2\sqrt[5]{2.....}}}}$ is 2^{e-1}

Reason (R): The value of $1+\frac{1}{2}\left(1+\frac{1}{3}\left(1+\frac{1}{4}(1+...)\right)\right)$

$=1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\cdots=e-1$

92. Assertion (A): The value of $\{\ell n\ (1+e)\}+\{\ell n\,(1+e^2)\}+\{\ell n\ (1+e^4)\}+\{\ell n\ (1+e^8)\}+$ ..... (where {x} is the fractional part of x), is $1-\ell n\ (e-1)$.

Reason (R): Since $\ell n(1+e^{2^k})$ is just larger then 2^k, its fractional part is $\ell n(1+e^{2^k})-\ell n\,e^{2^k}$ $=\ell n(1+e^{-2^k})$.

Now $\sum_{k=0}^{\infty}\ell n(1+e^{-2^k})=\ell n\left(\prod_{k=0}^{\infty}(1+e^{-2^k})\right)$

$=\ell n\left(\left(\frac{1-e^{-1}}{1-e^{-1}}\right)\prod_{k=0}^{\infty}(1+e^{-2^k})\right)=\ell n\frac{e}{e-1}$

93. Assertion (A): The last two digits of 7^{7^7} is 23.

Reason (R) : $7^7=(8-1)^7=4k+3$; $^{4k+3}$ $=7^3(7^4)^k=7^3(2400+1)^k=100m+7^3$, where k and m are integers.

94. Assertion (A) : The largest positive integer such that $\frac{2007!}{2007^n}$ is an integer is 9.

Reason (R) : We have $2007=3^2.223$. Using the fact that the number of times a prime p divides n! is given by $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+.....$, it follows that the answer is 9.

95. Assertion (A): The largest integer n such that $7^{2048}-1$ is divisible by 2^n is 12.

Reason (R) : $7^{2048}-1=(7-1)(7+1)(7^2+1)(7^4+1)$ ..... $(7^{1024}+1)$. For positive integer, $7^{2k}+1=(8-1)^{2k}+1=8m+2$, is divisible by 2 exactly once.

pdfworld.net

www.pdfworld.net

www.pdfworld.net

Match the Columns for JEE Advanced

96.

	Column - I	Column - II
(A)	The value of ${}^{2003}C_1 + {}^{2003}C_4 + {}^{2003}C_7 + \ldots\ldots {}^{2003}C_{2002}$ is	(P) 1002
(B)	The value of $\left(\frac{1}{1}+\frac{1}{2002}\right)\left(\frac{1}{2}+\frac{1}{2001}\right)\left(\frac{1}{3}+\frac{1}{2000}\right)\ldots\left(\frac{1}{1001}+\frac{1}{1002}\right)$ is	(Q) $2^{2003}-3$
(C)	The number of integers n between 4 and 2^{2003}, inclusive for which 2. 6. 10...(4n – 2) is divisible by $\lfloor n$ is	(R) $\frac{2^{2003}-2}{3}$
(D)	The value of $\frac{y_1y_2y_3}{(y_1-x_1)(y_2-x_2)(y_3-x_3)}$ when $(x_i + y_i)$, i = 1, 2, 3 satisfy both $x^3 - 3xy^2 = 2005$ and $y^3 - 3x^2y = 2004$, is	(S) $\frac{(2003)^{1001}}{\lfloor 2002}$

97.

	Column - I	Column - II
(A)	If λ be the number of terms in the expansion of $(5^{1/6} + 7^{1/9})^{1824}$ which are integers then, λ is divisible by	(P) 2
(B)	If λ be the number of terms in the expansion of $(5^{1/6} + 2^{1/8})^{100}$ which are rational, then λ is divisible by	(Q) 3
(C)	If λ be the number of terms in the expansion of $(3^{1/4} + 4^{1/3})^{99}$ which are irrational, then λ is divisible by	(R) 7
(D)	If in the expansion of $\left(x^3 - \frac{1}{x^2}\right)^{\lambda}$, $\lambda \in N$ sum of the coefficients of x^5 and x^{10} is zero, then λ is divisible by	(S) 13 (T) 17

98.

	Column - I	Column - II
(A)	If λ be the number of terms in the expansion of $(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)^{20}$ and if unit's place and ten's place digits in 3^{λ} are O and T, then	(P) O + T = 3
(B)	If λ be the number of terms in the expansion of $\left(x^2 + 1 + \frac{1}{x^3}\right)^{100}$ and if unit's place and ten's place digits in 7^{λ} are O and T, then	(Q) O + T = 7
(C)	If λ be the number of terms in the expansion of $(1 + x)^{101}(1 + x^2 - x)^{100}$ and if unit's place and ten's place digits in 9^{λ} are O and T, then	(R) O + T = 9
(D)	If last digit of the number 9^{9^9} is T and last digit of $2^{T^{100}}$ is O, then	(S) T – O = 7 (T) O – T = 7

www.pdfworld.net

www.pdfworld.net

99.

	Column - I	Column - II
(A)	If n be the degree of the polynomial $\sqrt{(3x^2+1)}\{(x+\sqrt{(3x^2+1)})^7-(x-\sqrt{(3x^2+1)})^7\}$, then n divisible by	(P) 2
(B)	In the expansion of $(x+a)^n$ there is only one middle term for $x=3$, $a=2$ and seventh term is numerically greatest term, then n divisible by	(Q) 4
(C)	The sum of the binomial coefficients in the expansion of $(x^{-3/4}+n\,x^{5/4})^m$, where m is positive integer, lies between 200 and 400 and the term independent of x equals 448. Then n^5 divisible by	(R) 8
(D)	If sum of the coefficients of the first, second and third terms in the expansion of $\left(x^2+\frac{1}{x}\right)^n$ is 46, then coefficient of the term that does not contain x is	(S) 16
		(T) 84

100.

	Column - I	Column - II
(A)	If last digit of the number 2^{999} is λ and last digit of $3^{5\lambda}$ is μ, then $\lambda+\mu=$	(P) 9
(B)	For a positive integer n, if the expansion of $\left(\frac{5}{x^2}+x^4\right)^n$ has term independent of x, then n can be	(Q) 15
(C)	If ${}^mC_3+{}^mC_4>{}^{m+1}C_3$, then least value of m is	(R) 18
(D)	Sum of the values of n satisfying the equation $\left(\frac{n!}{4!}\right)^2+\frac{(7!)(5!)}{4\cdot 4!}=240\left(\frac{n!}{4!}\right)$	(S) 13
		(T) 7

Review Exercises for JEE Advanced

1. The binomial coefficients of the third and tenth terms of the expansion of $(5x^{-3/2}-x^{1/3})^n$ are equal. Find out which term in the expansion does not contain x.

2. In the expansion of $\left(\frac{x\sqrt[3]{x}}{6}+x^{\frac{-28}{15}}\right)^n$ the sum of the coefficients for the last three terms equals 79. Find the term independent of x.

3. Show that the term independent of x in the expansion of $\left(1+x+\frac{6}{x}\right)^{10}$ is $1+\sum_{r=1}^{5}{}^{10}C_{2r}\,{}^{2r}C_r\,6^r$.

4. If n is an odd positive integer and a, b, c are distinct, find the number of distinct terms in the expansion of $(a+b+c)^n+(a+b-c)^n$.

5. Prove that the co-efficient of x^r in $\{(r-2)x^2+nx-r\}(x+1)^n$ is $n.{}^nC_{r-2}$, $n\in N$.

6. In the expansion of the expression $(x+a)^{15}$, if the eleventh term is the geometric mean of the eighth and twelth terms, which term in the expansion is the greatest?

www.pdfworld.net

www.pdfworld.net

7. If x denotes $\left(2+\sqrt{3}\right)^n$, $n \in N$ & [x] the integral part of x then find the value of $x - x^2 + x[x]$.

8. If p is an odd prime prove that the integral part of $(2+\sqrt{5})^p - 2^{p+1}$ is divisible by 20 p.

9. Find the

(i) last digit of the number $9^{(9^9)}$ and of the number $2^{(3^4)}$.

(ii) last two digits of the numbers 2^{999} and 3^{999}.

(iii) last two digits of the number $14^{(14^{14})}$.

(iv) last three digits of 17^{256}.

10. If ${}^nJ_r = \dfrac{(1-x^n)(1-x^{n-1})(1-x^{n-2})\ldots\ldots(1-x^{n-r+1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^r)}$, prove that ${}^nJ_{n-r} = {}^nJ_r$.

11. If $\sum_{r=0}^{n}(r^2+r+1)\,.\,r! = 2000\,.\,(2000)!$. Find the value of n.

12. If n is an integer greater than 1, show that ; $a - {}^nC_1(a-1) + {}^nC_2(a-2) - \ldots\ldots + (-1)^n(a-n) = 0$

13. With usual notations, prove that

$${}^2C_1 + {}^2C_2 \cdot \frac{2^3}{3} + {}^3C_3 \cdot \frac{2^4}{4} + \ldots + {}^nC_n \cdot \frac{2^{n+1}}{n+1} = n\,.\,3^n\left[\frac{3}{n+1} - \frac{1}{n}\right].$$

14. Sum the series S

$$= C_1^2 + \frac{1+2}{2}C_2^2 + \frac{1+2+3}{3}C_3^2 + \ldots \text{ upto n terms.}$$

15. Prove that the sum of the elements in the ninth row of Pascal's Triangle equals the sum of the elements of all previous rows, with 1 added.

16. What is the sum of all numbers of the form $\dfrac{12!}{a!b!c!}$ where a, b, c range over all non-negative integers satisfying $a + b + c = 12$?

17. Prove that $(n-1)^2\,.\,C_1 + (n-3)^2\,.\,C_3 + (n-5)^2\,.\,C_5 + \ldots\ldots\ldots = n(n+1)^{2n-3}$.

18. If the coefficient of x^n in the expansion of $\left(1 + \sum_{k=0}^{n} k.x^k\right)^2$ is 310, then find the value of n.

19. Prove that for $x > 1$ $px^q - qx^p - p + q > 0$ (p, q positive integers and $q > p$).

20. Prove the identity

$$(x^2+a^2)^n = \{x^n - {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 - \ldots.\}^2 + \{{}^nC_1x^{n-1}a - {}^nC_3x^{n-3}a^3 + \ldots\}^2.$$

21. Prove the equality

$$1^2 + 2^2 + \ldots. + n^2 = {}^{n+1}C_2 + 2({}^nC_2 + {}^{n-1}C_2 + \ldots\ldots + {}^2C_2).$$

22. Evaluate $\dfrac{C_0}{2} - \dfrac{C_1}{6} + \dfrac{C_2}{10} - \dfrac{C_3}{14} + \ldots. + \dfrac{(-1)^n C_n}{4n+2}$

23. Prove the inequality

$$\binom{2n+x}{n}\binom{2n-x}{n} \le \binom{2n}{n}^2$$

24. The expressions $1 + x$, $1 + x + x^2$, $1 + x + x^2 + x^3$,..... $1 + x + x^2 + \ldots\ldots + x^n$ are multiplied together and the terms of the product thus obtained are arranged in increasing powers of x in the form of $a_0 + a_1x + a_2x^2 + \ldots\ldots$, then

(a) how many terms are there in the product.

(b) show that the coefficients of the terms in the product, equidistant from the beginning and end are equal.

(c) show that the odd coefficients = the sum of the even coefficients = $\dfrac{(n+1)!}{2}$

www.pdfworld.net

www.pdfworld.net

25. If $S_n = {}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 \ldots\ldots\ldots\ldots + n.{}^nC_n$ then find $\sum_{n=1}^{n} S_n$ .Also prove that ${}^nC_1.({}^nC_2)^2.({}^nC_3)^3 \ldots\ldots\ldots ({}^nC_n)^n \le \left(\frac{2^n}{n+1}\right)^{{}^{n+1}C_2}$.

26. If $(1 + x + x^2)^{2n} = a_0 + a_1x + a_2x^2 + \ldots. + a_{4n}x^{4n}$, show that $a_0 + a_2 + a_4 + \ldots.. + a_{2n} = \frac{1}{4}(9^n + 1 + 2a_{2n})$

27. Prove that , $1 + \frac{2}{6} + \frac{2}{6}\cdot\frac{5}{12} + \frac{2}{6}\cdot\frac{5}{12}\cdot\frac{8}{18} +$ ..... ad infinitum $= \sqrt[3]{4}$.

28. Prove that the coefficient of x^r in the expansion of $(1-4x)(1-4x)^{-\frac{1}{2}}$ is $\frac{(2r)!}{(r!)^2}$

29. Find the coefficient of x^n in the expansion of

(i) $\frac{3x^2 - 2}{x + x^2}$

(ii) $(1 - 2x + 3x^2 - 4x^3 + \ldots.)^{-n}$.

30. Show that the middle term of $\left(x + \frac{1}{x}\right)^{4n}$ is equal to the coefficient of x^n in the expansion of $(1-4x)^{-\left(n+\frac{1}{2}\right)}$

31. If $x > 0$, which is the first negative term in the expansion of $(11 - 13x) / (1 - x)^3$?

32. Prove that : $\Sigma_1^{\infty} \frac{2n-1}{(n+3)(n!)} = \frac{1}{3}(43 - 15e)$.

33. Prove that $(6 + 6x + x^2)\log_e(1 + x) - (6x + 3x^2)$

$= x^5\left\{\frac{1.2}{3.4.5} - \frac{2.3}{4.5.6}x + \frac{3.4}{5.6.7}x^2 - \ldots\right\}$

34. Sum $\Sigma_{n=1}^{n=\infty} \frac{n^2+1}{n(n+2)}x^n$

35. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2 - 2)^2$ $P_3(x) = ((x-2)^2 - 2)^2 - 2)^2$..... . In general $P_k(x) = (P_{k-1}(x) - 2)^2$ then, find the constant term in $P_k(x)$.

Target Exercises for JEE Advanced

1. Find the value of x for which the fourth term in the expansion,

$\left(5^{\frac{2}{5}\log_5\sqrt{4^x + 44}} + \frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336 .

2. If a, b, c be three consecutive co–efficients in the expansion of some power of $(1 + x)$, prove that the power is $\frac{2ac + b(a+c)}{b^2 - ac}$. Also find that which term of the expansion has the coefficient 'a' in terms of a, b & c .

3. In the binomial expansion of, $\left(\sqrt{y} + \frac{1}{(2\sqrt[4]{y})}\right)^n$ the first three co-efficients form an A.P. in order. Find other terms in the expansion of which the power of 'y' is a natural number $n \in N$.

4. Find the remainder when $32^{32^{32}}$ is divided by 7.

5. Find the coefficient of x^{49} in the polynomial $\left(x - \frac{C_1}{C_0}\right)\left(x - 2^2\frac{C_2}{C_1}\right)\left(x - 3^2\frac{C_3}{C_2}\right)\ldots\ldots\ldots\ldots$ $\left(x - 50^2\frac{C_{50}}{C_{50}}\right)$ where $C_r = {}^{50}C_r$.

6. Prove that $[(\sqrt{7} + \sqrt{3})^{2n}] + 1 \pm$ where [.] represents the integral part, is divisible by 2^{2n} for all n.

7. Prove that $10^n - (5 + \sqrt{17})^n - (5 - \sqrt{17})^n$ is divisible by 2^{n+1}.

8. If n be a positive integer then prove that the integral part P of $(5 + 2\sqrt{6})^n$ is an odd integer. If f be the fractional part of $(5 + 2\sqrt{6})^n$, prove that $P = \frac{1}{1-f} - f$

9. Let $E = (2n+1)(2n+3)(2n+5)\ldots.(4n-3)(4n-1)$, $n > 1$. Then prove that 2^nE is divisible by ${}^{4n}C_{2n}$.

www.pdfworld.net

www.pdfworld.net

10. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ $(n \in N\ ;\ C_r = {}^nC_r)$ and
$S_1 = C_0 + C_3 + C_6 + ...$
$S_2 = C_1 + C_4 + C_7 + ...$
$S_3 = C_2 + C_5 + C_8 + ...$
Show that the values of S_1, S_2 & S_3 are respectively given by, $\frac{1}{3}\left(2^n + 2\cos\frac{r\pi}{3}\right)$ with $r = n$ for S_1 ; $r = n-2$ for S_2 and $r = (n+2)$ for S_3.

11. If $(x+1)(x+2).....(x+n-1)(x+n) = A_0 + A_1x + ...+ A_nx^n$. Prove that
(A) $A_1 + 2A_2 + ... + nA_n = (n+1)!\left[\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n+1}\right]$
(B) $A_1 - 2A_2 + ... + (-1)^{n-1}\,nA_n = (n-1)!$
(C) $A_1 = n!\left[1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right]$

12. By considering the expansion of, $(e^x+1)^n - (e^x-1)^n$ in two different ways, prove that
$(n-1)^2 . C_1 + (n-3)^2 . C_3 + (n-5)^2 . C_5 + ... = n(n+1)2^{n-3}$, where $C_1, C_3, C_5, ...$ are the combinatorial co-efficients in the expansion of $(1+x)^n$.

13. If $(1+x+x^2)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + ... + C_{2n}x^{2n}$, prove that
$C_0C_2 - C_1C_3 + C_2C_4 - .... + C_{2n-2}C_{2n} - = C_{n+1}$

14. If $(1+x+x^2+x^3)^n = C_0 + C_1x + C_2x^2 + ... C_{3n}x^{3n}$.
(i) Find the value of $C_0 + C_1 + C_2 + ... + C_{3n}$
(ii) Prove that $C_0 + C_2 + C_4 + C_6 .... = C_1 + C_3 + C_5 + C_7 + ...$
(iii) Prove that $C_0 = C_{3n}$, $C_1 = C_{3n-1}$, $C_2 = C_{3n-2}$, etc.

15. Find the sum of : $3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18{}^nC_3$ + upto $(n+1)$ terms.

16. Evaluate : $\sum_{r=0}^{n}\frac{1}{(r+1)(r+3)}.{}^nC_rx^r$.

17. Prove that $\sum_{k=1}^{n-1}k(n-k)({}^nC_k)^2 = n^2\,{}^{2n-2}C_n$

18. Prove that $\frac{C_0}{1} - \frac{C_1}{5} + \frac{C_2}{9} - \frac{C_3}{13} + ...$
$(-1)^n\frac{C_n}{4n+1} = \frac{4^n.n!}{1.5.9.13.....(4n-3)(4n+1)}$.

19. If $(1+2x+2x^2)^n = a_0 + a_1x + a_2x^2 + ... a_nx^{2n}$ prove that $a_2 = 4n^2 - 2^{2-n}.n\,[1 + 3\,{}^{n-1}C_2 + ... + (2n-1)\,{}^{n-1}C_{n-1}]$

20. Show that for $m \le n$, ${}^mC_1.{}^nC_m - {}^mC_2.{}^{2n}C_m + {}^mC_3 - ..... + (-1)^{m-1}.{}^{mn}C_m = (-1)^{m-1}n^m$.

21. Prove that $\sum_{k=0}^{n}\left[\frac{n-2k}{n}.{}^{2n}C_k\right]^2 = \frac{2}{n}.{}^{2n-2}C_{n-1}$

22. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, prove that
$a_0 + a_3 + a_6 + ... = a_1 + a_4 + a_7 + ... = a_2 + a_5 + a_8 + ...$

23. By expanding $\left(2+x+\frac{1}{2}\right)^m$ in two different ways prove that $\sum_{r=0}^{m}\frac{2^{2m-2r}}{(2m-2r)!(r!)^2} = \frac{(4m)!}{(2m!)^3}$

24. Prove $\sum_{r=1}^{k}2^r\,{}^nC_r\,{}^{k-1}C_{r-1} = \sum_{r=1}^{k}{}^nC_r\,{}^{n+k-r-1}C_{n-1}$ (where $n \ge k$)

25. Find the value of $\sum_{m=p}^{n}{}^nC_m.{}^mC_p$.

26. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then show that :
$C_1(1-x) - \frac{C_2}{2}(1-x)^2 + \frac{C_3}{3}(1-x)^3 - .... + (-1)^{n-1}\frac{1}{n}(1-x)^n = (1-x) + \frac{1}{2}(1-x^2) + \frac{1}{3}(1-x^3) + ...+\frac{1}{n}(1-x^n)$

www.pdfworld.net

www.pdfworld.net

27. If P_r is the coefficient of x^r in the expansion of $(1+x)^2\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{2^2}\right)^2$ ..... then prove that $P_r = \frac{2^2}{2^r-1}(P_{r-1} + P_{r-2})$. Also show that $P_4 = \frac{1072}{315}$

28. Show that $\left(\frac{{}^nC_1}{{}^nC_0}\right)^2 - \left(\frac{2.{}^nC_2}{{}^nC_1}\right)^2 + \left(\frac{3.{}^nC_3}{{}^nC_2}\right)^2 - .....$ upto n terms $= \frac{n}{6}(14n^2 - 9n + 1)$

29. If $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$ and $\sum_{r=0}^{n} \frac{1}{{}^nC_r} = a$, then find the value of $\sum\sum_{0\le i<j\le n}\left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j}\right)$ in terms of 'a' and 'n'

30. Prove that

$${}^{2n+1}C_0^2 + {}^{2n+1}C_2^2 + {}^{2n+1}C_4^2 + ... + {}^{2n+1}C_{2n}^2$$
$$= {}^{2n+1}C_1^2 + {}^{2n+1}C_3^2 + {}^{2n+1}C_5^2 + ... + {}^{2n+1}C_{2n+1}^2$$

31. If $(1+2x+2x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, $n \in N$ then prove that

(i) $a_r = {}^nC_0 . {}^{2n}C_r + {}^nC_1 . {}^{2n-2}C_{r-2} + {}^nC_2 . {}^{2n-4}C_{r-4} + ...$

(ii) $a_r = 2^{r-n}({}^{2n}C_r + {}^nC_1 . {}^{2n-2}C_r + {}^nC_2 . {}^{2n-4}C_r + ...)$

32. Evaluate $\sum_{r=1}^{n} \frac{s_r}{2r+1} . {}^nC_r$ where s_r is the sum of the squares of the first r natural numbers.

33. If $(6 - 12x + 12x^2)^n = \sum_{r=0}^{2n} a_r x^r$, prove that $a_r = (-1)^r . 3^n . 2^r \{{}^{2n}C_r + {}^nC_1 . {}^{2n-2}C_r + {}^nC_2 . {}^{2n-4}C_r + ...\}$

34. Prove that : $\Sigma_1^{\infty} \frac{n-1}{(n+2)(n!)}x^n = \frac{1}{x^2}\left\{x^2 - 3x + 3)e^x + \frac{1}{2}x^2 - 3\right\}$

35. Sum $\Sigma_{n=1}^{n=\infty} \frac{(n+1)^3}{n(n+3)}x^n$

Previous Years' Questions (JEE Advanced)

A. Fill in the blanks

1. The larger of $99^{50} + 100^{50}$ and 101^{50} is ..... **[IIT - 1982]**

2. The sum of the coefficients of the polynomial $(1 + x - 3x^2)^{2163}$ is ..... **[IIT - 1982]**

3. If $(1+ax)^n = 1 + 8x + 24x^2 + ...$ then a = ... and n=..... **[IIT - 1983]**

4. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is ..... **[IIT - 1997]**

B. Mutiple Choice Questions with ONE correct answer

5. Given positive integers r > 1, n > 2 and that the coefficient of (3r)th and (r + 2)th terms in the binomial expansion of $(1 + x)^{2n}$ are equal. Then **[IIT - 1983]**

(A) n = 2r (B) n = 2r + 1
(C) n = 3r (D) None of these

6. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is **[IIT - 1983]**

(A) $\frac{405}{256}$ (B) $\frac{504}{259}$
(C) $\frac{450}{263}$ (D) None of these

pdfworld.net

www.pdfworld.net

www.pdfworld.net

7. If C_r stands for ${}^nC_r$, then the sum of the series

$$\frac{2.\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)}{n!} \text{ x } [C_0{}^2 - 2C_1{}^2 + 3C_2{}^2 - \ldots\ldots$$

$+(-1)n(n+1)\,C_n{}^2]$ where n is a positive integer, is equal to **[IIT - 1986]**

(A) 0 (B) $(-1)^{n/2}(n+1)$

(C) $(-1)^{n/2}(n+2)$ (D) $(-1)^n n$

8. The coefficient of x^{17} in the expansion of $[(x-1)(x-2)(x-3)\ldots\ldots\ldots(x-18)]$ is **[IIT - 1990]**

(A) 342 (B) 171/2

(C) –171 (D) 684

9. If the sum of the coefficients in the expansion of $(\alpha^2x^2 - 2\alpha x + 1)^{51}$ vanishes, then the value of α is **[IIT - 1991]**

(A) 2 (B) –1

(C) 1 (D) – 2

10. The coefficient of x^{53} in the expansion, $\sum_{m=0}^{100} {}^{100}C_m\ (x-3)^{100-m}\,2^m$ is **[IIT - 1992]**

(A) ${}^{100}C_{47}$ (B) ${}^{100}C_{53}$

(C) $-{}^{100}C_{53}$ (D) ${}^{-100}C_{100}$

11. The expansion $[x+(x^3-1)^{1/2}]^5 + [x-(x^3-1)^{1/2}]^5$ is a polynomial of degree **[IIT - 1992]**

(A) 5 (B) 6

(C) 7 (D) 8

12. The value of $C_0 + 3C_1 + 5C_2 + 7C_3 + \ldots\ldots + (2n+1)C_n$ is equal to **[IIT - 1993]**

(A) 2 (B) $2^n + n.2^{n-1}$

(C) $2^n.(n+1)$ (D) None of these

13. The largest term in the expansion of $(3+2x)^{50}$ where x = 1/5 is **[IIT - 1993]**

(A) 5^{th} (B) 51^{th}

(C) 7^{th} (D) 6^{th}

14. If $\sum_{r=0}^{n} \frac{1}{{}^nC_r} = a_n$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals **[IIT - 1998]**

(A) $(n-1)a_n$ (B) na_n

(C) $\frac{1}{2}na_n$ (D) None of these

15. Let n be an odd integer, if $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r\theta$ for every value of θ, then **[IIT - 1998]**

(A) $b_0 = 1, b_1 = 3$

(B) $b_0 = 0, b_1 = n$

(C) $b_0 = -1, b_1 = n]$

(D) $b_0 = 0, b_1 = n^2 - 3n + 3$

16. If in the expansion of $(1+x)^m\,(1-x)^n$, the coefficients of x and x^2 are 3 and –6 respectively, then m is **[IIT - 1999]**

(A) 6 (B) 9

(C) 12 (D) 24

17. For $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$ **[IIT - 2000]**

(A) $\binom{n+1}{r-1}$ (B) $2\binom{n+1}{r+1}$

(C) $2\binom{n+2}{r}$ (D) $\binom{n+2}{r}$

18. In the binomial expansion of $(a-b)^n$, $n \ge 5$, the sum of the 5th and 6th terms is zero. Then a/b equals **[IIT - 2001]**

(A) $\frac{n-5}{6}$ (B) $\frac{n-4}{5}$

(C) $\frac{5}{n-4}$ (D) $\frac{6}{n-5}$

19. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$ where $\binom{p}{q} = 0$, if $p < q$, is maximum when 'm' is **[IIT - 2002]**

(A) 5 (B) 10

(C) 15 (D) 20

www.pdfworld.net

www.pdfworld.net

20. Coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is **[IIT - 2003]**

(A) ${}^{12}C_6$ (B) ${}^{12}C_6+1$
(C) ${}^{12}C_6$ (D) ${}^{12}C_6+2$

21. If ${}^{n-1}C_r = (k^2-3)\,{}^nC_{r+1}$, then $k \in$ **[IIT - 2004]**
(A) $(-\infty, -2]$ (B) $[2, \infty)$
(C) $[-\sqrt{3}, \sqrt{3}]$ (D) $(\sqrt{3}, 2]$ **[IIT - 2010]**

22. The value of $\binom{30}{0}\binom{30}{10}-\binom{30}{1}\binom{30}{11}$
$+\binom{30}{2}\binom{30}{12}....+\binom{30}{20}\binom{30}{30}$ is where $\binom{n}{r}={}^nC_r$
[IIT - 2005]

(A) $\binom{30}{10}$ (B) $\binom{30}{15}$
(C) $\binom{60}{30}$ (D) $\binom{31}{10}$

23. For r = 0, 1, ..., 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$, and $(1+x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

(A) $B_{10}-C_{10}$ (B) $A_{10}\,(B^2_{10}-C_{10}A_{10})$
(C) 0 (D) $C_{10}-B_{10}$

C. Multiple Choice Questions with ONE or MORE THAN ONE correct answers

24. In the expansion of $\left(x+\frac{\alpha}{x^2}\right)^n$, $\alpha \neq 0$, if no term is independent of x, then n is - **[IIT - 2001]**
(A) 10 (B) 12
(C) 16 (D) 20

D. Subjective Problems

25. Given that $C_1 + 2C_2x + 3C_3x^2 + ...+ 2nC_{2n}x^{2n-1}$
$= 2n(1+x)^{2n-1}$ where $C_r = \frac{(2n)!}{r!(2n-r)!}$
r = 0, 1, 2,.....,2n prove that ${C_1}^2 - 2{C_2}^2 + 3{C_3}^2 - .....-2n{C_{2n}}^2 = (-1)^n nC_n$. **[IIT - 1979]**

26. Prove that $7^{2n} + (2^{3n-3})(3^{n-1})$ is divisible by 25 for any natural number n. **[IIT - 1982]**

27. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ...+C_nx^n$ then show that the sum of the products of the C_i's taken two at a time, represented by $\sum_{0\le i<j\le n}\sum C_iC_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2(n!)^2}$
[IIT - 1983]

28. Given $s_n = 1 + q + q^2 + ... + q^n$; $S_n = 1 + \frac{q+1}{2}+\left(\frac{q+1}{2}\right)^2+.....+\left(\frac{q+1}{2}\right)^n$ $q \neq 1$
Prove that ${}^{n+1}C_1 + {}^{n+1}C_2s_1 + {}^{n+1}C_3s_2 +.....$
$+ {}^{n+1}C_ns_n = 2^nS_n$ **[IIT - 1984]**

29. Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+4}$. **[IIT - 1988]**

30. Prove that $C_0 - 2^2C_1 + 3^2C_2 - ..... + (-1)^n (n+1)^2 C_n = 0, n > 2$, where $C_r = {}^nC_r$. **[IIT - 1989]**

31. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$, then show that $b_n = {}^{2n+1}C_{n+1}$. **[IIT - 1992]**

32. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and is an even positive integer. **[IIT - 1993]**

33. Let n be a positive integer and $(1+x+x^2)^n = a_0 + a_1x + ...+a_{2n}x^{2n}$. Show that ${a_0}^2 - {a_1}^2 .....+{a_{2n}}^2 = a_n$ **[IIT - 1994]**

34. Prove that $\frac{3!}{2(n+3)} = \sum_{r=0}^{n}(-1)^r\left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$
[IIT - 1997]

www.pdfworld.net

www.pdfworld.net

35. For any positive integers m, n(with $n \geq m$), let $\binom{m}{n} = {}^{n}C_{m}$. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \binom{m}{m} = \binom{n+1}{m+1}.$$

Hence or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + ... + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}. \quad \textbf{[IIT - 2000]}$$

36. Prove that $2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - ..... (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$ **[IIT - 2003]**

ANSWERS

Practice Problems—A

1. (i) 35 (ii) 56
2. No solution **3.** 10 **4.** 7
5. 9 **6.** No solution **7.** 17
8. x = 6, y = 2 **9.** x=18, y= 8
10. x = 6, y = 2

Practice Problems—B

1. 9
2. (i) $x^8 + 8x^6y + 24x^4y^2 + 32x^2y^3 + 16y^4$
(ii) $a^5 - 10a^4 + 40a^3 - 80a^2 + 80a - 32$
(iii) $64a^6 + 192a^5b + 240a^4b^2 + 160a^3b^3 + 60a^2b^4 + 12ab^5 + b^6$
3. (i) $a^{32} - 16a^{30}b^2 + 120a^{28}b^4 + ...$
(ii) $a^9 + 27a^8b + 324a^7b^2 + ...$
(iii) $y^7 + 7y^5 + 21y^3 + ...$
4. $\frac{64x^6}{729} - \frac{32x^4}{27} + \frac{20x^2}{3} - 20 + \frac{135}{4x^2} - \frac{243}{8x^4} + \frac{729}{64x^6}$
5. (a) $109\sqrt{2} - 89\sqrt{3}$; (b) 64
6. $x^8 + 4x^7 + 10x^6 + 16x^5 + 25x^4 + 16x^3 + 4x^2 + 4x + 1$
7. $1 + 8x + 20x^2 + 8x^3 - 26x^4 - 8x^5 + 20x^6 - 8x^7 + x^8$.
8. y = 5

Practice Problems—C

1. (i) $240\ a^4b^2$ (ii) $-\frac{429a^{14}}{16}$
(iii) $720\ a^2b^8$
2. (i) $-2099520\ x^3$; (ii) $126\sqrt[3]{a}$
3. (i) $40a^7b^3$ (ii) 18564
(iii) $12012\sqrt{2}x^5$
4. (i) $110565\ a^4$ (ii) ${}^{11}C_5 \frac{3^6}{5^5}$
(iii) $\frac{2^5}{3^6} \times {}^{11}C_6$ **5.** 11
6. (i) $-{}^{15}C_9 \cdot 3^9$ (ii) 7
7. (i) $(-1)^n \frac{(3n)!}{n!\,(2n)!}$ (ii) 120
8. $(-1)^p \frac{2n+1}{(p+1)!\,(2n-p)!} x^{2p-2n+1}$.
9. 15
10. (i) 990 (ii) 3660
11. x = 2

Practice Problems—D

1. (i) $252\sqrt[3]{x^2}$ (ii) $1716a^3\sqrt{ab^2}$
2. $90720x^{28}y^{12}$ **3.** $\frac{189a^{17}}{8}, -\frac{21}{16}a^{19}$
4. 6 **5.** ${}^{24}C_{14}.3^2.2^2$ **6.** 26

www.pdfworld.net

www.pdfworld.net

Practice Problems—E

1. 12 **3.** $T_3 = {}^{18}C_2\, x^{6.5} = 153x^{6.5}$
4. Eleventh term
5. T_4 and T_5 are equal and the greatest.
6. (i) $\frac{1}{2^{100}}\,{}^{100}C_{50}$ (ii) $\frac{4}{3}\left(\frac{9}{10}\right)^{12}$
(iii) $\frac{2^{21}}{3^{77}}\,{}^{105}C_{28}$ (iv) $\sqrt{10}.50^3\,{}^{20}C_7$
7. $5/8 < x < 20/21$
8. 1/2 **9.** $x = \frac{n-k}{n}$

Practice Problems—F

1. 4 **3.** 3, 03, 803 **7.** 9 and 2
8. 67

Practice Problems—G

3. 2^{2n+1} **4.** 415

Practice Problems—H

1. $(101!)^{100}$
3. (i) 1 (ii) –1.
4. 2^{39} **5.** $T_4 = \binom{10}{3}x^{11}$
6. (i) $2^n - 2(n+1)$ (ii) 0
(iii) 2^{2n-1} (iv) 2^{2n-1}

Practice Problems—I

1. 212993 **2.** $\frac{np}{2}(p+1)^n$

Practice Problems—K

3. n = 2, 4 **6.** 1

Practice Problems—L

3. (i) $1+\frac{3}{2}x+\frac{3}{8}x^2-\frac{1}{16}x^3$

(ii) $1-x+\frac{3}{2}x^2-\frac{5}{2}x^3$

(iii) $1-x+\frac{2}{3}x^2-\frac{10}{27}x^3$

4. $-\frac{429}{16}x^7$ **5.** $\frac{(r+1)(r+2)(r+3)}{1\,.\,2\,.\,3}x^r.$
6. $\frac{15015}{16}$
7. (i) $1-\frac{23x}{6}$ (ii) $\frac{2}{3}\left(1+\frac{x}{24}\right)$
8. $1 - 4x + 13x^2$ **10.** 8th term
11. 11th term **12.** 2

Practice Problems—M

1. 210 **2.** 6!
3. (i) 3360 (ii) 30
5. (a) 28, (b) 35

Practice Problems—N

2. $\frac{1}{2e}$ **3.** $1 - e^{-1}$ **4.** 2e
5. e **6.** e – 1 **7.** 3e / 2
8. 1 – 2/e **9.** 15 e **10.** –1/e
11. $\frac{1}{x}[(x-1)e^x + x + 1]$

Practice Problems—O

3. $c-\frac{b}{2}+\frac{1}{3};\ \frac{-c}{2}+\frac{b}{3}-\frac{1}{4};\ b=1,$
$c=\frac{1}{6};\ a=\frac{1}{2}$
4. $\log_e 2$ **5.** $2\log_e 2 - 1$
6. $2 - 2\log_e 2$
7. $\frac{1}{2\sqrt{x}}\log\frac{1+\sqrt{x}}{1-\sqrt{x}}+\frac{1}{2x}\log(1-x)$
8. $\frac{1}{4}\left[6+\left(x-\frac{1}{x}\right)\log\frac{1+x}{1-x}\right]$

Objective Exercises

1. B	**2.** C	**3.** B
4. B	**5.** B	**6.** D
7. C	**8.** B	**9.** B
10. D	**11.** B	**12.** D
13. C	**14.** C	**15.** C

pdfworld.net

www.pdfworld.net

www.pdfworld.net

16. A **17.** B **18.** A
19. B **20.** D **21.** B
22. D **23.** B **24.** B
25. C **26.** C **27.** A
28. A **29.** A **30.** B
31. B **32.** C **33.** B
34. A **35.** D **36.** B
37. C **38.** B **39.** B
40. C **41.** B **42.** A
43. B **44.** A **45.** A
46. B **47.** C **48.** B
49. D **50.** D **51.** ABCD
52. ABD **53.** ABCD **54.** AB
55. AB **56.** BCD **57.** BCD
58. ABCD **59.** ACD **60.** BC
61. AC **62.** ABC **63.** AD
64. ABC **65.** BD **66.** AD
67. BCD **68.** AC **69.** CD
70. ABCD **71.** A **72.** B
73. B **74.** B **75.** A
76. B **77.** C **78.** B
79. A **80.** B **81.** D
82. A **83.** D **84.** C
85. C **86.** A **87.** B
88. A **89.** B **90.** D
91. D **92.** A **93.** D
94. A **95.** D
96. (A) – R; (B) – S; (C) – Q; (D) – P
97. (A) – PQT; (B) – PT; (C) – RS; (D) – Q
98. (A) – P; (B) – QT; (C) – RS; (D) – S
99. (A) – PQR; (B) – PQRS; (C) – PQRST; (D) – T
100. (A) – P; (B) – PQR; (C) – T; (D)–S

Review Exercises for JEE Advanced

1. –1375 **2.** $\frac{11}{3888}$

4. $\frac{(n+1)(n+3)}{4}$ **6.** T_8 **7.** 1

9. (i) 9, 2 (ii) 89, 67
(iii) 36 (iv) 681

11. 1999 **14.** $\frac{1}{2}\left[{}^{2n}C_n\left(1+\frac{n}{2}\right)-1\right]$

16. 3^{12} **18.** 12

22. $2^{2n-1}\cdot\frac{(n!)^2}{2n!}$

24. (a) $\frac{n^2+n+2}{2}$ **25.** $(n-1)2^n+1$

27. $n=-\frac{2}{3}$; $x=-\frac{1}{2}$

29. (i) $(-1)^{n-1}$ (ii) $\frac{2n!}{n!\,n!}$

31. 11th term

34. $\frac{5}{4x}(x+2)+\frac{x}{1-x}+\frac{1}{2x^2}(5-x^2)\log(1-x)$

35. 4

Target Exercises for JEE Advanced

1. x = 0 or 1

2. $n=\frac{2ac+b(a+c)}{b^2-ac}$ and $r=\frac{a(b+c)}{b^2-ac}$

3. If $n=4$, then $T_1=y^2$. If $n=8$, $T_1=y^4$, $T_7=(35/8)\,y$

4. 4 **5.** 924 **14.** (i) 2^{2n}

15. 0

16. $\frac{(1+x)^{n+2}C_1x^{-n+2}C_0}{(n+1)(n+2)x^2}$

$-\frac{(1+x)^{n+3}\,{}^{-n+3}C_2x^2\,{}^{-n+3}C_1x^{-n+3}C_0}{(n+1)(n+2)(n+3)x^3}$

25. ${}^nC_r\,(3^{n-r}-2^{n-r})$ **32.** $n(n+3)\,2^{n-2}$

35. $\frac{4}{9x^2}(6+3x+2x^2)+$

$\frac{2x-x^2}{(1-x)^2}+\frac{1}{3x^3}(8-x^3)\log(1-x)$

Previous Years' Questions (JEE Advanced)

1. $(101)^{50}$ **2.** –1
3. $a=2$, $n=4$ **4.** 41 **5.** A
6. A **7.** C **8.** C
9. C **10.** C **11.** C
12. C **13.** B **14.** C
15. B **16.** C **17.** D
18. B **19.** C **20.** D
21. D **22.** A **23.** D
24. ACD

www.pdfworld.net